Quantum Theory of Condensed Matter

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Room H33
Wednesdays at 16:15

Sheet 2

1. Bosonic commutation relations

Refresh the physics of the simple harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2},$$

which can be written in "second quantized" form, by expressing $\hat{x}$ and $\hat{p}$ in terms of *boson* creation and annihilation operators:

$$\hat{H} = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right), \quad a^\dagger = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, \hat{x} - \mathrm{i} \frac{\hat{p}}{\sqrt{m\omega}} \right).$$

From the canonical commutation relations between position and momentum operators, it follows immediately (do you remember it?) that the basis commutation relations hold:

$$[a, a^\dagger] = 1, \quad [a, a] = 0, \quad a\,|0\rangle = 0$$

where $[A, B] = AB - BA$, $|0\rangle$ is the vacuum, and $\dagger$ indicates the Hilbert space adjoint.

1. Show that for two non commuting bosonic operators A, and B it holds

$$[A, B^n] = \sum_{k=0}^{n-1} B^k \,[A, B]\, B^{n-1-k}.$$

2. Prove —using Ex. 1.1— one of the following relations valid for bosonic operators b, $b^\dagger$

$$\begin{aligned} \left[b, (b^\dagger)^n\right] &= n(b^\dagger)^{n-1} = \frac{\partial (b^\dagger)^n}{\partial b^\dagger}, \\ \left[b^\dagger, b^n\right] &= -nb^{n-1} = -\frac{\partial b^n}{\partial b}, \end{aligned}$$

2. Exponential of bosonic operators

A particular role is played in quantum mechanics by exponential operators. Time evolution, spatial translation and any transformation associated to a continuum symmetry group is represented by an exponential operator. Thus we dedicate a special exercise to them.

1. Using the previous arguments (Ex. 1.2) show that the following relation hold

$$g_1(\alpha; b, b^\dagger) = \mathrm{e}^{-\alpha b^\dagger} b \mathrm{e}^{\alpha b^\dagger} = b + \alpha$$

2. Simplify the following expression

$$g_2(\alpha; b, b^\dagger) = \mathrm{e}^{-(\alpha^* b^\dagger - \alpha b)} b \mathrm{e}^{(\alpha^* b^\dagger - \alpha b)}.$$

Hint: Introduce a "dummy" variable λ, consider the auxiliary function:

$$\tilde{g}_2(\lambda, \alpha; b, b^\dagger) = \mathrm{e}^{-\lambda(\alpha^* b^\dagger - \alpha b)} b \mathrm{e}^{\lambda(\alpha^* b^\dagger - \alpha b)}$$

and calculate the derivative $\partial \tilde{g}_2(\lambda, \alpha; b, b^\dagger)/\partial\lambda$. Notice that:

$$\begin{aligned} \tilde{g}_2(1, \alpha; b, b^\dagger) &= g_2(\alpha; b, b^\dagger) \\ \tilde{g}_2(0, \alpha; b, b^\dagger) &= b. \end{aligned}$$

3. Calculating with fermion operators

The basis commutation relations for fermion creation and annihilation operators are

$$[c, c^\dagger]_+ = 1, \quad [c, c]_+ = 0, \quad c|0\rangle = 0,$$

where $[A, B]_+ = AB + BA$, $|0\rangle$ the vacuum, and $\dagger$ indicates the Hilbert space adjoint. Similarly to exercise 2, simplify the following expressions involving, this time, the fermionic operators c, and $c^\dagger$

$$g(\alpha; c, c^\dagger) = \mathrm{e}^{-(\alpha^* c^\dagger - \alpha c)} c \mathrm{e}^{(\alpha^* c^\dagger - \alpha c)}, \quad h(\alpha; c, c^\dagger) = \mathrm{e}^{-\alpha c^\dagger c} c \mathrm{e}^{\alpha c^\dagger c}.$$

Frohes Schaffen!

Useful relations

We give here a list of useful identities. They are not particularly useful for the exercises of this Sheet. They will be used during the course. Some of them should be known by heart...
We denote with A, B and C generic operators, while with b and c bosonic and fermionic operators respectively.

$$
\begin{aligned}
[A, B^n] &= \sum_{k=0}^{n-1} B^k [A,B] B^{n-1-k}, \\
[AB, C] &= A[B,C] + [A,C]B, \\
[AB, C] &= A[B,C]_+ - [A,C]_+ B, \\
\mathrm{e}^A B \mathrm{e}^{-A} &= \sum_{m=0}^{\infty} \frac{1}{m!}[A,B]_m,
\end{aligned}
$$

with $[A,B]_m = [A,[A,B]_{m-1}]$ and $[A,B]_0 = B$.

$$
[b, b^\dagger] = 1, \quad [b,b] = 0, \quad b\,|0\rangle = 0,
$$

$$
\begin{aligned}
b^\dagger b|n\rangle &= n|n\rangle, \quad n = 0,1,2,\ldots \\
b|n\rangle &= \sqrt{n}|n-1\rangle, \\
b^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\
\left[b, f(b^\dagger)\right] &= \frac{\partial f(b^\dagger)}{\partial b^\dagger}, \\
\left[b^\dagger, f(b)\right] &= -\frac{\partial f(b)}{\partial b}, \\
\mathrm{e}^{-\alpha b^\dagger} b \mathrm{e}^{\alpha b^\dagger} &= b + \alpha, \\
\mathrm{e}^{-\alpha b} b^\dagger \mathrm{e}^{\alpha b} &= b^\dagger - \alpha,
\end{aligned}
$$

Baker-Campbell-Hausdorff formula

$$
\mathrm{e}^{\alpha b^\dagger}\mathrm{e}^{\beta b} = \exp\left(\alpha b^\dagger + \beta b - \frac{\alpha\beta}{2}\right),
$$

$$
[c, c^\dagger]_+ = 1, \quad [c,c]_+ = 0, \quad c\,|0\rangle = 0,
$$

$$
\begin{aligned}
c^\dagger c|n\rangle &= n|n\rangle, \quad n = 0,1, \\
c^\dagger|0\rangle &= |1\rangle, \\
c^\dagger|1\rangle &= 0, \\
\left[c, f(c^\dagger)\right]_+ &= \frac{\partial f(c^\dagger)}{\partial c^\dagger} = f(0), \\
\left[c^\dagger, f(c)\right]_+ &= \frac{\partial f(c)}{\partial c} = f(0), \\
\left[c, c^\dagger c\right] &= c, \\
\left[c^\dagger, c^\dagger c\right] &= -c^\dagger.
\end{aligned}
$$