

Quantum Theory of Condensed Matter

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Room H33
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Sheet 3

1. Occupation number representation

Let us consider a fermionic system with two single particle states $|\phi_1\rangle$ and $|\phi_2\rangle$ that span the (two-dimensional) *one*-particle Hilbert space.

1. Which dimension has the *two*-particle Hilbert space? Which dimension has the Fock space? Write down the form of the basis of the Fock space explicitly as Slater determinants of the wave functions $\phi_1(\mathbf{r})$, $\phi_2(\mathbf{r})$ and in the occupation number representation.
2. Calculate in the Fock basis the matrix representation of the creation and annihilation operators c_i , c_i^\dagger ($i = 1, 2$) and also of the occupation operators $n_i = c_i^\dagger c_i$
3. Verify the anticommutator relations

$$[c_i, c_j]_+ = [c_i^\dagger, c_j^\dagger]_+ = 0, \quad [c_i, c_j^\dagger]_+ = \delta_{ij}$$

explicitly using matrix multiplication of the matrices calculated at point 2.

4. Consider a Hamilton operator

$$\hat{H} = \hat{T} + \hat{V},$$

where \hat{T} is a single particle operator and \hat{V} a two particle one. With respect to the single particle basis $|\phi_i\rangle$ the matrix elements are:

$$\begin{aligned} \langle \phi_i | \hat{T} | \phi_i \rangle &= \epsilon, & \langle \phi_i | \hat{T} | \phi_j \rangle &= t \text{ for } i \neq j \\ \langle \phi_1, \phi_2 | \hat{V} | \phi_1, \phi_2 \rangle &= U, & \langle \phi_1, \phi_2 | \hat{V} | \phi_2, \phi_1 \rangle &= J \end{aligned}$$

where the notation is such that, *e.g.*:

$$\langle \phi_1, \phi_2 | \hat{V} | \phi_2, \phi_1 \rangle \equiv \int d\mathbf{r}_1 d\mathbf{r}_2 \phi_1^*(\mathbf{r}_1) \phi_2^*(\mathbf{r}_2) V(\mathbf{r}_1, \mathbf{r}_2) \phi_1(\mathbf{r}_2) \phi_2(\mathbf{r}_1).$$

Remember that in second quantization a single and two particle operators are respectively written as:

$$\hat{T} = \sum_{\lambda, \mu} c_\lambda^\dagger \langle \phi_\lambda | \hat{T} | \phi_\mu \rangle c_\mu, \quad \hat{V} = \frac{1}{2} \sum_{\lambda\mu\lambda'\mu'} c_\lambda^\dagger c_\mu^\dagger \langle \phi_\lambda, \phi_\mu | \hat{V} | \phi_{\lambda'}, \phi_{\mu'} \rangle c_{\mu'} c_{\lambda'},$$

where $|\phi_\lambda\rangle$ represent a generic single particle basis and c_λ^\dagger the corresponding creation operator. Write the operator \hat{H} in second quantization and in the matrix representation (starting from the single particle basis introduced). Calculate the eigenvalues and eigenvectors for \hat{H} .

5. (Optional) Again, write \hat{H} in second quantization, but this time as a single particle basis use the eigenvectors of \hat{T} . Which is the connection between this creation and annihilation operators and the ones considered in the points 1.-4.? Is this a unitary transformation?

Frohes Schaffen!