

Quantum Theory of Condensed Matter

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Room H33
Wednesdays at 16:15

Sheet 8

1. Wick's theorem

1. Show that, for a system of non-interacting fermions described by the Hamiltonian in the eigenvalue basis

$$H = \sum_{\alpha} \epsilon_{\alpha} c_{\alpha}^{\dagger} c_{\alpha},$$

the following relation for the many-body grandcanonical expectation values holds:

$$\langle c_{\alpha_1}^{\dagger} c_{\alpha_2}^{\dagger} c_{\alpha_3} c_{\alpha_4} \rangle = \langle c_{\alpha_1}^{\dagger} c_{\alpha_4} \rangle \langle c_{\alpha_2}^{\dagger} c_{\alpha_3} \rangle \delta_{\alpha_1 \alpha_4} \delta_{\alpha_2 \alpha_3} - \langle c_{\alpha_1}^{\dagger} c_{\alpha_3} \rangle \langle c_{\alpha_2}^{\dagger} c_{\alpha_4} \rangle \delta_{\alpha_1 \alpha_3} \delta_{\alpha_2 \alpha_4},$$

where

$$\langle c_{\alpha_1}^{\dagger} c_{\alpha_2}^{\dagger} c_{\alpha_3} c_{\alpha_4} \rangle \equiv \frac{1}{Z} \text{Tr} \{ c_{\alpha_1}^{\dagger} c_{\alpha_2}^{\dagger} c_{\alpha_3} c_{\alpha_4} \exp[-\beta(H - \mu N)] \}$$

and Z is the grandcanonical partition function. The trace is taken over the full Fock space.

2. Derive from point (a) that, for non-interacting fermions, in every other given single particle basis $\{|n\rangle\}$ the following relation holds:

$$\langle c_{n_1}^{\dagger} c_{n_2}^{\dagger} c_{n_3} c_{n_4} \rangle = \langle c_{n_1}^{\dagger} c_{n_4} \rangle \langle c_{n_2}^{\dagger} c_{n_3} \rangle - \langle c_{n_1}^{\dagger} c_{n_3} \rangle \langle c_{n_2}^{\dagger} c_{n_4} \rangle.$$

Note that this is valid even if in this basis the Hamiltonian

$$H = \sum_{n,m} t_{nm} c_n^{\dagger} c_m$$

would contain non-diagonal terms t_{nm} for $n \neq m$.

Hint: Diagonalize first H using a unitary transformation $c_n = \sum_{\alpha} u_{n\alpha} c_{\alpha}$. Apply then the equation proved in point (a). Finally perform the canonical transformation in the opposite direction.

2. The discrete jellium model

Consider a set of spinless fermions on a crystal lattice described by the Hamiltonian in the Wannier basis:

$$H = -t \sum_{\langle i,j \rangle} c_i^{\dagger} c_j + U \sum_{\langle i,j \rangle} \hat{n}_i \hat{n}_j$$

where $\langle i, j \rangle$ are the nearest neighbours sites i and j , \hat{n}_i is the counting operator $c_i^{\dagger} c_i$. Prove that, in the same spirit of the continuous jellium model, the Hamiltonian should be modified in order to include charge neutrality and thus should read:

$$H = -t \sum_{\langle i,j \rangle} c_i^{\dagger} c_j + U \sum_{\langle i,j \rangle} (\hat{n}_i - 1)(\hat{n}_j - 1)$$

Frohes Schaffen!