University of Regensburg Summer Semester 2007

Quantum Theory of Condensed Matter

Prof. Milena Grifoni Room H33
Dr. Andrea Donarini Wednesdays at 13:15

Sheet 1

1. Rules of the game

Nothing else than "hands-on" approach helps to fix ideas, we will give than a lot of importance to the written exercises proposed on a weekly basis. In particular:

1. An exercise sheet is given at every Wednesday lecture. **Clearly written** solutions of the exercise sheets should be handed in by Tuesday of the following week at 12:00 in the post box of the course located in "*Treppenhaus 1. Etage* (*Bibliothek Physik*)". The exercise solutions will be then discussed during Wednesday exercise class.

2. One week of June is dedicated to the computational laboratory. This particular exercise sheet will consist of a task to be solved with a computer simulation. You will be asked to develop a simple code that allows you to calculate and visualize some of the results obtained in the course. Linux CIP pool will be available for coding and running the simulations. Group sizes of 2-3 people are encouraged. Any program language can be used for the calculation but assistance can be ensured only for MATLAB®.

2. Bosonic commutation relations

Refresh the physics of the simple harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2},$$

which can be written in "second quantized" form, by expressing $\hat{x}$ and $\hat{p}$ in terms of *boson* creation and annihilation operators:

$$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right), \quad a^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\hat{x} - \mathrm{i}\frac{\hat{p}}{\sqrt{m\omega}}\right).$$

From the canonical commutation relations between position and momentum operators, it follows immediately (do you remember it?) that the basis commutation relations hold:

$$[a, a^\dagger] = 1, \quad [a, a] = 0, \quad a\,|0\rangle = 0$$

where $[A, B] = AB - BA$, $|0\rangle$ is the vacuum, and $\dagger$ indicates the Hilbert space adjoint.

1. Show that for two non commuting bosonic operators A, and B it holds

$$[A, B^n] = \sum_{k=0}^{n-1} B^k\,[A, B]\,B^{n-1-k}.$$

2. Prove —using Ex. 1.1— one of the following relations valid for bosonic operators b, $b^\dagger$

$$\begin{aligned}\left[b, (b^\dagger)^n\right] &= n(b^\dagger)^{n-1} = \frac{\partial (b^\dagger)^n}{\partial b^\dagger}, \\ \left[b^\dagger, b^n\right] &= -nb^{n-1} = -\frac{\partial b^n}{\partial b},\end{aligned}$$

3. Exponential of bosonic operators

A particular role is played in quantum mechanics by exponential operators. Time evolution, spatial translation and any transformation associated to a continuum symmetry group is represented by an exponential operator. Thus we dedicate a special exercise to them.

1. Using the previous arguments (Ex. 2.2) show that the following relation hold

$$g_1(\alpha; b, b^\dagger) = \mathrm{e}^{-\alpha b^\dagger} b \mathrm{e}^{\alpha b^\dagger} = b + \alpha$$

2. Simplify the following expression

$$g_2(\alpha; b, b^\dagger) = \mathrm{e}^{-(\alpha^* b^\dagger - \alpha b)} b \mathrm{e}^{(\alpha^* b^\dagger - \alpha b)}.$$

   Hint: Introduce a "dummy" variable λ, consider the auxiliary function:

$$\tilde{g}_2(\lambda, \alpha; b, b^\dagger) = \mathrm{e}^{-\lambda(\alpha^* b^\dagger - \alpha b)} b \mathrm{e}^{\lambda(\alpha^* b^\dagger - \alpha b)}$$

   and calculate the derivative $\partial \tilde{g}_2(\lambda, \alpha; b, b^\dagger)/\partial\lambda$. Notice that:

$$\begin{aligned} \tilde{g}_2(1, \alpha; b, b^\dagger) &= g_2(\alpha; b, b^\dagger) \\ \tilde{g}_2(0, \alpha; b, b^\dagger) &= b. \end{aligned}$$

3. Prove the identity (**Campbell-Baker-Hausdorff identity**)

$$\mathrm{e}^{\alpha b^\dagger} \mathrm{e}^{\beta b} = \exp\left(\alpha b^\dagger + \beta b - \frac{\alpha\beta}{2}\right)$$

   Hint: Show first that the quantity

$$f(\lambda) = \mathrm{e}^{\lambda\alpha b^\dagger} \mathrm{e}^{\lambda\beta b} \mathrm{e}^{\lambda^2 \frac{\alpha\beta}{2}}$$

   satisfies the following relation

$$\frac{\partial f(\lambda)}{\partial\lambda} = \left(\alpha b^\dagger + \beta b\right) f(\lambda).$$

Frohes Schaffen!