Canonical Decomposition of L-functions at Nonpositive Integers

By

Julio de Mello Bezerra

Advisor: Prof. Dr. Guido Kings
Co-advisor: Dr. Johannes Sprang

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Notation

Throughout this work, we let

$\mathbb{N} = \mathbb{Z}_{\geq 0}$ be the set of natural numbers *including zero*.

F is a totally real field extension of $\mathbb{Q}$ of degree g.

I is the set of complex embeddings τ of F into $\mathbb{C}$ (where $\mathrm{im}(\tau) \subset \mathbb{R}$ since F is totally real).

For any subset $X \subset F$, we denote by X_+ the set of totally real elements of X, that is, the elements $x \in X$ such that $\tau(x) > 0$ for all $\tau \in I$.

We denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ once and for all.

We denote by $\overline{F}$ a fixed Galois closure of F with respect to $\mathbb{Q}$ and fix an embedding $\overline{F} \hookrightarrow \mathbb{C}$.

Elements are written in bold to emphasize the fact that they are tuples. When α is used, we also underline it to indicate that it is a tuple.

Introduction

The goal of this thesis, as the title suggests, is to canonically express the values of Hecke L-functions associated to nontrivial finite Hecke characters at nonpositive integers in terms of equivariant cohomology classes. These values will sometimes be referred to as "special values" of the L-function. We mostly follow the papers [8] and [9] by Bannai, Hagihara, Yamada and Yamamoto.

We begin the thesis by introducing different types of zeta functions and briefly showing how the values of these zeta functions at nonpositive integers may be obtained by differentiating a certain generating function associated to each zeta function. This is used as motivation for the development of the theory that follows, and is a common theme throughout the thesis. In the end, we show how the special values of the Hecke L-functions we are seeking are actually generated by differentiating a canonical class in equivariant cohomology.

Throughout the thesis we assume that the Hecke L-function is defined over a totally real number field F, since for number fields which are not totally real, the values of the Hecke L-functions in which we are interested are all zero. In chapters 1 to 3 we work only on the special case in which the narrow class number of the field F equals 1, but this assumption is only made for simplicity and is removed in chapter 4, where we work on the general case.

We finish chapter 1 by providing a non-canonical decomposition of the Hecke L-function in terms of a linear combination of certain zeta functions, so we may differentiate the generating functions associated to these zeta functions to obtain the special values of the Hecke L-functions we are after. This result is classic and based on the work of Shintani in [14].

In chapter 2 we introduce the theory of equivariant sheaf cohomology on G-schemes and apply it to the algebraic torus $\mathbb{T}$ associated to the number field F. We then show that there exists a canonical class in equivariant cohomology coming from the generating functions associated to the zeta functions we used in the decomposition of the L-function in chapter 1. This canonical class is called the Shintani generating class.

In the third chapter we prove that the special values of the Hecke L-functions can be canonically obtained from differentiating the Shintani generating class, after we specialize it to the torsion points of the algebraic torus $\mathbb{T}$. This shows that the "right" object to be considered is not a generating function, but a generating class in equivariant cohomology. Some notions from group (co)homology and algebraic topology are used in order to obtain the special values from the class in equivariant cohomology.

In the fourth and last chapter we generalize the methods of the previous chapters in order to obtain the special values canonically in the general case when the narrow class number of F may be greater than one. The main idea consists of considering instead infinitely many copies of tori $\mathbb{T}^{\mathfrak{a}}$ associated to the fractional ideals $\mathfrak{a}$ of F.

We finish this introduction by remarking that these special values are all algebraic and satisfy interesting congruence relations, giving rise to p-adic L-functions interpolating them. Equivariant cohomology is successfully used in the construction of these p-adic L-functions in [11] and [9], which highlights the importance of the theory developed in the present thesis. Unfortunately, these p-adic L-functions lie outside the scope of this thesis.

Chapter One

Hecke L-functions at Nonpositive Integers

1.1 Zeta Functions at Nonpositive Integers

We begin this section by briefly discussing some properties of the *Riemann zeta function* ζ and the *Gamma function* Γ, defined respectively as

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad \Gamma(s) := \int_0^{\infty} e^{-u} u^{s-1} du,$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and respectively for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. Both of these functions can be continued to a meromorphic function in $\mathbb{C}$ (see [1, Théorème VII.2.1] and Lemma 1.1.1 below).

A simple way to relate these two functions is by noting that, for $\lambda \in \mathbb{R}_+$,

$$\lambda^{-s}\Gamma(s) = \int_0^{\infty} e^{-t\lambda} t^{s-1} dt \tag{1.1}$$

after using the change of variables $u = \lambda t$. Through a simple contour integration argument, this can be extended to $\lambda \in \mathbb{C}$, $\mathrm{Re}(\lambda) > 0$. Then one can set $\lambda = n$ and sum over all natural numbers, and after justifying the interchange of sum and integral, we get

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nt} t^{s-1} dt.$$

Thus, by setting $G(t) := \frac{e^{-t}}{1-e^{-t}} = \sum_{n=1}^{\infty} e^{-nt}$, we have:

$$\zeta(s)\Gamma(s) = \int_0^\infty G(t)t^{s-1}dt. \tag{1.2}$$

More conceptually, the integral on the right hand side is defined as the *Mellin transform* of the function $G(t)$. The following Lemma can be used to give an analytic continuation of ζ and to evaluate it at negative integers:

Lemma 1.1.1. *Suppose $f \in C^\infty(\mathbb{R}_{\geq 0})$ and all of its derivatives rapidly decrease (that is, they are $O(|x|^{-N})$ as $x \to \infty$ for all $N \in \mathbb{N}$). Then $M(f,s) := \frac{1}{\Gamma(s)}\int_0^\infty f(t)t^{s-1}dt$, defined for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, admits an analytic continuation to all of $\mathbb{C}$, and for $k \in \mathbb{N}$, $M(f,-k) = (-1)^k(\frac{d}{dt})^k f(t)\big|_{t=0}$.*

Proof. See [1, Proposition VII.2.6]. □

Note that we cannot apply this Lemma directly, since $G(t)$ is not well defined at $t = 0$. Instead, we use $F(t) := tG(t)$ and the identity

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty G(t)t^{s-1}dt = \frac{1}{(s-1)\Gamma(s-1)}\int_0^\infty F(t)t^{s-2}dt = \frac{1}{s-1}M(F,s-1). \tag{1.3}$$

Now we may use Lemma 1.1.1 to deduce that $\zeta(-k) = \frac{(-1)^k B_{k+1}}{k+1}$, where $B_k \in \mathbb{Q}$ is the *k-th Bernoulli number*, defined by

$$\sum_{n=0}^\infty B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} = \frac{te^{-t}}{1 - e^{-t}} = F(t).$$

The above discussion can be generalized to *Dirichlet L-functions*, defined for a multiplicative character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ as

$$L(s,\chi) := \sum_{n=1}^\infty \chi(n)n^{-s},$$

where $\chi(n) := \chi(n \bmod N)$ if $\gcd(n,N) = 1$ and $\chi(n) = 0$ otherwise. This series is absolutely convergent for $\mathrm{Re}(s) > 1$.

In order to proceed with the generalization it is convenient to introduce another zeta function, the *Hurwitz zeta function*, defined as

$$\zeta(s,x) := \sum_{n=0}^{\infty}(n+x)^{-s}$$

for $\mathrm{Re}(s) > 1$ and $0 < x \leq 1$. Now we observe that we can obtain the Dirichlet L-function as a linear combination of Hurwitz zeta functions in the following manner:

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s} = \sum_{a=1}^{N}\chi(a)\sum_{n=1}^{\infty}(Nn+a)^{-s} = \sum_{a=1}^{N}\chi(a)N^{-s}\sum_{n=1}^{\infty}\left(n+\frac{a}{N}\right)^{-s} = \sum_{a=1}^{N}\chi(a)N^{-s}\zeta\left(s,\frac{a}{N}\right).$$

We have thus reduced the problem of analytically continuing $L(s,\chi)$ and determining the values of $L(-k,\chi)$ to doing so for $\zeta(s,x)$.

Now we may proceed as in the case of the Riemann zeta, obtaining

$$\Gamma(s)\zeta(s,x) = \int_0^{\infty} G(t,x)t^{s-1}dt$$

for $G(t,x) := \frac{e^{-xt}}{1-e^{-t}} = \sum_{n=0}^{\infty} e^{-(n+x)t}$. Applying the same reasoning as in (1.3) and Lemma 1.1.1 we get the desired analytic continuation, as well as

$$\zeta(-k,x) = -\frac{B_{k+1}(x)}{k+1} \quad \text{and} \quad L(-k,\chi) = -\sum_{a=1}^{N}\chi(a)N^k\frac{B_{k+1}(a/N)}{k+1},$$

where $B_n(x) \in \mathbb{Q}[x]$ is the *n-th Bernoulli polynomial*, defined by

$$\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} = t\frac{e^{tx}}{e^t-1} = tG(t,1-x) = tG(-t,x)$$

and satisfying

$$B_n(x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}B_jx^{n-j}$$

with B_j the previously defined j-th Bernoulli number (see [3, Proposition 4.9]).

We also remark that both in the case of the Riemann zeta as well as in the case of the Hurwitz zeta, the meromorphic continuation may be achieved after converting the corresponding integral to a contour integral which avoids the unique singularity of G at zero (see [10, Chapter 2]).

The next step would be to generalize this method to Hecke L-functions, but this is a much more difficult task. In his breakthrough article [14], Shintani showed how such a generalization could be obtained for Hecke L-functions with finite Hecke characters associated to totally real number fields, to be introduced in the next sections. Following Shintani's work, it is possible to decompose the L-function as a linear combination of now-called Shintani zeta functions, similarly to the case of the Hurwitz zeta functions previously, which in fact occur as special cases. Unlike in the case of the Dirichlet L-functions above, though, this decomposition does not follow from a simple algebraic manipulation, but from a remarkable result of geometric nature called Shintani's Unit Theorem. This Theorem, as well as the decomposition of the L-function, will be presented in the next sections. For now, we finish this section by introducing the Shintani zeta function within our familiar framework, and therefore obtaining its values at negative integers.

Let $A = (a_{ij})$ be a complex $r \times m$ matrix with $Re(a_{ij}) > 0$ for all i and j; $\chi \in \mathbb{C}^r$ with $|\chi_i| \leq 1$ for all i; and $\boldsymbol{x} \in \mathbb{R}^r$ such that $0 \leq x_i \leq 1$ for all i, but not all x_i are 0. We define linear forms L_i on $\mathbb{C}^m$ and L_j^* on $\mathbb{C}^r$ by

$$L_i(\boldsymbol{z}) = \sum_{k=1}^{m} a_{ik} z_k, \qquad L_j^*(\boldsymbol{w}) = \sum_{k=1}^{r} a_{kj} w_k \quad \text{for} \quad \boldsymbol{z} = (z_1, \ldots, z_m), \ \boldsymbol{w} = (w_1, \ldots, w_r).$$

$L_i(\boldsymbol{z})$ can be seen as the i-th row of $A\boldsymbol{z}$ and $L_j^*(\boldsymbol{w})$ as the j-th column of $\boldsymbol{w}^T A$. Then we define the *Shintani zeta function* of $\boldsymbol{s} \in \mathbb{C}^m$ as

$$\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi) = \sum_{\boldsymbol{n} \in \mathbb{N}^r} \chi^{\boldsymbol{n}} \prod_{j=1}^{m} L_j^*(\boldsymbol{n} + \boldsymbol{x})^{-s_j}.$$

Note that when $A = 1$ and $\chi = 1$, $\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi) = \zeta(s, x)$ the Hurwitz zeta function. We also note that the Shintani zeta function converges absolutely when $\mathrm{Re}(s_i) > \frac{r}{m}$ for all i.

We proceed as before, introducing the function

$$G(\boldsymbol{t}, A, \boldsymbol{x}, \chi) := \sum_{\boldsymbol{n}\in\mathbb{N}^r} \chi^{\boldsymbol{n}} e^{-\sum_{j=1}^m L_j^*(\boldsymbol{n}+\boldsymbol{x})t_j}$$

for $\boldsymbol{t} \in \mathbb{R}_+^m$ and noting that

$$G(\boldsymbol{t}, A, \boldsymbol{x}, \chi) = \prod_{i=1}^r \frac{e^{-x_i L_i(\boldsymbol{t})}}{1-\chi_i e^{-L_i(\boldsymbol{t})}}.$$

It then follows as before that

$$\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi) = \int_0^\infty \cdots \int_0^\infty G(\boldsymbol{t}, A, \boldsymbol{x}, \chi) \prod_{j=1}^m \frac{t_j^{s_j-1}}{\Gamma(s_j)} dt_j. \tag{1.4}$$

Converting this integral into a contour integral convergent for all $\boldsymbol{s} \in \mathbb{C}^m$ cannot be done in a straightforward manner as before, but first requires a partition of $\mathbb{R}_+^m$ into m special regions together with a change of variables in each region, a method ingeniously devised by Shintani in [14]. In this manner, it is possible to obtain a meromorphic continuation of $\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi)$ to all of $\mathbb{C}^m$.

In the case of $\chi_i \neq 1$ for all $i \leq m$, we are able to use the following generalization of 1.1.1 to obtain the analytic continuation and the values of the Shintani zeta function at negative integers:

Lemma 1.1.2. *Suppose $f \in C^\infty(\mathbb{R}_{\geq 0}^m)$ rapidly decreases, as well as all of its partial derivatives. Then $M(f, \boldsymbol{s}) := \int_0^\infty \cdots \int_0^\infty f(\boldsymbol{t}) \prod_{j=1}^m \frac{t_j^{s_j-1}}{\Gamma(s_j)} dt_j$, defined for all $\boldsymbol{s} \in \mathbb{C}^m$ with $Re(s_j) > 0$ for all j, admits an analytic continuation to all of $\mathbb{C}^m$, and for $\boldsymbol{k} \in \mathbb{N}^m$, $M(f, -\boldsymbol{k}) = \prod_{i=1}^n (-\frac{\partial}{\partial t_i})^{k_i} f(t)\big|_{t=0}$.*

Proof. This follows easily from 1.1.1. Also appears in [2, Lemme 3.2]. □

Indeed, the function $G(\boldsymbol{t}, A, \boldsymbol{x}, \chi)$ satisfies the hypotheses of the Lemma for $\chi_i \neq 1$ $\forall i \leq m$. In fact, it has no singularities in a neighbourhood of zero, so we may obtain its power series expansion around 0 as

$$G(\boldsymbol{t}, A, \boldsymbol{x}, \chi) = \sum_{\boldsymbol{n}\in\mathbb{N}^m} B_{\boldsymbol{n}+1}(\boldsymbol{x}) \prod_{i=1}^m \frac{t_i^{n_i}}{(n_i+1)!}.$$

We may call the $B_{\boldsymbol{n}}(\boldsymbol{x})$ *generalized Bernoulli polynomials*, since they reduce to Bernoulli polynomials when $m = 1$, and remark that they have rational coefficients. Just like before, we conclude that

$$\zeta((-k, \ldots, -k), A, \boldsymbol{x}, \chi) = (-1)^{mk} \frac{B_{\boldsymbol{k}+1}(\boldsymbol{x})}{(k+1)^m} \quad \text{for } k \in \mathbb{N}.$$

1.2 Geometric Shintani Zeta Function

In this section we introduce zeta functions associated to certain geometric cones, and obtain its values at negative integers after a decomposition into Shintani zeta functions.

First, let F be a totally real number field of degree g over $\mathbb{Q}$, and let $\mathcal{O}_F$ be its ring of integers. We denote by $\mathcal{O}_{F+}$ the set of totally positive integers and by $\Delta := \mathcal{O}_{F+}^{\times}$ the set of totally positive units of F. Furthermore, we call I the set of embeddings $\tau : F \hookrightarrow \mathbb{C}$ and set $\alpha^{\tau} := \tau(\alpha)$. Note that we have a canonical $\mathbb{R}$-linear isomorphism

$$F \otimes \mathbb{R} \cong \mathbb{R}^I := \prod_{\tau \in I} \mathbb{R}, \qquad \alpha \otimes 1 \mapsto (\alpha^{\tau}). \tag{1.5}$$

In what follows, we fix a numbering $I = \{\tau_1, \dots, \tau_g\}$ of elements in I, and use $\mathbb{R}^I = \mathbb{R}^g$ interchangeably. We will next define the geometric Shintani zeta function associated to a cone.

Definition 1.2.1. A rational closed polyhedral cone in $\mathbb{R}^I_+ \cup \{0\}$, which we simply call a cone, is any set of the form

$$\sigma_{\underline{\alpha}} := \{x_1\alpha_1 + \ldots + x_m\alpha_m \mid x_1, \dots, x_m \in \mathbb{R}_{\geq 0}\}$$

for some $\underline{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathcal{O}_{F+}^m$, where $\alpha \in \mathcal{O}_{F+}$ is seen as an element of $\mathbb{R}^I$ through the isomorphism (1.5). In this case, we say that $\underline{\alpha}$ is a generator of $\sigma_{\underline{\alpha}}$. By considering the case $m = 0$, we see that $\sigma = \{0\}$ is a cone.

We define the dimension $\dim \sigma$ of a cone σ to be the dimension of the $\mathbb{R}$-vector space generated by σ. For any subset $R \subset \mathbb{R}^I_+$, we let

$$\breve{R} := \{(u_{\tau_1}, \dots, u_{\tau_g}) \in \mathbb{R}^I_+ \mid \exists \delta > 0,\ 0 < \forall \delta' < \delta, (u_{\tau_1}, \dots, u_{\tau_{g-1}}, u_{\tau_g} - \delta') \in R\}.$$

Note that by definition, if $\dim \sigma < g$, then we have $\breve{\sigma} = \emptyset$, where $\breve{\sigma}$ is the cone without the $(g-1)$-dimensional lower face.

Definition 1.2.2. Let σ be a cone, and let $\phi\colon \mathcal{O}_F \to \mathbb{C}^\times$ be a finite order additive character on $\mathcal{O}_F$ which factors through $\mathcal{O}_F/\mathfrak{f}$ for some nonzero ideal $\mathfrak{f} \subset \mathcal{O}_F$ (note that this implies $|\phi(\alpha)| = 1$ for all $\alpha \in \mathcal{O}_F$). We define the *geometric Shintani zeta function* $\zeta_\sigma(\phi, \boldsymbol{s})$ associated to a cone σ and character ϕ by the series

$$\zeta_\sigma(\phi, \boldsymbol{s}) \coloneqq \sum_{\alpha \in \breve{\sigma} \cap \mathcal{O}_F} \phi(\alpha) \prod_{j=1}^{g} (\alpha^{\tau_j})^{-s_j}, \tag{1.6}$$

where $\boldsymbol{s} = (s_j) \in \mathbb{C}^g$. The series (1.6) converges if $\mathrm{Re}(s_j) > 1$ for all $j \leq g$.

If we let $\boldsymbol{s} = (s, \ldots, s)$ for $s \in \mathbb{C}$, then we have

$$\zeta_\sigma(\phi, (s, \ldots, s)) = \sum_{\alpha \in \breve{\sigma} \cap \mathcal{O}_F} \phi(\alpha) N(\alpha)^{-s}. \tag{1.7}$$

Proposition 1.2.3. *The geometric Shintani zeta function associated to a g-dimensional cone σ and character ϕ can be written as a linear combination of Shintani zeta functions.*

Proof. Let $\sigma = \sigma_{\underline{\alpha}}$ be a g-dimensional cone generated by $\underline{\boldsymbol{\alpha}} = (\alpha_1, \ldots, \alpha_g) \in \mathcal{O}_{F+}^g$, and we let $P_{\underline{\alpha}} \coloneqq \{x_1\alpha_1 + \ldots + x_g\alpha_g \mid \forall i \ 0 \leq x_i \leq 1\}$ be the parallelepiped spanned by $\alpha_1, \ldots, \alpha_g$. First, note that $\breve{P}_{\underline{\alpha}} \cap \mathcal{O}_F$ is finite since $P_{\underline{\alpha}}$ is compact and $\mathcal{O}_F$ is a lattice when embedded into $F \otimes \mathbb{R} \cong \mathbb{R}^g$.

Now, let $R^\sigma_{\underline{\alpha}}$ be the set of $\boldsymbol{x} \in \mathbb{R}^g$ such that $\boldsymbol{x} \cdot \underline{\boldsymbol{\alpha}} = (x_1\alpha_1, \ldots, x_g\alpha_g) \in \breve{\sigma} \cap \mathcal{O}_F$ and $R^P_{\underline{\alpha}}$ the set of $\boldsymbol{x} \in \mathbb{R}^g$ such that $\boldsymbol{x} \cdot \underline{\boldsymbol{\alpha}} \in \breve{P}_{\underline{\alpha}} \cap \mathcal{O}_F$. Then we claim that $R^\sigma_{\underline{\alpha}} = R^P_{\underline{\alpha}} \oplus \mathbb{N}^g$. Indeed, given $\boldsymbol{x} \in R^\sigma_{\underline{\alpha}}$, it can be uniquely written as $(\lfloor x_1 \rfloor, \ldots, \lfloor x_g \rfloor) + (\{x_1\}, \ldots, \{x_g\})$, where $(\lfloor x_1 \rfloor, \ldots, \lfloor x_g \rfloor) \in \mathbb{N}^g$ and $(\{x_1\}, \ldots, \{x_g\}) \in R^P_{\underline{\alpha}}$. Note also that $\phi(x_1\alpha_1 + \cdots + x_g\alpha_g) = \phi(\{x_1\}\alpha_1 + \cdots + \{x_g\}\alpha_g) \prod_{i=1}^g \phi(\alpha_i)^{\lfloor x_i \rfloor}$.

Furthermore, since σ is g-dimensional, $\alpha_1, \ldots, \alpha_g$ form a basis of F over $\mathbb{Q}$, so $\boldsymbol{x} \in R^\sigma_{\underline{\alpha}}$ implies that $\boldsymbol{x} \in \mathbb{Q}^g$, thus we may conclude that for every $\tau \in I$, $\tau(x_1\alpha_1 + \ldots + x_g\alpha_g) = x_1\tau(\alpha_1) + \ldots + x_g\tau(\alpha_g)$. We may then rewrite $\zeta_\sigma(\phi, \boldsymbol{s})$ as

$$\zeta_\sigma(\phi, \boldsymbol{s}) = \sum_{\beta \in \breve{\sigma} \cap \mathcal{O}_F} \phi(\beta) \prod_{j=1}^{g} (\beta^{\tau_j})^{-s_j} = \sum_{\boldsymbol{x} \in R^\sigma_{\underline{\alpha}}} \left[\phi(x_1\alpha_1 + \cdots + x_g\alpha_g) \prod_{j=1}^{g} \left(\sum_{i=1}^{g} x_i \alpha_i^{\tau_j} \right)^{-s_j} \right] =$$

$$= \sum_{\boldsymbol{x} \in R^P_{\underline{\alpha}}} \left[\phi(\{x_1\}\alpha_1 + \cdots + \{x_g\}\alpha_g) \sum_{\boldsymbol{n} \in \mathbb{N}^g} \left(\prod_{m=1}^{g} \phi(\alpha_l)^{n_l} \prod_{j=1}^{g} \left(\sum_{i=1}^{g} (x_i + n_i) \alpha_i^{\tau_j} \right)^{-s_j} \right) \right] =$$

$$= \sum_{\boldsymbol{x} \in R^P_{\underline{\alpha}}} \phi(\{x_1\}\alpha_1 + \cdots + \{x_g\}\alpha_g)\zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi),$$

where

$$A := \begin{pmatrix} \alpha_1^{\tau_1} & \cdots & \alpha_1^{\tau_g} \\ \vdots & \ddots & \vdots \\ \alpha_g^{\tau_1} & \cdots & \alpha_g^{\tau_g} \end{pmatrix} \quad \text{and} \quad \chi := (\phi(\alpha_1), \ldots, \phi(\alpha_g)).$$

The claim now follows from the finiteness of $R^P_{\underline{\alpha}}$.

□

Definition 1.2.4. We define $\mathscr{G}_\sigma(t)$ to be the meromorphic function on $F \otimes \mathbb{C} \cong \mathbb{C}^I$ given by

$$\mathscr{G}_\sigma(t) := \frac{\sum_{\alpha \in \check{P}_{\underline{\alpha}} \cap O_F} e^{2\pi i \mathrm{Tr}(\alpha t)}}{\left(1 - e^{2\pi i \mathrm{Tr}(\alpha_1 t)}\right) \cdots \left(1 - e^{2\pi i \mathrm{Tr}(\alpha_g t)}\right)},$$

where $\mathrm{Tr}(\alpha t) := \sum_{\tau \in I} \alpha^\tau t_\tau$ for any $\alpha \in O_F$, and $\check{P}_{\underline{\alpha}} \cap O_F$ was defined in the proof of Proposition 1.2.3. The definition of $\mathscr{G}_\sigma(t)$ depends only on the cone and is independent of the choice of the generator $\underline{\alpha}$, which will be clear by Corollary 1.2.6 below.

Remark 1.2.5. If $F = \mathbb{Q}$ and $\sigma = \mathbb{R}_{\geq 0}$, then we have $\mathscr{G}_\sigma(t) = \frac{e^{2\pi i t}}{1 - e^{2\pi i t}}$, which is exactly the previously defined $G(t)$ from (1.2) after the normalization $t \mapsto -2\pi i t$.

For $\boldsymbol{k} = (k_\tau) \in \mathbb{N}^I$, we denote $\partial^{\boldsymbol{k}} := \prod_{\tau \in I} \partial_\tau^{k_\tau}$, where $\partial_\tau := \frac{1}{2\pi i} \frac{\partial}{\partial z_\tau}$. For $u \in F$, we let ξ_u be the finite additive character on O_F defined by $\xi_u(\alpha) := e^{2\pi i \mathrm{Tr}(\alpha u)}$. We note that any additive character ϕ on O_F with values in $\mathbb{C}^\times$ of finite order is of this form for some $u \in F$ (see (2.5) in Chapter 2).

We can now use (1.4) and 1.1.2 to obtain the following Corollary from Proposition 1.2.3.

Corollary 1.2.6. *Let $\underline{\alpha}$ and σ be as in Definition 1.2.4. For any $u \in F$ satisfying $\xi_u(\alpha_j) \neq 1$ for $j = 1, \ldots, g$, we have*

$$\partial^{\boldsymbol{k}} \mathscr{G}_\sigma(t)\big|_{t = u \otimes 1} = \zeta_\sigma(\xi_u, -\boldsymbol{k}).$$

Proof. We fix $u \in F$ and set $\phi = \xi_u$ and $\boldsymbol{x}_\alpha$ as the unique $\boldsymbol{x} \in R^P_{\underline{\alpha}}$ determined by $\alpha \in \check{P}_{\underline{\alpha}} \cap O_F$. Let now $\zeta(\boldsymbol{s}, A, \boldsymbol{x}_\alpha, \chi)$ be the zeta function from the proof of Proposition 1.2.3. Let $G(t, A, \boldsymbol{x}_\alpha, \chi)$ be the corresponding generating function, given by

$$G(\boldsymbol{t}, A, \boldsymbol{x}_\alpha, \chi) = \frac{e^{-\mathrm{Tr}(\alpha t)}}{(1 - e^{-\mathrm{Tr}(\alpha_1 t)} e^{2\pi i \mathrm{Tr}(\alpha_1 u)}) \cdots (1 - e^{-\mathrm{Tr}(\alpha_g t)} e^{2\pi i \mathrm{Tr}(\alpha_g u)})}$$

where we note that the condition $\xi_u(\alpha_j) \neq 1$ for $j = 1, \ldots, g$ means that $\chi_i \neq 1$ for all i. Now, from Proposition 1.2.3, we have that

$$\zeta_\sigma(\xi_u, \boldsymbol{s}) = \sum_{\boldsymbol{x} \in R^P_{\underline{\alpha}}} \prod_{m=1}^{g} (e^{2\pi i \mathrm{Tr}(\alpha_m u)})^{x_m} \zeta(\boldsymbol{s}, A, \boldsymbol{x}, \chi) = \sum_{\alpha \in \check{P}_{\underline{\alpha}} \cap O_F} (e^{2\pi i \mathrm{Tr}(\alpha u)}) \zeta(\boldsymbol{s}, A, \boldsymbol{x}_\alpha, \chi).$$

Evaluating at $-\boldsymbol{k}$ and using Lemma 1.1.2 gives

$$\zeta_\sigma(\xi_u, -\boldsymbol{k}) = \prod_{i=1}^{g} (-\frac{\partial}{\partial t_i})^{k_i} \frac{\sum_{\alpha \in \check{P}_{\underline{\alpha}} \cap O_F} (e^{2\pi i \mathrm{Tr}(\alpha u)}) e^{-\mathrm{Tr}(\alpha t)}}{(1 - e^{-\mathrm{Tr}(\alpha_1 t)} e^{2\pi i \mathrm{Tr}(\alpha_1 u)}) \cdots (1 - e^{-\mathrm{Tr}(\alpha_g t)} e^{2\pi i \mathrm{Tr}(\alpha_g u)})} \Bigg|_{t=0},$$

and finally the variable change $t \mapsto -2\pi i t$ gives

$$\zeta_\sigma(\xi_u, -\boldsymbol{k}) = \partial^{\boldsymbol{k}} \mathscr{G}_\sigma(t + u)\big|_{t=0} = \partial^{\boldsymbol{k}} \mathscr{G}_\sigma(t)\big|_{t=u \otimes 1}.$$

□

1.3 Lerch Zeta Function

Before giving the decomposition of the Hecke L-functions, we introduce yet another zeta function, called Lerch zeta function, which best captures the multiplicative action of Δ on O_F. Let $\mathbb{T}(\mathbb{C}) := \mathrm{Hom}_{\mathbb{Z}}(O_F, \mathbb{C}^\times)$ be the abelian group of $\mathbb{Z}$-module morphisms and $\xi \in \mathbb{T}(\mathbb{C})$ be a $\mathbb{C}^\times$-valued character of O_F of finite order. For $\varepsilon \in \Delta$ we write $\xi^\varepsilon(\alpha) := \xi(\varepsilon\alpha)$ for any $\alpha \in O_F$, and denote by $\Delta_\xi := \{\varepsilon \in \Delta \mid \xi^\varepsilon = \xi\}$ the isotropic (or stabilizer) subgroup of ξ. Now we prove the following Lemma.

Lemma 1.3.1. *Let ξ be a character of finite order. Then Δ_ξ is a subgroup of Δ of finite index.*

Proof. Let $\Delta \cdot \xi$ be the Δ-orbit of ξ, that is, the set of all ξ^ε for $\varepsilon \in \Delta$. Since $\xi\colon O_F/\mathfrak{f} \to \mathbb{C}^\times$ for some ideal $\mathfrak{f}$ of O_F, we have that $\Delta \cdot \xi \subset \mathrm{Hom}_{\mathbb{Z}}(O_F/\mathfrak{f}, \mathbb{C}^\times)$, which is a group of finite order equal to $N(\mathfrak{f})$. Therefore by the orbit-stabilizer theorem we deduce the claim.

□

We define a function $\xi\Delta$ on $\mathcal{O}_F$ to be the sum over the Δ-orbit of ξ:

$$\xi\Delta := \sum_{\varepsilon \in \Delta/\Delta_\xi} \xi^\varepsilon.$$

By definition, $\xi\Delta$ satisfies $\xi^\varepsilon\Delta = \xi\Delta$ for any $\varepsilon \in \Delta$ and defines a map $\xi\Delta\colon \Delta\backslash\mathcal{O}_F \to \mathbb{C}$, where $\Delta\backslash\mathcal{O}_F$ denotes the quotient of $\mathcal{O}_F$ by the equivalence relation induced by the multiplicative action of Δ on $\mathcal{O}_F$.

We define the *Lerch zeta function* by the series

$$\mathcal{L}(\xi\Delta, s) := \sum_{\alpha \in \Delta_\xi \backslash \mathcal{O}_{F+}} \xi(\alpha) N(\alpha)^{-s}. \tag{1.8}$$

This sum converges absolutely for $\mathrm{Re}(s) > 1$ and may be continued meromorphically to the whole complex plane, being entire if $\xi \neq 1$.

Remark 1.3.2. Note that we have

$$\mathcal{L}(\xi\Delta, s) = \sum_{\alpha \in \Delta\backslash\mathcal{O}_{F+}} \sum_{\varepsilon \in \Delta_\xi\backslash\Delta} \xi(\varepsilon\alpha) N(\alpha)^{-s} = \sum_{\alpha \in \Delta\backslash\mathcal{O}_{F+}} \xi\Delta(\alpha) N(\alpha)^{-s}.$$

The importance of $\mathcal{L}(\xi\Delta, s)$ is in its relation to the Hecke L-functions of F. Let $\mathfrak{f}$ be a nonzero integral ideal of $\mathcal{O}_F$. For a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ and $\alpha \in F$, we write $\alpha \equiv 1 \bmod^\times \mathfrak{p}^n$ if $\alpha \in 1 + \mathfrak{p}^n\mathcal{O}_{F,\mathfrak{p}}$, for $\mathcal{O}_{F,\mathfrak{p}}$ the ring localized at $\mathfrak{p}$. Then if $\mathfrak{f} = \prod_{i=1}^n \mathfrak{p}_i^{n_i}$, we write $\alpha \equiv 1 \bmod^\times \mathfrak{f}$ if $\alpha \equiv 1 \bmod^\times \mathfrak{p}_i^{n_i}$ for all $i \leq n$. This is equivalent to requiring that $\beta \equiv \gamma \bmod \mathfrak{f}$ for $\alpha = \frac{\beta}{\gamma}$, $\beta, \gamma \in \mathcal{O}_F$, as long as $\overline{\beta}, \overline{\gamma} \in (\mathcal{O}_F/\mathfrak{f})^\times$. We denote by $\mathrm{Cl}_F^+(\mathfrak{f}) := \mathfrak{I}_\mathfrak{f}/P_\mathfrak{f}^+$ the *strict ray class group modulo* $\mathfrak{f}$ *of* F, where $\mathfrak{I}_\mathfrak{f}$ is the group of fractional ideals of F prime to $\mathfrak{f}$ and $P_\mathfrak{f}^+ := \{(\alpha) \mid \alpha \in F_+, \alpha \equiv 1 \bmod^\times \mathfrak{f}\}$. A *finite Hecke character of* F *of conductor* $\mathfrak{f}$ is a character

$$\chi\colon \mathrm{Cl}_F^+(\mathfrak{f}) \to \mathbb{C}^\times.$$

Then the *Hecke L-function of the field* F *associated to* χ is

$$L(\chi, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s}$$

where $\chi(\mathfrak{a}) = 0$ if $\mathfrak{a} \notin \mathfrak{I}_\mathfrak{f}$ and the sum is over all integral ideals of $\mathcal{O}_F$. For $\mathfrak{f} = (1)$ and χ trivial, this reduces to the *Dedekind zeta function* of F

$$L(\chi, s) = \zeta_F(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} N(\mathfrak{a})^{-s}.$$

These functions can be analytically continued to the whole complex plane, except for a simple pole at $s = 1$ in the case of the Dedekind zeta function (see [13, Chapter VII §8 Corollary 8.6] or [10, §2.7 Theorem 2] for a proof using Shintani zeta functions).

We now briefly introduce some groups from algebraic number theory and show that $\mathrm{Cl}_F^+(\mathfrak{f})$ is finite. For the field F, its *narrow class group* is defined to be $\mathfrak{I}/P^+$, where $\mathfrak{I}$ is the group of fractional ideals and P^+ the subgroup of totally positive principal fractional ideals. Let $\mathfrak{I}/P$ be the class group of F, which is clearly a subgroup of $\mathfrak{I}/P^+$, whose orders are called class number and narrow class number, respectively. For a fractional ideal $\mathfrak{f}$, it is easy to show that there is an ideal prime to $\mathfrak{f}$ in every class of $\mathfrak{I}/P$, so $\mathfrak{I}/P \cong \mathfrak{I}_\mathfrak{f}/P(\mathfrak{f})$, where $P(\mathfrak{f})$ denotes the subgroup of $\mathfrak{I}_\mathfrak{f}$ of principal ideals. We then obtain the exact sequence

$$0 \longrightarrow P(\mathfrak{f})/P^+(\mathfrak{f}) \longrightarrow \mathfrak{I}_\mathfrak{f}/P^+(\mathfrak{f}) \longrightarrow \mathfrak{I}_\mathfrak{f}/P(\mathfrak{f}) \longrightarrow 0$$

where we note that $P(\mathfrak{f})/P^+(\mathfrak{f})$ has order dividing 2^g. Since the class group is finite, we conclude that the narrow class group is also finite, with order equal to the class number times a power of 2. Now, we may obtain the following exact sequence for the strict ray class group:

$$0 \longrightarrow P^+(\mathfrak{f})/P^+_\mathfrak{f} \longrightarrow \mathfrak{I}_\mathfrak{f}/P^+_\mathfrak{f} \longrightarrow \mathfrak{I}_\mathfrak{f}/P^+(\mathfrak{f}) \longrightarrow 0. \tag{1.9}$$

Since we have showed above that the narrow class number is finite, we may restrict our attention to the kernel $P^+(\mathfrak{f})/P^+_\mathfrak{f}$. Since every ideal $\mathfrak{a} \in P^+(\mathfrak{f})$ is determined by an element $\alpha \in F_+^\times$ up to multiplication by Δ, and $\mathfrak{a} \in P^+_\mathfrak{f}$ precisely when $\alpha \equiv 1 \bmod^\times \mathfrak{f}$, we have an identification $P^+(\mathfrak{f})/P^+_\mathfrak{f} \cong (\mathcal{O}_F/\mathfrak{f})^\times/\Delta_\mathfrak{f}$ where $\Delta_\mathfrak{f}$ denotes $\Delta \bmod^\times \mathfrak{f}$. From this it is clear that $\mathrm{Cl}_F^+(\mathfrak{f})$ is finite and that there exists a unique character $\chi_{\mathrm{fin}} \colon (\mathcal{O}_F/\mathfrak{f})^\times \to \mathbb{C}^\times$ associated to χ such that $\chi((\alpha)) = \chi_{\mathrm{fin}}(\alpha)$

for any $\alpha \in \mathcal{O}_{F+}$ prime to $\mathfrak{f}$. In particular, we have $\chi_{\mathrm{fin}}(\varepsilon) = 1$ for any $\varepsilon \in \Delta$. Extending by *zero*, we regard χ_{fin} as functions on $\mathcal{O}_F/\mathfrak{f}$ and $\mathcal{O}_F$ with values in $\mathbb{C}$.

In what follows, we let $\mathbb{T}[\mathfrak{f}] := \mathrm{Hom}(\mathcal{O}_F/\mathfrak{f}, \mathbb{C}^\times)$ be the set of $\mathfrak{f}$-torsion points of $\mathbb{T}(\mathbb{C})$. We say that a character χ, χ_{fin} or $\xi \in \mathbb{T}[\mathfrak{f}]$ is *primitive*, if it does not factor respectively through $\mathrm{Cl}^+_F(\mathfrak{f}')$, $(\mathcal{O}_F/\mathfrak{f}')^\times$ or $\mathcal{O}_F/\mathfrak{f}'$ for any integral ideal $\mathfrak{f}' \neq \mathfrak{f}$ such that $\mathfrak{f}'|\mathfrak{f}$. Then we have the following finite Fourier transform.

Lemma 1.3.3. *For any $\xi \in \mathbb{T}[\mathfrak{f}]$, let*

$$c_\chi(\xi) := \frac{1}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\beta)\xi(-\beta).$$

Then we have

$$\chi_{\mathrm{fin}}(\alpha) = \sum_{\xi \in \mathbb{T}[\mathfrak{f}]} c_\chi(\xi)\xi(\alpha).$$

Moreover, if χ_{fin} is primitive, then we have $c_\chi(\xi) = 0$ for any non-primitive ξ.

Proof. The first statement follows from

$$\sum_{\xi \in \mathbb{T}[\mathfrak{f}]} c_\chi(\xi)\xi(\alpha) = \frac{1}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\beta)\left(\sum_{\xi \in \mathbb{T}[\mathfrak{f}]} \xi(\alpha - \beta)\right) = \chi_{\mathrm{fin}}(\alpha),$$

where the last equality follows from the fact that $\sum_{\xi \in \mathbb{T}[\mathfrak{f}]} \xi(\alpha) = N(\mathfrak{f})$ if $\alpha \equiv 0 \pmod{\mathfrak{f}}$ and $\sum_{\xi \in \mathbb{T}[\mathfrak{f}]} \xi(\alpha) = 0$ if $\alpha \not\equiv 0 \pmod{\mathfrak{f}}$, since this is the sum of all n-th roots of unity, for some n dividing $N(\mathfrak{f})$. Next, suppose χ_{fin} is primitive, and let $\mathfrak{f}' \neq \mathfrak{f}$ be an integral ideal of F such that $\mathfrak{f}'|\mathfrak{f}$ and $\xi \in \mathbb{T}[\mathfrak{f}']$. Since χ_{fin} is primitive, it does not factor through $\mathcal{O}_F/\mathfrak{f}'$, hence there exists an element $\gamma \in \mathcal{O}_F$ prime to $\mathfrak{f}$ such that $\gamma \equiv 1 \pmod{\mathfrak{f}'}$ and $\chi_{\mathrm{fin}}(\gamma) \neq 1$. Then since $\xi \in \mathbb{T}[\mathfrak{f}']$, we have $\xi(\gamma\alpha) = \xi(\alpha)$ for any $\alpha \in \mathcal{O}_F$. This gives

$$\begin{aligned} c_\chi(\xi) = \frac{1}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\beta)\xi(-\beta) &= \frac{1}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\beta)\xi(-\gamma\beta) \\ &= \frac{\overline{\chi}_{\mathrm{fin}}(\gamma)}{\boldsymbol{N}(\mathfrak{f})} \sum_{\beta \in \mathcal{O}_F/\mathfrak{f}} \chi_{\mathrm{fin}}(\gamma\beta)\xi(-\gamma\beta) = \overline{\chi}_{\mathrm{fin}}(\gamma)c_\chi(\xi). \end{aligned}$$

Since $\overline{\chi}_{\mathrm{fin}}(\gamma) \neq 1$, we have $c_\chi(\xi) = 0$ as desired. $\square$

Note that since multiplication by $\varepsilon \in \Delta$ is bijective on $O_F/\mathfrak{f}$ and since $\chi_{\text{fin}}(\varepsilon) = 1$, we have $c_\chi(\xi^\varepsilon) = c_\chi(\xi)$.

We now obtain the following decomposition of Hecke L-functions as Lerch zeta functions, assuming the narrow class number of F equals one.

Proposition 1.3.4. *Assume that the narrow class number of F is one, and let $\chi\colon \mathrm{Cl}_F^+(\mathfrak{f}) \to \mathbb{C}^\times$ be a finite primitive Hecke character of F of conductor $\mathfrak{f} \neq (1)$. Then for $U[\mathfrak{f}] := \mathbb{T}[\mathfrak{f}] \setminus \{1\}$, we have*

$$L(\chi, s) = \sum_{\xi \in U[\mathfrak{f}]/\Delta} c_\chi(\xi)\mathcal{L}(\xi\Delta, s).$$

Proof. By definition and Lemma 1.3.3, we have

$$\begin{aligned}
\sum_{\xi \in \mathbb{T}[\mathfrak{f}]/\Delta} c_\chi(\xi)\mathcal{L}(\xi\Delta, s) &= \sum_{\xi \in \mathbb{T}[\mathfrak{f}]/\Delta} \sum_{\alpha \in \Delta\backslash O_{F+}} \sum_{\varepsilon \in \Delta_\xi\backslash\Delta} c_\chi(\xi)\xi(\varepsilon\alpha)N(\alpha)^{-s} \\
&= \sum_{\alpha \in \Delta\backslash O_{F+}} \sum_{\xi \in \mathbb{T}[\mathfrak{f}]/\Delta} \sum_{\varepsilon \in \Delta_\xi\backslash\Delta} c_\chi(\xi^\varepsilon)\xi^\varepsilon(\alpha)N(\alpha)^{-s} \\
&= \sum_{\alpha \in \Delta\backslash O_{F+}} \sum_{\xi \in \mathbb{T}[\mathfrak{f}]} c_\chi(\xi)\xi(\alpha)N(\alpha)^{-s} \\
&= \sum_{\alpha \in \Delta\backslash O_{F+}} \chi_{\text{fin}}(\alpha)N(\alpha)^{-s} = \sum_{\mathfrak{a} \subset O_F} \chi(\mathfrak{a})N\mathfrak{a}^{-s}.
\end{aligned}$$

The last equality follows from the fact that $\mathrm{Cl}_F^+(\mathfrak{f}) \cong (O_F/\mathfrak{f})^\times/\Delta_\mathfrak{f}$ and the assertion from the fact that $c_\chi(\xi) = 0$ for $\xi = 1$. $\square$

Remark 1.3.5. We assumed the condition on the narrow class number for simplicity. In section §4.1 we give a generalization for any narrow class number.

1.4 Non-canonical Decomposition: Special Case

As promised, in this section we decompose certain Hecke L-functions as a linear combination of Shintani zeta functions, thus determining their values at negative integers. It should be noted that this decomposition is non-canonical, a problem which will be addressed in the following chapters.

In [14] Shintani constructively proved what is now known as Shintani's Unit Theorem, which states that the fundamental domain of the action of a subgroup of finite index of Δ on $(F \otimes \mathbb{R})_+ \cong \mathbb{R}_+^I$ is given by a finite set of rational polyhedral cones. This implies that the Lerch zeta function $\mathcal{L}(\xi\Delta, s)$ may be expressed as a finite sum of functions $\zeta_\sigma(\xi, (s, \ldots, s))$ using a Shintani decomposition, as we will show below.

We say that a cone σ is *simplicial*, if there exists a generator of σ that is linearly independent over $\mathbb{R}$. Any cone generated by a subset of such a generator is called a *face* of σ. A simplicial fan Φ is a set of simplicial cones such that for any $\sigma \in \Phi$, any face of σ is also in Φ, and for any cones $\sigma, \sigma' \in \Phi$, the intersection $\sigma \cap \sigma'$ is a common face of σ and σ'.

Definition 1.4.1. A *Shintani decomposition* is a simplicial fan Φ satisfying the following properties.

1. $\mathbb{R}_+^I \cup \{0\} = \coprod_{\sigma \in \Phi} \sigma^\circ$, where σ° is the relative interior of σ, i.e., the interior of σ in the $\mathbb{R}$-linear span of σ.

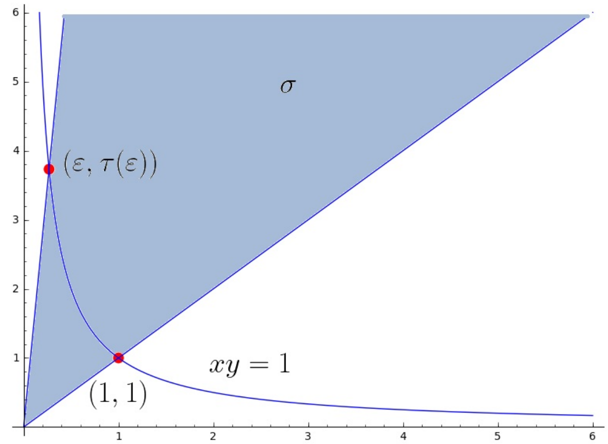
2. For any $\sigma \in \Phi$ and $\varepsilon \in \Delta$, we have $\varepsilon\sigma \in \Phi$.

3. The quotient $\Delta \backslash \Phi$ is a finite set.

We may obtain such decomposition by slightly modifying the construction of Shintani [14, Theorem 1] (see also [10, §2.7 Theorem 1], [15, Theorem 4.1]). For any integer $q \geq 0$, we denote by Φ_{q+1} the subset of Φ consisting of cones of dimension $q+1$. Note that by [15, Proposition 5.6], Φ_g satisfies

$$\mathbb{R}_+^I = \coprod_{\sigma \in \Phi_g} \breve{\sigma}. \tag{1.10}$$

Remark 1.4.2. Let F be a real quadratic extension of $\mathbb{Q}$. Then by Dirichlet's Unit Theorem (see [4, Theorem 10.7]), Δ is generated by a single fundamental unit ε, and the cone generated by $(1, \varepsilon)$ can be visualized as



where all the units in Δ are obviously contained in the line $xy = 1$, and the Δ action on σ clearly covers the whole positive quadrant. For this image, $F = \mathbb{Q}(\sqrt{3})$ and $\varepsilon = 2 + \sqrt{3}$.

We have the following result.

Proposition 1.4.3. *Let* $\xi \colon \mathcal{O}_F \to \mathbb{C}^\times$ *be a character of finite order, and* $\Delta_\xi \subset \Delta$ *its isotropic subgroup. If* Φ *is a Shintani decomposition, then we have*

$$\mathcal{L}(\xi\Delta, s) = \sum_{\sigma \in \Delta_\xi \backslash \Phi_g} \zeta_\sigma(\xi, (s, \dots, s)). \tag{1.11}$$

Proof. By (1.10), if C is a set of representatives of $\Delta_\xi \backslash \Phi_g$, then $\coprod_{\sigma \in C} \breve{\sigma}$ is a representative of the set $\Delta_\xi \backslash \mathbb{R}^I_+$. Our result follows from the definition of the Lerch zeta function and (1.7). $\square$

The expression (1.11) is non-canonical, since it depends on the choice of the Shintani decomposition. In fact, in the decomposition of the Hecke L-function given by combining (1.11) with 1.3.4, this is the only step which is non-canonical. The goal of the following chapters is to introduce tools from algebraic geometry in order to obtain the negative integer values of the Lerch zeta function canonically.

We conclude that we can obtain the Hecke L-function as a linear combination of geometric Shintani zeta functions in the Proposition below.

Proposition 1.4.4. *Assume that the narrow class number of F is one, and let $\chi\colon \mathrm{Cl}_F^+(\mathfrak{f}) \to \mathbb{C}^\times$ be a finite primitive Hecke character of F of conductor $\mathfrak{f} \neq (1)$. Then, for $U[\mathfrak{f}] := \mathbb{T}[\mathfrak{f}] \setminus \{1\}$ and Φ a Shintani decomposition, we have*

$$L(\chi, s) = \sum_{\xi \in U[\mathfrak{f}]/\Delta} c_\chi(\xi) \sum_{\sigma \in \Delta_\xi \backslash \Phi_g} \zeta_\sigma(\xi, (s, \ldots, s)).$$

This gives the Hecke L-function as a linear combination of geometric Shintani zeta functions.

Proof. After combining (1.11) with 1.3.4, we only need to show that $\Delta_\xi \backslash \Phi_g$ is finite. It follows from Shintani's Unit Theorem, or equivalently from property (3) of the definition 1.4.1 of a Shintani decomposition given above, that $\Delta_\xi \backslash \Phi_g$ is finite when Δ_ξ is a subgroup of finite index of Δ, which was proven in Lemma 1.3.1. □

Before we end this chapter, we finish with a few remarks regarding the negative integer values of Hecke L-functions associated to finite Hecke characters. First, we note that our hypothesis of the field F being totally real is not restrictive, since if the number field K is not totally real, then $L(\chi, -k) = 0$ for all $0 \leq k \in \mathbb{Z}$, except for the case when χ is the trivial character, for which $L(\chi, 0) \neq 0$. This result follows from the functional equation that the Hecke L-functions satisfy. The decomposition of the Hecke L-function given above assumes $\mathfrak{f} \neq (1)$ and χ non-trivial, therefore it does not include the Dedekind zeta functions. Despite that, it is possible to give a decomposition of Hecke L-functions which includes the case of the Dedekind zeta, as well as to deduce its values at negative integers. For a totally real field F, the values $\zeta_F(-k)$ are all in $\mathbb{Q}$ and are given by associated generalized Bernoulli polynomials. They are equal to zero if $0 < k$ is even, and non-zero if k is odd. Note that this includes the case of $F = \mathbb{Q}$, that is, the Riemann zeta. All of these claims are proven in [13, Chapter VII §9].

Chapter Two

Equivariant Cohomology

2.1 Equivariant Cohomology of the Algebraic Torus

Let F be a number field of degree g as before. In order to study the characters of the ring of integers $\mathcal{O}_F$ more systematically and to use the tools from algebraic geometry, we need to define them in a more general context, namely as points of group schemes to be defined below.

Definition 2.1.1. Let S be a scheme. A group scheme over S is a pair (G, m) where G is a S-scheme and $m : G \times_S G \to G$ is a morphism of S-schemes such that, for all schemes T over S, $(G(T), m)$ is a group.

We begin by introducing the *multiplicative group scheme* $\mathbb{G}_m$, which represents the functor that associates a ring R to its multiplicative group R^*, that is, the R-valued points of $\mathbb{G}_m$ are $\mathbb{G}_m(R) = R^*$. First, we claim that $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}] = \operatorname{Spec} \mathbb{Z}[x, y]/(xy - 1)$. Indeed

$$\mathbb{G}_m(R) = \operatorname{Hom}_{Sch}(\operatorname{Spec} R, \operatorname{Spec} \mathbb{Z}[t, t^{-1}]) \cong \operatorname{Hom}_{Rings}(\mathbb{Z}[t, t^{-1}], R) \cong R^*$$

where the last isomorphism is given by observing that ring homomorphisms from $\mathbb{Z}[t, t^{-1}]$ to R are uniquely given by assigning t to an invertible element of R. The multiplication in $\mathbb{G}_m$ is given by

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \mathbb{Z}[t, t^{-1}] \otimes \mathbb{Z}[t, t^{-1}] & \longleftarrow & \mathbb{Z}[t, t^{-1}] \\ t \otimes t & \longleftarrow\!\shortmid & t. \end{array}$$

A group scheme of central importance to this work associated to $\mathcal{O}_F$ is the *algebraic torus* $\mathbb{T}^{\mathcal{O}_F} = \mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{G}_m)$ as an abelian group, which represents the functor that associates a commutative ring R to the group $\mathbb{T}(R) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, R^\times)$ of $\mathbb{Z}$-module homomorphisms. In fact, as a group,

$$\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{G}_m) \cong \mathbb{G}_m^g \cong \mathbb{G}_m \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{Z}) = \mathbb{G}_m \otimes \mathcal{O}_F^\vee.$$

We still need to describe $\mathbb{T}$ as a scheme, but first we shall define morphisms of group schemes:

Definition 2.1.2. A *morphism of group schemes* $\psi\colon (G, m), \to (G', m')$ over S is a morphism of schemes over S such that for every scheme T over S the induced map $\psi\colon G(T) \to G'(T)$ is a group homomorphism.

For any ring R and any $\alpha \in \mathcal{O}_F$, the natural morphism $\mathbb{T}(R) \to \mathbb{G}_m(R)$ defined by mapping $\xi \in \mathbb{T}(R) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, R^\times)$ to $\xi(\alpha) \in \mathbb{G}_m(R) = R^\times$ induces a morphism of group schemes $t^\alpha\colon \mathbb{T} \to \mathbb{G}_m$ by Yoneda's Lemma, which gives a rational function of $\mathbb{T}$. Then we have

$$\mathbb{T} = \mathrm{Spec}\, \mathbb{Z}[t^\alpha \mid \alpha \in \mathcal{O}_F],$$

where $t^\alpha, t^{\alpha'}$ satisfies the relation $t^\alpha t^{\alpha'} = t^{\alpha+\alpha'}$ for any $\alpha, \alpha' \in \mathcal{O}_F$. If we take a basis $\alpha_1, \ldots, \alpha_g$ of $\mathcal{O}_F$ as a $\mathbb{Z}$-module, then we have

$$\mathrm{Spec}\, \mathbb{Z}[t^\alpha \mid \alpha \in \mathcal{O}_F] = \mathrm{Spec}\, \mathbb{Z}[t^{\pm\alpha_1}, \ldots, t^{\pm\alpha_g}] \cong \mathbb{G}_m^g,$$

which is consistent with the previous group isomorphism $\mathbb{T} \cong \mathbb{G}_m^g$. One is also tempted to use the characterization $\mathbb{T} \cong \mathbb{G}_m \otimes \mathcal{O}_F^\vee$ to produce a right action of Δ on $\mathbb{T}$, but in order to make this precise, we need first the following definition:

Definition 2.1.3. Let G be a group with identity e. A G-scheme is a scheme X equipped with a right action of G. We denote by $[u]\colon X \to X$ the action of $u \in G$, so that $[uv] = [v] \circ [u]$ for any $u, v \in G$ holds.

As hinted above, the action of Δ on O_F by multiplication induces an action of Δ on $\mathbb{T}$. Explicitly:

$$\begin{aligned} [\varepsilon]\colon \mathbb{T} &\longrightarrow \mathbb{T} \\ \mathbb{Z}[t^\alpha \mid \alpha \in O_F] &\longleftarrow \mathbb{Z}[t^\alpha \mid \alpha \in O_F] \colon [\varepsilon]_\sharp \\ t^{\varepsilon\alpha} &\longleftarrow\!\shortmid t^\alpha. \end{aligned}$$

We are now ready to define equivariant sheaves. Assume X is a G-scheme from now on:

Definition 2.1.4. A *G-equivariant structure* on an $\mathscr{O}_X$-module $\mathscr{F}$ is a family of isomorphisms

$$\iota_u\colon [u]^*\mathscr{F} \xrightarrow{\cong} \mathscr{F}$$

for $u \in G$, such that $\iota_e = \mathrm{id}_{\mathscr{F}}$ and the diagram

$$\begin{tikzcd} {[uv]^*\mathscr{F}} \arrow[r, "\iota_{uv}"] \arrow[d, equal] & \mathscr{F} \\ {[u]^*[v]^*\mathscr{F}} \arrow[r, "{[u]^*\iota_v}"] & {[u]^*\mathscr{F}} \arrow[u, "\iota_u"'] \end{tikzcd}$$

is commutative. We call $\mathscr{F}$ equipped with a G-equivariant structure a *G-equivariant sheaf.*

Remark 2.1.5. Note that the bottom part of the diagram consists of the pullback functor $[u]^*$ applied to $\iota_v\colon [v]^*\mathscr{F} \xrightarrow{\cong} \mathscr{F}$. The pullback is necessary in the definition since G is acting simultaneously on the $\mathscr{O}_X(U)$-module $\mathscr{F}(U)$ and on the underlying ring $\mathscr{O}_X(U)$ for each U open of X.

For any G-equivariant sheaf $\mathscr{F}$ on X, we define the equivariant global section by $\Gamma(X/G, \mathscr{F}) := \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \Gamma(X, \mathscr{F})) = \Gamma(X, \mathscr{F})^G$. Then the equivariant cohomology $H^m(X/G, -)$ is defined to be the m-th right derived functor of $\Gamma(X/G, -)$.

Note that the structure sheaf $\mathscr{O}_X$ is naturally a G-equivariant sheaf. We will introduce below a Δ-equivariant sheaf consisting of the structure sheaf of $\mathbb{T}_{\bar{F}}$ with a twist, where $\mathbb{T}_{\bar{F}}$ denotes the base change of $\mathbb{T}$ to the Galois closure $\bar{F}$ of F with respect to $\mathbb{Q}$, which is fixed, and we also denote by $I = \mathrm{Hom}(F, \overline{F})$ the set of embeddings.

Definition 2.1.6. For any $\boldsymbol{k} = (k_\tau) \in \mathbb{Z}^I$, we define a Δ-equivariant sheaf $\mathcal{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k})$ on $\mathbb{T}_{\bar{F}}$ as follows. As an $\mathcal{O}_{\mathbb{T}_{\bar{F}}}$-module we let $\mathcal{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k}) := \mathcal{O}_{\mathbb{T}_{\bar{F}}}$. The Δ-equivariant structure

$$\iota_\varepsilon \colon [\varepsilon]^* \mathcal{O}_{\mathbb{T}_{\bar{F}}} \xrightarrow{\cong} \mathcal{O}_{\mathbb{T}_{\bar{F}}}$$

is given by multiplication by $\varepsilon^{-\boldsymbol{k}} := \prod_{\tau \in I} (\varepsilon^\tau)^{-k_\tau}$ for any $\varepsilon \in \Delta$. Note that for $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{Z}^I$, we have $\mathcal{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k}) \otimes \mathcal{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k}') = \mathcal{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k} + \boldsymbol{k}')$. For the case $\boldsymbol{k} = (k, \dots, k)$, we have $\varepsilon^{-\boldsymbol{k}} = N(\varepsilon)^{-k} = 1$ for any $\varepsilon \in \Delta$, hence $\mathcal{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k}) = \mathcal{O}_{\mathbb{T}_{\bar{F}}}$.

Lemma 2.1.7. *$\mathcal{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k})$ is a Δ-equivariant sheaf.*

Proof. First, remember that since $\mathbb{T}_{\bar{F}}$ is affine, there is an equivalence of categories between the category of quasi-coherent $\mathcal{O}_{\mathbb{T}_{\bar{F}}}$-modules $\mathscr{F}$ and the category of R-modules for $R := \bar{F}[t^\alpha \mid \alpha \in O_F]$, given by the R-module of global sections $\mathscr{F}(\mathbb{T}_{\bar{F}})$.

Through this equivalence, $\mathcal{O}_{\mathbb{T}_{\bar{F}}}$ corresponds to R and $[\varepsilon]^* \mathcal{O}_{\mathbb{T}_{\bar{F}}}$ corresponds to $R \otimes_{[\varepsilon]_\sharp R} R$ where $\otimes \colon R \times R \to R \otimes_{[\varepsilon]_\sharp R} R$ is given by the map $[\varepsilon]_\sharp$ on the left and the identity on the right. This is viewed as an R-module through multiplication on the second component. Note that we may identify $R \cong R \otimes_{[\varepsilon]_\sharp R} R$ by sending $r \mapsto r \otimes 1$.

This way the map $\iota_\varepsilon \colon [\varepsilon]^* \mathcal{O}_{\mathbb{T}_{\bar{F}}} \xrightarrow{\cong} \mathcal{O}_{\mathbb{T}_{\bar{F}}}$ becomes $R \otimes_{[\varepsilon]_\sharp R} R \to R$, where $r \otimes 1 \mapsto \varepsilon^{-\boldsymbol{k}} r$, on R-modules. Now that the Δ-equivariant structure is explicit, for $\varepsilon, \delta \in \Delta$ as in Definition 2.1.4, the diagram becomes

$$\begin{tikzcd}
R \otimes_{[\varepsilon\delta]_\sharp R} R \arrow[rr] & & R \\
R \otimes_{[\varepsilon]_\sharp R} R \otimes_{[\delta]_\sharp R} R \arrow[u, equal] \arrow[rr] & & R \otimes_{[\varepsilon]_\sharp R} R \arrow[u]
\end{tikzcd}$$

and an element $r \otimes 1 \otimes 1$ of $R \otimes_{[\varepsilon]_\sharp R} R \otimes_{[\delta]_\sharp R} R$ is sent to $(\varepsilon\delta)^{-\boldsymbol{k}} r \in R$. $\square$

The open subscheme $U := \mathbb{T} \setminus \{1\}$ also carries the Δ-scheme structure from $\mathbb{T}$. We will now construct the *equivariant Čech complex*, which may be used to express the cohomology of U with coefficients in a Δ-equivariant quasi-coherent $\mathscr{O}_U$-module $\mathscr{F}$. For any $\alpha \in O_F$, we let $U_\alpha := \mathbb{T} \setminus \{t^\alpha = 1\}$. If we let $B := \mathbb{Z}[t^\alpha \mid \alpha \in O_{F+}]$, then $\Gamma(U_\alpha, \mathscr{O}_\mathbb{T}) = B_\alpha = B[\frac{1}{1-t^\alpha}]$. Then any $\varepsilon \in \Delta$ induces an isomorphism $[\varepsilon] \colon U_{\varepsilon\alpha} \to U_\alpha$.

We say that $\alpha \in O_{F+}$ is *primitive* if $\alpha/N \notin O_{F+}$ for any integer $N > 1$.

Lemma 2.1.8. *Let $A \subset O_{F+}$ be the set of primitive elements of O_{F+}. Then*

1. *$\varepsilon A = A$ for any $\varepsilon \in \Delta$.*
2. *The set $\mathfrak{U} := \{U_\alpha\}_{\alpha \in A}$ gives an affine open covering of U.*

Proof. (1) fix $\varepsilon \in \Delta$. We will show that $\varepsilon A \subset A$. For $\alpha \in O_F$, let $\varepsilon\alpha = N\beta$, for $1 < N \in \mathbb{N}$, $\beta \in O_F$. Then $\alpha/N = \varepsilon^{-1}\beta \in O_F$, and the claim follows from the contrapositive. The other inclusion follows from $\varepsilon^{-1}A \subset A$.
(2) $\cup_{\alpha \in A} U_\alpha = \mathbb{T} \setminus \cap_{\alpha \in A}\{t^\alpha = 1\} = \mathbb{T} \setminus \cap_{\alpha \in O_F}\{t^\alpha = 1\} = \mathbb{T} \setminus \{1\} = U$. □

We note that for any simplicial cone σ of dimension m, there exists a generator $\underline{\alpha} \in A^m$, unique up to permutation of the components.

Let q be an integer ≥ 0. For any $\underline{\alpha} = (\alpha_0, \ldots, \alpha_q) \in A^{q+1}$, we let $U_{\underline{\alpha}} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$, and we denote by $j_{\underline{\alpha}} \colon U_{\underline{\alpha}} \hookrightarrow U$ the inclusion. The sheaf $j_{\underline{\alpha}*} j^*_{\underline{\alpha}} \mathscr{F}$ simply consists of the sheaf $\mathscr{F}$ restricted to $U_{\underline{\alpha}}$ and then defined on U again, that is, $j_{\underline{\alpha}*} j^*_{\underline{\alpha}} \mathscr{F}(V) = \mathscr{F}(V \cap U_{\underline{\alpha}})$ for all V open of U. We let

$$\mathscr{C}^q(\mathfrak{U}, \mathscr{F}) := \prod^{\text{alt}}_{\underline{\alpha} \in A^{q+1}} j_{\underline{\alpha}*} j^*_{\underline{\alpha}} \mathscr{F}$$

be the subsheaf of $\prod_{\underline{\alpha} \in A^{q+1}} j_{\underline{\alpha}*} j^*_{\underline{\alpha}} \mathscr{F}$ consisting of sections $\boldsymbol{s} = (s_{\underline{\alpha}})$ such that $s_{\rho(\underline{\alpha})} = \operatorname{sgn}(\rho) s_{\underline{\alpha}}$ for any $\rho \in \mathfrak{S}_{q+1}$ and $s_{\underline{\alpha}} = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$. We define the differential $d^q \colon \mathscr{C}^q(\mathfrak{U}, \mathscr{F}) \to$

$\mathscr{C}^{q+1}(\mathfrak{U}, \mathscr{F})$ to be the usual alternating sum

$$(d^q f)_{\alpha_0\cdots\alpha_{q+1}} := \sum_{j=0}^{q+1} (-1)^j f_{\alpha_0\cdots\check{\alpha}_j\cdots\alpha_{q+1}}\big|_{U_{(\alpha_0,\ldots,\alpha_{q+1})}\cap V} \tag{2.1}$$

for any section $(f_{\underline{\alpha}})$ of $\mathscr{C}^q(\mathfrak{U}, \mathscr{F})$ of each open set $V \subset U$. If we let $\mathscr{F} \hookrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F})$ be the natural inclusion, then we have the following exact sequence:

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^0} \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^1} \cdots \xrightarrow{d^{q-1}} \mathscr{C}^q(\mathfrak{U}, \mathscr{F}) \xrightarrow{d^q} \cdots$$

where exactness follows from the permutation property in the definition of the subsheaf $\mathscr{C}^q(\mathfrak{U}, \mathscr{F})$.

We next consider the action of Δ. For any $\underline{\alpha} \in A^{q+1}$ and $\varepsilon \in \Delta$, we have a commutative diagram

$$\begin{tikzcd} U_{\varepsilon\underline{\alpha}} \arrow[r, hook, "j_{\varepsilon\underline{\alpha}}"] \arrow[d, "{[\varepsilon]}"', "\cong"] & U \arrow[d, "{[\varepsilon]}"', "\cong"] \\ U_{\underline{\alpha}} \arrow[r, hook, "j_{\underline{\alpha}}"] & U, \end{tikzcd} \tag{2.2}$$

where $\varepsilon\underline{\alpha} := (\varepsilon\alpha_0, \ldots, \varepsilon\alpha_q)$. Then we have isomorphisms

$$[\varepsilon]^* j_{\underline{\alpha}*} j^*_{\underline{\alpha}} \mathscr{F} \cong j_{\varepsilon\underline{\alpha}*} j^*_{\varepsilon\underline{\alpha}} [\varepsilon]^* \mathscr{F} \xrightarrow{\cong} j_{\varepsilon\underline{\alpha}*} j^*_{\varepsilon\underline{\alpha}} \mathscr{F}, \tag{2.3}$$

where the last isomorphism is induced by the Δ-equivariant structure $\iota_\varepsilon \colon [\varepsilon]^*\mathscr{F} \cong \mathscr{F}$. Since the pullback $[\varepsilon]^*$ commutes with products for $[\varepsilon]$ an isomorphism, this induces an isomorphism $\iota_\varepsilon \colon [\varepsilon]^*\mathscr{C}^q(\mathfrak{U}, \mathscr{F}) \xrightarrow{\cong} \mathscr{C}^q(\mathfrak{U}, \mathscr{F})$. We now claim that this isomorphism is compatible with the differential (2.1):

Lemma 2.1.9. *$\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is a complex of Δ-equivariant sheaves on U.*

Proof. We need to show that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & {[\varepsilon]^*\mathscr{F}} \arrow[r] \arrow[d, "\iota_\varepsilon"] & {[\varepsilon]^*\mathscr{C}^0(\mathfrak{U}, \mathscr{F})} \arrow[r, "{[\varepsilon]^*d^0}"] \arrow[d, "\iota_\varepsilon"] & {[\varepsilon]^*\mathscr{C}^1(\mathfrak{U}, \mathscr{F})} \arrow[r, "{[\varepsilon]^*d^1}"] \arrow[d, "\iota_\varepsilon"] & \cdots \arrow[r, "{[\varepsilon]^*d^{q-1}}"] \arrow[d, "\iota_\varepsilon"] & {[\varepsilon]^*\mathscr{C}^q(\mathfrak{U}, \mathscr{F})} \arrow[r, "{[\varepsilon]^*d^q}"] \arrow[d, "\iota_\varepsilon"] & \cdots \\ 0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \arrow[r, "d^0"] & \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \arrow[r, "d^1"] & \cdots \arrow[r, "d^{q-1}"] & \mathscr{C}^q(\mathfrak{U}, \mathscr{F}) \arrow[r, "d^q"] & \cdots \end{tikzcd}$$

By fixing an open V of U and $q \in \mathbb{N}$ we restrict ourselves to a single square diagram of modules, which after applying the first isomorphism of (2.3) becomes:

$$\begin{tikzcd}
\prod^{\text{alt}}_{\underline{\alpha}\in A^{q+1}} j_{\varepsilon\underline{\alpha}*} j^*_{\varepsilon\underline{\alpha}} [\varepsilon]^*\mathscr{F}(V) \arrow[r, "{[\varepsilon]^* d^q}"] \arrow[d, "\iota_\varepsilon"] & \prod^{\text{alt}}_{\underline{\alpha}\in A^{q+2}} j_{\varepsilon\underline{\alpha}*} j^*_{\varepsilon\underline{\alpha}} [\varepsilon]^*\mathscr{F}(V) \arrow[d, "\iota_\varepsilon"] \\
\prod^{\text{alt}}_{\underline{\alpha}\in A^{q+1}} j_{\varepsilon\underline{\alpha}*} j^*_{\varepsilon\underline{\alpha}} \mathscr{F}(V) \arrow[r, "d^q"] & \prod^{\text{alt}}_{\underline{\alpha}\in A^{q+2}} j_{\varepsilon\underline{\alpha}*} j^*_{\varepsilon\underline{\alpha}} \mathscr{F}(V)
\end{tikzcd}$$

Now let $f \in \mathscr{F}(V)$, and $[\varepsilon]^*(f) \in [\varepsilon]^*\mathscr{F}(V)$. Then by defining $\tilde{f}$ as the image of $[\varepsilon]^*(f)$ under $\iota_\varepsilon\colon [\varepsilon]^*\mathscr{F} \xrightarrow{\cong} \mathscr{F}$, the previous diagram becomes element wise

$$\begin{tikzcd}
\left(([\varepsilon]^*(f))_{\varepsilon\underline{\alpha}}\right) \arrow[r, mapsto] \arrow[d, mapsto] & \left(\sum_{j=0}^{q+1}(-1)^j ([\varepsilon]^*(f))_{\varepsilon\alpha_0\cdots\varepsilon\check{\alpha}_j\cdots\varepsilon\alpha_{q+1}}\big|_{U_{\varepsilon\underline{\alpha}}\cap V}\right) \arrow[d, mapsto] \\
\left(\tilde{f}_{\varepsilon\underline{\alpha}}\right) \arrow[r, mapsto] & \left(\sum_{j=0}^{q+1}(-1)^j \tilde{f}_{\varepsilon\alpha_0\cdots\varepsilon\check{\alpha}_j\cdots\varepsilon\alpha_{q+1}}\big|_{U_{\varepsilon\underline{\alpha}}\cap V}\right)
\end{tikzcd}$$

which is clearly commutative. It's clear that the previous diagram commutes with the restriction maps of each sheaf, therefore we are done. $\square$

Proposition 2.1.10. *The sheaf $\mathscr{C}^q(\mathfrak{U},\mathscr{F})$ is acyclic with respect to the functor $\Gamma(U/\Delta,-)$.*

Proof. By definition, the functor $\Gamma(U/\Delta,-)$ is the composite of the left-exact functors $\Gamma(U,-)$ and $\mathrm{Hom}_{\mathbb{Z}[\Delta]}(\mathbb{Z},-)$. Following [7, Théorème 5.2.1] we have a spectral sequence converging in the following manner

$$E_2^{a,b} = H^a\big(\Delta, H^b(U,\mathscr{C}^q(\mathfrak{U},\mathscr{F}))\big) \Rightarrow H^{a+b}(U/\Delta,\mathscr{C}^q(\mathfrak{U},\mathscr{F})).$$

We now prove that $H^b(U,\mathscr{C}^q(\mathfrak{U},\mathscr{F})) = 0$ if $b \neq 0$. Notice that

$$\mathscr{C}^q(\mathfrak{U},\mathscr{F}) \cong \prod_{\alpha_0<\cdots<\alpha_q} j_{\underline{\alpha}*} j^*_{\underline{\alpha}}\mathscr{F}$$

after fixing a total order on the set $A \subset \mathbb{R}$. Indeed, for any $\underline{\alpha} \in A^{q+1}$, if $\alpha_i = \alpha_j$ for $i \neq j$, then $(s)_{\underline{\alpha}} = 0$ for $(s) \in \prod^{\text{alt}}_{\underline{\alpha}\in A^{q+1}} j_{\underline{\alpha}*} j^*_{\underline{\alpha}}\mathscr{F}$. Otherwise, we may re-arrange the coordinates of $\underline{\alpha}$ such that $\rho(\underline{\alpha}) = \underline{\tilde{\alpha}} = (\tilde{\alpha}_0,\ldots,\tilde{\alpha}_q)$ satisfies $\tilde{\alpha}_0 < \ldots < \tilde{\alpha}_q$. Then $\big(s_{\underline{\alpha}}\big) = \big(\mathrm{sgn}(\rho) s_{\underline{\tilde{\alpha}}}\big)$, so

$(s_{\underline{\tilde{\alpha}}})$ determines $(s_{\underline{\alpha}})$.

We know that the pullback of a quasi-coherent sheaf is quasi-coherent, so $j^*_{\underline{\alpha}}\mathscr{F}$ is a quasi-coherent sheaf of $U_{\underline{\alpha}}$, which is an affine scheme, and therefore $H^b(U_{\underline{\alpha}}, j^*_{\underline{\alpha}}\mathscr{F}) = 0$ for $b > 0$. We know that the product of quasi-coherent sheaves on an affine scheme is quasi-coherent, therefore we also have $H^b(U_{\underline{\alpha}}, \prod_{\alpha_0<\cdots<\alpha_q} j^*_{\underline{\alpha}}\mathscr{F}) = 0$ for $b > 0$. Now, since $j_{\underline{\alpha}}$ is the inclusion of an affine scheme into U, it is an affine morphism, so we may use Lemma 2.1.11 below to deduce that $H^b(U_{\underline{\alpha}}, \prod_{\alpha_0<\cdots<\alpha_q} j^*_{\underline{\alpha}}\mathscr{F}) = H^b(U, j_{\underline{\alpha}*}\prod_{\alpha_0<\cdots<\alpha_q} j^*_{\underline{\alpha}}\mathscr{F}) = 0$ if $b > 0$. Lastly, since the pushforward is a right adjoint, it commutes with products, and so the claim follows.

It is now sufficient to prove that $H^a\big(\Delta, H^0(U, \mathscr{C}^q(\mathfrak{U}, \mathscr{F}))\big) = 0$ for any integer $a \neq 0$, where

$$H^0(U, \mathscr{C}^q(\mathfrak{U}, \mathscr{F})) = \prod_{\underline{\alpha}\in A^{q+1}}^{\text{alt}} \Gamma(U, j_{\underline{\alpha}*}j^*_{\underline{\alpha}}\mathscr{F}) \cong \prod_{\alpha_0<\cdots<\alpha_q} \Gamma(U_{\underline{\alpha}}, \mathscr{F}).$$

Assume that the total order is preserved by the action of Δ, that is, $\varepsilon\alpha > \varepsilon\beta \Leftrightarrow \alpha > \beta$ (for example, we may take the order on $\mathbb{R}$ through an embedding $\tau\colon A \hookrightarrow \mathbb{R}$ for some $\tau \in I$). Let B be the subset of A^{q+1} consisting of elements $\underline{\alpha} = (\alpha_0, \dots, \alpha_q)$ such that $\alpha_0 < \cdots < \alpha_q$. Since the action of Δ on B is free, we denote by B_0 a subset of B representing the set $\Delta\backslash B$, so that any $\underline{\alpha} \in B$ may be written uniquely as $\underline{\alpha} = \varepsilon\underline{\alpha}_0$ for some $\varepsilon \in \Delta$ and $\underline{\alpha}_0 \in B_0$. We let

$$M \coloneqq \prod_{\underline{\alpha}\in B_0} \Gamma(U_{\underline{\alpha}}, \mathscr{F}),$$

and we let $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\Delta], M)$ be the coinduced module of M, with the action of Δ given for any $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\Delta], M)$ by $\varepsilon\varphi(u) = \varphi(u\varepsilon)$ for any $u \in \mathbb{Z}[\Delta]$ and $\varepsilon \in \Delta$. Then we have a $\mathbb{Z}[\Delta]$-linear isomorphism

$$H^0(U, \mathscr{C}^q(\mathfrak{U}, \mathscr{F})) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\Delta], M) \tag{2.4}$$

given by mapping any $(s_{\underline{\alpha}}) \in H^0(U, \mathscr{C}^q(\mathfrak{U}, \mathscr{F}))$ to the $\mathbb{Z}$-linear homomorphism

$$\varphi_{(s_{\underline{\alpha}})}(\delta) \coloneqq \Big(\iota_\delta\big([\delta]^* s_{\delta^{-1}\underline{\alpha}_0}\big)\Big) \in M$$

for any $\delta \in \Delta$. The compatibility of (2.4) with the action of Δ is seen as follows. By definition, the action of $\varepsilon \in \Delta$ on $(s_{\underline{\alpha}}) \in H^0(U, \mathscr{C}^q(\mathfrak{U}, \mathscr{F}))$ is given by $\varepsilon\big((s_{\underline{\alpha}})\big) = \big(\iota_\varepsilon([\varepsilon]^* s_{\varepsilon^{-1}\underline{\alpha}})\big)$. Hence noting that

$$\iota_\delta \circ [\delta]^* \iota_\varepsilon = \iota_{\delta\varepsilon} \colon \Gamma(U_{\underline{\alpha}}, [\delta\varepsilon]^* \mathscr{F}) \to \Gamma(U_{\underline{\alpha}}, \mathscr{F})$$

and $[\delta]^* \circ \iota_\varepsilon = [\delta]^* \iota_\varepsilon \circ [\delta]^* \colon \Gamma(U_{\underline{\alpha}}, [\varepsilon]^* \mathscr{F}) \to \Gamma(U_{\underline{\alpha}}, [\delta]^* \mathscr{F})$ for any $\delta \in \Delta$, we have

$$\begin{aligned} \varphi_{\varepsilon(s_{\underline{\alpha}})}(\delta) &= \Big(\iota_\delta\big([\delta]^*(\iota_\varepsilon([\varepsilon]^* s_{\varepsilon^{-1}\delta^{-1}\underline{\alpha}_0}))\big)\Big) = \Big((\iota_\delta \circ [\delta]^* \iota_\varepsilon)\big([\delta\varepsilon]^* s_{\varepsilon^{-1}\delta^{-1}\underline{\alpha}_0}\big)\Big) \\ &= \big(\iota_{\delta\varepsilon}([\delta\varepsilon]^* s_{\varepsilon^{-1}\delta^{-1}\underline{\alpha}_0})\big) = \varphi_{(s_{\underline{\alpha}})}(\delta\varepsilon) \end{aligned}$$

as desired. The fact that (2.4) is an isomorphism follows from the fact that B_0 is a representative of $\Delta \backslash B$. By (2.4) and Shapiro's Lemma (see [5, Proposition 1.6.4]), we have $H^a(\Delta, H^0(U, \mathscr{C}^q(\mathfrak{U}, \mathscr{F}))) \cong H^a(\{1\}, M) = 0$ for $a \neq 0$ as desired. □

The following Lemma was used in the proof of Proposition 2.1.10 above.

Lemma 2.1.11. *Let $f : X \to Y$ be an affine morphism of schemes, and $\mathscr{F}$ a quasi-coherent sheaf on X. Then for any integer $m \geq 0$, we have a natural isomorphism*

$$H^m(X, \mathscr{F}) \cong H^m(Y, f_* \mathscr{F}).$$

Proof. See [6, Chapitre III corollaire 1.3.3].

□

Proposition 2.1.10 gives the following Corollary.

Corollary 2.1.12. *We let $C^\bullet(\mathfrak{U}/\Delta, \mathscr{F}) := \Gamma(U/\Delta, \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$. Then for any integer $m \geq 0$, the equivariant cohomology $H^m(U/\Delta, \mathscr{F})$ is given as*

$$H^m(U/\Delta, \mathscr{F}) = H^m(C^\bullet(\mathfrak{U}/\Delta, \mathscr{F})).$$

By definition, for any integer $q \in \mathbb{Z}$, we have

$$C^q(\mathfrak{U}/\Delta, \mathscr{F}) = \Bigg(\prod^{\text{alt}}_{\underline{\alpha} \in A^{q+1}} \Gamma(U_{\underline{\alpha}}, \mathscr{F})\Bigg)^\Delta.$$

2.2 Shintani Generating Class

In this section, we will use the description of equivariant cohomology of Corollary 2.1.12 to define the Shintani generating class as a class in $H^{g-1}(U/\Delta, \mathscr{O}_{\mathbb{T}})$. We will then consider the action of the differential operators ∂_τ on this class. We first interpret the generating functions $\mathcal{G}_\sigma(z)$ of Definition 1.2.4 as rational functions on $\mathbb{T}$.

Proposition 2.2.1. *Let $\mathfrak{D}^{-1} := \{u \in F \mid \mathrm{Tr}_{F/\mathbb{Q}}(u\mathcal{O}_F) \subset \mathbb{Z}\}$ be the inverse different of F. Then there exists an isomorphism*

$$(F \otimes \mathbb{C})/\mathfrak{D}^{-1} \xrightarrow{\cong} \mathbb{T}(\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{C}^\times), \qquad z \mapsto \xi_z \tag{2.5}$$

given by mapping any $z \in F \otimes \mathbb{C}$ to the character $\xi_z(\alpha) := e^{2\pi i \mathrm{Tr}(\alpha z)}$ in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{C}^\times)$.

Proof. We know that the trace pairing is perfect, that is, we have an isomorphism $F \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{Q})$, sending $\alpha \mapsto (\beta \mapsto \mathrm{Tr}(\alpha\beta))$. Therefore we also have the isomorphisms $F \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{Q})$ and $F \otimes \mathbb{C} \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{C})$, which sends $z \mapsto (\alpha \mapsto \mathrm{Tr}(\alpha z))$, where $\mathrm{Tr}_{F\otimes\mathbb{C}/\mathbb{C}} \colon F \otimes \mathbb{C} \to \mathbb{C}$ comes from the fact that $\mathbb{C} \hookrightarrow F \otimes \mathbb{C}$ is a finite locally free morphism of rings.

Now, by composing each $\mathrm{Tr}((-)z) \colon \mathcal{O}_F \to \mathbb{C}$ with $e^{2\pi i(-)} \colon \mathbb{C} \to \mathbb{C}^\times$, we have a surjection $F \otimes \mathbb{C} \to \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{C}^\times)$, whose kernel is precisely the preimage of $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_F, \mathbb{Z})$, which is $\mathfrak{D}^{-1}$ by definition.

□

The function t^α on $\mathbb{T}(\mathbb{C})$ corresponds through the isomorphism (2.5) to the function $e^{2\pi i \mathrm{Tr}(\alpha z)}$ for any $\alpha \in \mathcal{O}_F$. Thus the following holds.

Corollary 2.2.2. *For $\underline{\alpha} = (\alpha_1, \ldots, \alpha_g) \in A^g$ and $\sigma := \sigma_{\underline{\alpha}}$, consider the rational function*

$$\mathcal{G}_\sigma(t) := \frac{\sum_{\alpha \in \breve{P}_{\underline{\alpha}} \cap \mathcal{O}_F} t^\alpha}{\left(1 - t^{\alpha_1}\right) \cdots \left(1 - t^{\alpha_g}\right)}$$

on $\mathbb{T}$, where $P_{\underline{\alpha}}$ is again the parallelepiped spanned by $\alpha_1, \ldots, \alpha_g$. Then $\mathcal{G}_\sigma(t)$ corresponds to the function $\mathscr{G}_\sigma(z)$ of Definition 1.2.4 through the uniformization (2.5). Note that by definition, if we *let* $B := \mathbb{Z}[t^\alpha \mid \alpha \in O_{F+}]$, *then we have*

$$\mathcal{G}_\sigma(t) \in B_{\underline{\alpha}} := B\Big[\frac{1}{1-t^{\alpha_1}}, \dots, \frac{1}{1-t^{\alpha_g}}\Big]. \tag{2.6}$$

Again, we fix an ordering $I = \{\tau_1, \dots, \tau_g\}$. For any $\underline{\boldsymbol{\alpha}} = (\alpha_1, \dots, \alpha_g) \in O_{F+}^g$, let $\big(\alpha_i^{\tau_j}\big)$ be the same $g \times g$ matrix as in the proof of 1.2.3, whose (i, j)-component is $\alpha_i^{\tau_j}$, and let $\operatorname{sgn}(\underline{\boldsymbol{\alpha}}) \in \{0, \pm 1\}$ be the signature of $\det\big(\alpha_i^{\tau_j}\big)$.

We define the *Shintani generating class* $\mathcal{G}$ as follows.

Proposition 2.2.3. *For any* $\underline{\boldsymbol{\alpha}} = (\alpha_1, \dots, \alpha_g) \in A^g$, *we let*

$$\mathcal{G}_{\underline{\boldsymbol{\alpha}}} := \operatorname{sgn}(\underline{\boldsymbol{\alpha}}) \mathcal{G}_{\sigma_{\underline{\boldsymbol{\alpha}}}}(t) \in \Gamma(U_{\underline{\boldsymbol{\alpha}}}, \mathscr{O}_{\mathbb{T}}).$$

Then we have $(\mathcal{G}_{\underline{\boldsymbol{\alpha}}}) \in C^{g-1}(\mathfrak{U}, \mathscr{O}_{\mathbb{T}})$. *Moreover,* $(\mathcal{G}_{\underline{\boldsymbol{\alpha}}})$ *satisfies the cocycle condition* $d^{g-1}(\mathcal{G}_{\underline{\boldsymbol{\alpha}}}) = 0$, *hence defines a class*

$$\mathcal{G} := [\mathcal{G}_{\underline{\boldsymbol{\alpha}}}] \in H^{g-1}(U/\Delta, \mathscr{O}_{\mathbb{T}}).$$

We call this class the Shintani generating class.

Proof. By construction, $(\mathcal{G}_{\underline{\boldsymbol{\alpha}}})$ defines an element in $\Gamma(U, \mathscr{C}^{g-1}(\mathfrak{U}, \mathscr{O}_{\mathbb{T}})) = \prod^{\mathrm{alt}}_{\underline{\boldsymbol{\alpha}} \in A^g} \Gamma(U_{\underline{\boldsymbol{\alpha}}}, \mathscr{O}_{\mathbb{T}})$, since a permutation ρ of $\underline{\boldsymbol{\alpha}}$ still generates the same cone, and $\operatorname{sgn}(\rho(\underline{\boldsymbol{\alpha}})) = \operatorname{sgn}(\rho)\operatorname{sgn}(\underline{\boldsymbol{\alpha}})$. Since $[\varepsilon]^* \mathcal{G}_{\underline{\boldsymbol{\alpha}}} = \mathcal{G}_{\varepsilon\underline{\boldsymbol{\alpha}}}$ for any $\varepsilon \in \Delta$, we have

$$(\mathcal{G}_{\underline{\boldsymbol{\alpha}}}) \in \Gamma\big(U, \mathscr{C}^{g-1}(\mathfrak{U}, \mathscr{O}_{\mathbb{T}})\big)^{\Delta} = C^{g-1}(\mathfrak{U}/\Delta, \mathscr{O}_{\mathbb{T}}).$$

To prove the cocycle condition $d^{g-1}(\mathcal{G}_{\underline{\boldsymbol{\alpha}}}) = 0$, it is sufficient to check that

$$\sum_{j=0}^{g} (-1)^j \mathcal{G}_{(\alpha_0, \dots, \check{\alpha}_j, \dots \alpha_g)} = 0 \tag{2.7}$$

for any $\alpha_0, \dots, \alpha_g \in A$. Now, after remembering that $\sigma_{\underline{\boldsymbol{\alpha}}} = P_{\underline{\boldsymbol{\alpha}}} \bigoplus_{i=1}^g \alpha_i \mathbb{N}$ and rewriting

$$\mathcal{G}_\sigma(t) := \frac{\sum_{\alpha \in \breve{P}_{\underline{\boldsymbol{\alpha}}} \cap O_F} t^\alpha}{\big(1 - t^{\alpha_1}\big) \cdots \big(1 - t^{\alpha_g}\big)} = \sum_{\alpha \in \breve{P}_{\underline{\boldsymbol{\alpha}}} \cap O_F} t^\alpha \sum_{n_1 \in \mathbb{N}} t^{\alpha_1 n_1} \cdots \sum_{n_g \in \mathbb{N}} t^{\alpha_g n_g},$$

it becomes clear that the rational function $\mathcal{G}_{\sigma_{\underline{\alpha}}}(t)$ maps to the formal power series

$$\mathcal{G}_{\sigma_{\underline{\alpha}}}(t) \mapsto \sum_{\alpha \in \breve{\sigma}_{\underline{\alpha}} \cap O_F} t^{\alpha}$$

by taking the formal completion $B_{\underline{\alpha}} \hookrightarrow \mathbb{Z}[\![t^{\alpha_1}, \ldots, t^{\alpha_g}]\!]$, where $B_{\underline{\alpha}}$ is the ring defined in (2.6). Since the map taking the formal completion is injective, it is sufficient to check (2.7) for the associated formal power series. By [15, Proposition 6.2], we have

$$\sum_{j=0}^{g} (-1)^j \operatorname{sgn}(\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_g) \mathbf{1}_{\breve{\sigma}_{(\alpha_0, \ldots, \check{\alpha}_j, \ldots \alpha_g)}} \equiv 0$$

as a function on $\mathbb{R}_+^I$, where $\mathbf{1}_R$ is the characteristic function of $R \subset \mathbb{R}_+^I$ satisfying $\mathbf{1}_R(x) = 1$ if $x \in R$ and $\mathbf{1}_R(x) = 0$ if $x \notin R$. Then

$$\sum_{j=1}^{g} (-1)^j \mathcal{G}_{(\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_g)} = \sum_{j=1}^{g} (-1)^j \operatorname{sgn}(\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_g) \sum_{\alpha \in \breve{\sigma}_{(\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_g)} \cap O_F} t^{\alpha} =$$

$$= \sum_{\alpha \in O_F} t^{\alpha} \left(\sum_{j=0}^{g} (-1)^j \operatorname{sgn}(\alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_g) \mathbf{1}_{\breve{\sigma}_{(\alpha_0, \ldots, \check{\alpha}_j, \ldots \alpha_g)}}(\alpha) \right) = 0$$

as desired. □

We will next define differential operators ∂_τ for $\tau \in I$ on equivariant cohomology. Since $t^{\alpha} = e^{2\pi i \operatorname{Tr}(\alpha z)}$ through (2.5) for any $\alpha \in O_F$, we have

$$\frac{dt^{\alpha}}{t^{\alpha}} = \sum_{\tau \in I} 2\pi i \alpha^{\tau} dz_{\tau}. \tag{2.8}$$

Let $\alpha_1, \ldots, \alpha_g$ be a basis of O_F. For any $\tau \in I$, we let ∂_τ be the differential operator

$$\partial_\tau := \sum_{j=1}^{g} \alpha_j^{\tau} t^{\alpha_j} \frac{\partial}{\partial t^{\alpha_j}}.$$

By (2.8), we see that ∂_τ corresponds to the differential operator $\frac{1}{2\pi i} \frac{\partial}{\partial z_\tau}$ through the uniformization (2.5), and hence is independent of the choice of the basis $\alpha_1, \ldots, \alpha_g$. By Theorem 1.2.6 and Corollary 2.2.2, we have the following result.

Proposition 2.2.4. *Let* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_g) \in A^g$ *and* $\sigma = \sigma_{\underline{\alpha}}$. *For any* $\boldsymbol{k} = (k_\tau) \in \mathbb{N}^I$ *and* $\partial^{\boldsymbol{k}} := \prod_{\tau \in I} \partial_\tau^{k_\tau}$, *we have*

$$\partial^{\boldsymbol{k}} \mathcal{G}_\sigma(\xi) = \zeta_\sigma(\xi, -\boldsymbol{k})$$

for any torsion point $\xi \in U_{\underline{\alpha}}$.

The differential operator ∂_τ gives a morphism of abelian sheaves

$$\partial_\tau \colon \mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k}) \to \mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k} - 1_\tau)$$

where $\mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k})$ is the Δ-equivariant sheaf from 2.1.6. We claim the following.

Lemma 2.2.5. *The morphism of abelian sheaves* ∂_τ *defined above is compatible with the action of* Δ *for any* $\boldsymbol{k} \in \mathbb{Z}^I$.

Proof. The claim is that the following diagram is commutative

$$\begin{tikzcd}
\mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k}) \arrow[r] \arrow[d, "\partial_\tau"] & {[\varepsilon]^*\mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k})} \arrow[r, "\iota_\varepsilon"] & \mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k}) \arrow[d, "\partial_\tau"] \\
\mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k} - 1_\tau) \arrow[r] & {[\varepsilon]^*\mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k} - 1_\tau)} \arrow[r, "\iota_\varepsilon"] & \mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k} - 1_\tau).
\end{tikzcd}$$

Indeed, using the linearity of the differential operator ∂_τ and the explicit description of the Δ-equivariant structure given in the proof of Lemma 2.1.7, this amounts to showing that

$$\varepsilon^{-\boldsymbol{k}} \partial_\tau [\varepsilon]_\sharp (t^\alpha) = \varepsilon^{-\boldsymbol{k}+1_\tau} [\varepsilon]_\sharp (\partial_\tau t^\alpha).$$

Simplifying, we need to show that

$$\partial_\tau [\varepsilon]_\sharp (t^\alpha) = \tau(\varepsilon) [\varepsilon]_\sharp (\partial_\tau t^\alpha),$$

which follows from the definitions. $\square$

This induces a homomorphism

$$\partial_\tau \colon H^m(U_{\bar{F}}/\Delta, \mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k})) \to H^m(U_{\bar{F}}/\Delta, \mathscr{O}_{\mathbb{T}_{\bar{F}}}(\boldsymbol{k} - 1_\tau))$$

on equivariant cohomology.

Lemma 2.2.6. *The operators ∂_τ for $\tau \in I$ are commutative with each other. Moreover, the composite*

$$\partial \coloneqq \prod_{\tau \in I} \partial_\tau \colon \mathscr{O}_{\mathbb{T}_{\bar{F}}} \to \mathscr{O}_{\mathbb{T}_{\bar{F}}}(-1, \dots, -1) = \mathscr{O}_{\mathbb{T}_{\bar{F}}}$$

is defined over $\mathbb{Q}$, that is, it is a base change to $\bar{F}$ of a morphism of abelian sheaves $\partial \colon \mathscr{O}_{\mathbb{T}} \to \mathscr{O}_{\mathbb{T}}$. In particular, ∂ induces a homomorphism

$$\partial \colon H^m(U/\Delta, \mathscr{O}_{\mathbb{T}}) \to H^m(U/\Delta, \mathscr{O}_{\mathbb{T}}). \tag{2.9}$$

Proof. The commutativity is clear from the definition. Since the group $\mathrm{Gal}(\bar{F}/\mathbb{Q})$ permutes the operators ∂_τ, the operator ∂ is invariant under this action. This gives our assertion. □

In the following chapter, we obtain a canonical decomposition of the Hecke L-functions through the specialization of the classes

$$\partial^k \mathcal{G} \in H^{g-1}\big(U/\Delta, \mathscr{O}_{\mathbb{T}}\big) \tag{2.10}$$

for $k \in \mathbb{N}$ at nontrivial torsion points of $\mathbb{T}$.

Chapter Three

Canonical Decomposition: Special Case

3.1 Specialization to Torsion Points

Suppose we have a group homomorphism $\pi\colon G \to H$. For a G-scheme X and an H-scheme Y, we say that a morphism $f\colon X \to Y$ of schemes is *equivariant* with respect to π, if we have $f \circ [u] = [\pi(u)] \circ f$ for any $u \in G$. Under these hypotheses, if $\mathscr{F}$ is a H-equivariant sheaf on Y, then $f^*\mathscr{F}$ is naturally an G-equivariant sheaf on X. Indeed, the equivariant structure $\iota_u : [u]^*(f^*\mathscr{F}) \xrightarrow{\cong} f^*\mathscr{F}$ for any $u \in G$ is given by noting that $[u]^*(f^*\mathscr{F}) = f^*([\pi(u)]^*\mathscr{F})$ by the π-equivariance of f and $f^*\iota_{\pi(u)}\colon f^*([\pi(u)]^*\mathscr{F}) \xrightarrow{\cong} f^*\mathscr{F}$ from the covariant functor f^* applied to $\iota_{\pi(u)}\colon ([\pi(u)]^*\mathscr{F}) \xrightarrow{\cong} \mathscr{F}$. Thus $\iota_u = f^*\iota_{\pi(u)}$. Consequently, f induces the pull-back homomorphism

$$f^*\colon H^m(Y/H, \mathscr{F}) \to H^m(X/G, f^*\mathscr{F}) \tag{3.1}$$

on equivariant cohomology.

For any nontrivial torsion point ξ of $\mathbb{T}$, let $\Delta_\xi \subset \Delta$ be the isotropic subgroup of ξ. Since $\mathbb{Q}(\xi)$ is the function field of $\xi \in \mathbb{T}$, we may view $\xi := \operatorname{Spec} \mathbb{Q}(\xi)$ as a Δ_ξ-scheme with a trivial action of Δ_ξ. Then the natural inclusion $\xi \to U$ for $U := \mathbb{T} \setminus \{1\}$ is equivariant with respect to the inclusion $\Delta_\xi \subset \Delta$, hence the pullback (3.1) induces the specialization map

$$\xi^*\colon H^m(U/\Delta, \mathscr{O}_{\mathbb{T}}) \to H^m(\xi/\Delta_\xi, \mathscr{O}_\xi). \tag{3.2}$$

The purpose of this chapter is to prove the following Theorem, which provides a canonical decomposition of the Lerch zeta function for totally real fields with trivial narrow class group.

Theorem 3.1.1. *Let F be a totally real number field of degree g with trivial narrow class group, let ξ be a nontrivial torsion point of $\mathbb{T} = \mathbb{T}^{\mathcal{O}_F}$, and let k be an integer ≥ 0. If we let $\mathcal{G}$ be the Shintani generating class defined in Proposition 2.2.3, and if we let $\partial^k \mathcal{G}(\xi) \in H^{g-1}(\xi/\Delta_\xi, \mathcal{O}_\xi)$ be the image by the specialization map ξ^* of the class $\partial^k \mathcal{G}$ defined in (2.10), then we have*

$$\partial^k \mathcal{G}(\xi) = \mathcal{L}(\xi\Delta, -k)$$

through the canonical isomorphism $H^{g-1}(\xi/\Delta_\xi, \mathcal{O}_\xi) \cong \mathbb{Q}(\xi)$ given in Proposition 3.2.5 below.

We will prove Theorem 3.1.1 at the end of this chapter. The specialization map can be expressed explicitly in terms of cocycles as follows. We let $V_\alpha := U_\alpha \cap \xi$ for any $\alpha \in A$. Then $\mathfrak{V} := \{V_\alpha\}_{\alpha \in A}$ is an affine open covering of ξ. For any integer $q \geq 0$ and $\underline{\alpha} = (\alpha_0, \dots, \alpha_q) \in A^{q+1}$, we let $V_{\underline{\alpha}} := V_{\alpha_0} \cap \cdots \cap V_{\alpha_q}$ and

$$C^q\big(\mathfrak{V}/\Delta_\xi, \mathcal{O}_\xi\big) := \Bigg(\prod_{\underline{\alpha} \in A^{q+1}}^{\text{alt}} \Gamma(V_{\underline{\alpha}}, \mathcal{O}_\xi)\Bigg)^{\Delta_\xi}. \tag{3.3}$$

Here, note that $\Gamma(V_{\underline{\alpha}}, \mathcal{O}_\xi) = \mathbb{Q}(\xi)$ if $V_{\underline{\alpha}} \neq \emptyset$ and $\Gamma(V_{\underline{\alpha}}, \mathcal{O}_\xi) = \{0\}$ otherwise. The same argument as that of Corollary 2.1.12 shows that we have

$$H^m(\xi/\Delta_\xi, \mathcal{O}_\xi) \cong H^m\big(C^\bullet\big(\mathfrak{V}/\Delta_\xi, \mathcal{O}_\xi\big)\big). \tag{3.4}$$

3.2 Canonical Identification through Group (Co)Homology

In this section, we will use tools from group (co)homology and duality arguments to obtain the canonical isomorphism $H^{g-1}(\xi/\Delta_\xi, \mathcal{O}_\xi) \cong \mathbb{Q}(\xi)$, concluding with the proof of Theorem 3.1.1.

We let A_ξ be the subset of elements $\alpha \in A$ satisfying $\xi \in U_\alpha$. This is equivalent to the condition that $\xi(\alpha) \neq 1$. We will next prove in Lemma 3.2.1 that the cochain complex $C^\bullet\big(\mathfrak{V}/\Delta_\xi, \mathcal{O}_\xi\big)$ of (3.3) is isomorphic to the dual of the chain complex $C_\bullet(A_\xi)$ defined as follows. For any integer $q \geq 0$, we let

$$C_q(A_\xi) := \bigoplus_{\underline{\alpha} \in A_\xi^{q+1}}^{\mathrm{alt}} \mathbb{Z}\underline{\alpha}$$

be the quotient of $\bigoplus_{\underline{\alpha} \in A_\xi^{q+1}} \mathbb{Z}\underline{\alpha}$ by the submodule generated by

$$\{\rho(\underline{\alpha}) - \mathrm{sgn}(\rho)\underline{\alpha} \mid \underline{\alpha} \in A_\xi^{q+1}, \rho \in \mathfrak{S}_{q+1}\} \cup \{\underline{\alpha} = (\alpha_0, \ldots, \alpha_q) \mid \alpha_i = \alpha_j \text{ for some } i \neq j\}.$$

We denote by $\langle \underline{\alpha} \rangle$ the class represented by $\underline{\alpha}$ in $C_q(A_\xi)$. We see that $C_q(A_\xi)$ has a natural action of Δ_ξ and is a free $\mathbb{Z}[\Delta_\xi]$-module. In fact, a basis of $C_q(A_\xi)$ may be constructed in a similar way to the construction of B_0 in the proof of Proposition 2.1.10. Then $C_\bullet(A_\xi)$ is a complex of $\mathbb{Z}[\Delta_\xi]$-modules with respect to the standard differential operator $d_q \colon C_q(A_\xi) \to C_{q-1}(A_\xi)$ given by

$$d_q(\langle \alpha_0, \ldots, \alpha_q \rangle) := \sum_{j=0}^{q} (-1)^j \langle \alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_q \rangle$$

for any $\underline{\alpha} = (\alpha_0, \ldots, \alpha_q) \in A_\xi^{q+1}$. If we view $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module with trivial Δ_ξ-action and let $d_0 \colon C_0(A_\xi) \to \mathbb{Z}$ be the unique $\mathbb{Z}$-linear homomorphism defined by $d_0(\langle \alpha \rangle) = 1$ for any $\alpha \in A_\xi$, then $C_\bullet(A_\xi)$ is a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module. As announced previously, the following is true.

Lemma 3.2.1. *There exists a natural isomorphism of complexes*

$$C^\bullet(\mathfrak{B}/\Delta_\xi, \mathscr{O}_\xi) \xrightarrow{\cong} \mathrm{Hom}_{\Delta_\xi}(C_\bullet(A_\xi), \mathbb{Q}(\xi)).$$

Proof. Firstly, we note that the differential d^q of the cochain complex $\mathrm{Hom}_{\Delta_\xi}(C_\bullet(A_\xi), \mathbb{Q}(\xi))$ is defined by

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}[\Delta_\xi]}(C_q(A_\xi), \mathbb{Q}(\xi)) &\xrightarrow{d^{q+1}} \mathrm{Hom}_{\mathbb{Z}[\Delta_\xi]}(C_{q+1}(A_\xi), \mathbb{Q}(\xi)) \\ f &\longmapsto f \circ d_{q+1} \end{aligned}$$

where d_{q+1} is the differential of $C_\bullet(A_\xi)$. We also note that from the equality $\prod_{\underline{\alpha}\in A^{q+1}} \Gamma(V_{\underline{\alpha}}, \mathscr{O}_\xi) = \prod_{\underline{\alpha}\in A_\xi^{q+1}} \mathbb{Q}(\xi)$ we have

$$C^q(\mathfrak{B}/\Delta_\xi, \mathscr{O}_\xi) = \left(\prod_{\underline{\alpha}\in A^{q+1}}^{\text{alt}} \Gamma(V_{\underline{\alpha}}, \mathscr{O}_\xi)\right)^{\Delta_\xi} = \left(\prod_{\substack{\alpha_0<\ldots<\alpha_q \\ \underline{\alpha}\in A_\xi^{q+1}}} \mathbb{Q}(\xi)\right)^{\Delta_\xi}.$$

For the rest of this proof, we assume $\alpha \in A_\xi$. In order to prove the claim, we need to show that the following generic square is commutative for each $q \in \mathbb{N}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}[\Delta_\xi]}(C_q(A_\xi), \mathbb{Q}(\xi)) \arrow[r, "d^{q+1}"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}[\Delta_\xi]}(C_{q+1}(A_\xi), \mathbb{Q}(\xi)) \arrow[d] \\
\left(\prod_{\alpha_0<\ldots<\alpha_q} \mathbb{Q}(\xi)\right)^{\Delta_\xi} \arrow[r, "d^{q+1}"] & \left(\prod_{\alpha_0<\ldots<\alpha_{q+1}} \mathbb{Q}(\xi)\right)^{\Delta_\xi}
\end{tikzcd}$$

with the vertical arrows being natural isomorphisms. We first remember the universal property of the direct sum, which states that for each morphism $f_{\underline{\alpha}} : \mathbb{Z}_{\underline{\alpha}} \to \mathbb{Q}_\xi$, there exists a unique $f : \bigoplus_{\alpha_0<\ldots<\alpha_q} \mathbb{Z}_{\underline{\alpha}} \to \mathbb{Q}_\xi$ such that $f_{\underline{\alpha}} = f \circ j_{\underline{\alpha}}$, where $j_{\underline{\alpha}} : \mathbb{Z}_{\underline{\alpha}} \hookrightarrow \bigoplus_{\alpha_0<\ldots<\alpha_q} \mathbb{Z}_{\underline{\alpha}}$ is the natural inclusion. With this notation fixed, we get the following canonical isomorphisms

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}[\Delta_\xi]}\left(\bigoplus_{\alpha_0<\ldots<\alpha_q} \mathbb{Z}_{\underline{\alpha}}, \mathbb{Q}(\xi)\right) \arrow[r, "\cong"] & \prod_{\alpha_0<\ldots<\alpha_q} \mathrm{Hom}_{\mathbb{Z}[\Delta_\xi]}(\mathbb{Z}_{\underline{\alpha}}, \mathbb{Q}_\xi) \arrow[r, "\cong"] & \left(\prod_{\alpha_0<\ldots<\alpha_q} \mathbb{Q}_\xi\right)^{\Delta_\xi} \\
f \arrow[r, mapsto] & \prod_{\alpha_0<\ldots<\alpha_q} f_{\underline{\alpha}} \arrow[r, mapsto] & \prod_{\alpha_0<\ldots<\alpha_q} f_{\underline{\alpha}}(\underline{\alpha})
\end{tikzcd}$$

which correspond to the vertical arrows of the previous diagram. Now, to check compatibility with the differentials, we write it element wise

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & f \circ d_{q+1} \arrow[d, mapsto] \\
\prod_{\alpha_0<\ldots<\alpha_q} f_{\underline{\alpha}}(\underline{\alpha}) \arrow[r, mapsto] & \prod_{\alpha_0<\ldots<\alpha_q} \sum_{j=0}^{q+1} (-1)^j f_{(\alpha_0,\ldots,\check{\alpha}_j,\ldots,\alpha_{q+1})}\langle \alpha_0, \ldots, \check{\alpha}_j, \ldots, \alpha_{q+1}\rangle
\end{tikzcd}$$

and see that it is clearly commutative, so we're done.

□

We will next use a Shintani decomposition (see Definition 1.4.1) to construct a complex which is quasi-isomorphic to the complex $C_\bullet(A_\xi)$.

Lemma 3.2.2. *Let ξ be as above. There exists a Shintani decomposition Φ such that each $\sigma \in \Phi$ is of the form $\sigma = \sigma_{\underline{\alpha}}$ for some $\underline{\alpha} \in A_\xi^{q+1}$.*

Proof. Let Φ be a Shintani decomposition. We will deform Φ to construct a Shintani decomposition satisfying our assertion. By the definition of a simplicial fan, for any $\underline{\alpha} = (\alpha_0, \dots, \alpha_q) \in A^{q+1}$ such that $\sigma_{\underline{\alpha}} \in \Phi_{q+1}$, the face $\sigma_{\alpha_i} \in \Phi_1$ for each $0 \le i \le q$. Therefore we may restrict our attention to Φ_1. Let Λ be a finite subset of A such that $\{\sigma_\alpha \mid \alpha \in \Lambda\}$ represents the quotient set $\Delta_\xi \backslash \Phi_1$. We can assume the finiteness of Λ following the same reasoning as in the proof of Proposition 1.4.4. If $\xi(\alpha) \neq 1$ for any $\alpha \in \Lambda$, then Φ satisfies our assertion since $\alpha \in A_\xi$ if and only if $\xi(\alpha) \neq 1$.

Suppose that there exists $\alpha \in \Lambda$ such that $\xi(\alpha) = 1$. Since ξ is a nontrivial character on $\mathcal{O}_F$, there exists $\beta \in \mathcal{O}_{F+}$ such that $\xi(\beta) \neq 1$. Then for any integer N, we have $\xi(N\alpha + \beta) = \xi(\alpha)^N \xi(\beta) \neq 1$. Let Φ' be the set of cones obtained by deforming $\sigma = \sigma_\alpha$ to $\sigma' := \sigma_{N\alpha+\beta}$ and $\varepsilon\sigma$ to $\varepsilon\sigma'$ for any $\varepsilon \in \Delta_\xi$. By taking N sufficiently large, the amount of deformation can be made arbitrarily small so that Φ' remains a fan. Indeed, we only need to check that the higher dimension cones $\sigma'_{\underline{\alpha}} \in \Phi_{q+1}$ remain simplicial, that is, that $\underline{\alpha}$ is still linearly independent after changing α to $N\alpha + \beta$. This can be achieved by taking N sufficiently big for $\sigma'_{\underline{\alpha}}$. The finiteness of Λ guarantees that there is a finite number N large enough so that all $\sigma'_{\underline{\alpha}} \in \Phi_{q+1}$ remain simplicial. By repeating this process for any $\alpha \in \Lambda$ such that $\xi(\alpha) = 1$, we obtain a Shintani decomposition satisfying the desired condition. □

In what follows, we fix a Shintani decomposition Φ satisfying the condition of Lemma 3.2.2. Let $N \colon \mathbb{R}_+^I \to \mathbb{R}_+$ be the norm map defined by $N((a_\tau)) := \prod_{\tau \in I} a_\tau$, and we let

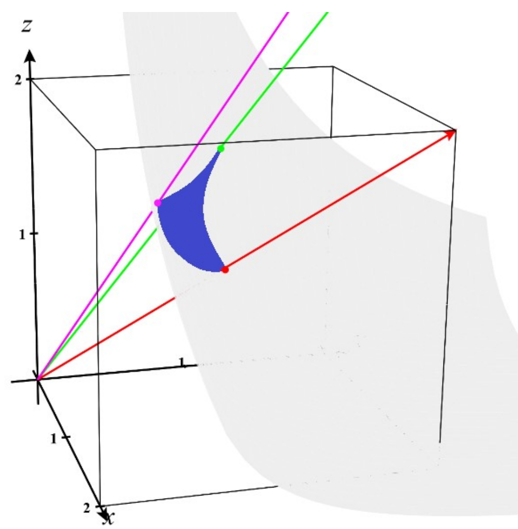
$$\mathbb{R}_1^I := \{(a_\tau) \in \mathbb{R}_+^I \mid N((a_\tau)) = 1\}$$

be the subset of $\mathbb{R}_+^I$ of norm one. Note that $\Delta \subset \mathbb{R}_1^I$ through the natural inclusion $\Delta \subset \mathbb{R}_+^I$. We also let

$$L \colon \mathbb{R}_+^I \to \mathbb{R}^g$$

be the homeomorphism defined by $(x_\tau) \mapsto (\log x_{\tau_i})$. If we define $W := \{(y_{\tau_i}) \in \mathbb{R}^g \mid \sum_{i=1}^{g} y_{\tau_i} = 0\}$, then W is an $\mathbb{R}$-linear subspace of $\mathbb{R}^g$ of dimension $g-1$, and L gives a homeomorphism $\mathbb{R}^I_1 \cong W \cong \mathbb{R}^{g-1}$. Therefore $\mathbb{R}^I_1$ is a real manifold homeomorphic to $\mathbb{R}^{g-1}$ and for any $\sigma \in \Phi_{q+1}$ the intersection $\sigma \cap \mathbb{R}^I_1$ is a subset of $\mathbb{R}^I_1$ which is homeomorphic to a simplex of dimension q. Recall that by Definition 1.4.1, Φ is a simplicial fan, therefore the set $\{\sigma \cap \mathbb{R}^I_1 \mid \sigma \in \Phi\}$ gives a simplicial decomposition of the topological space $\mathbb{R}^I_1$ that is compatible with the action of Δ_ξ.

Remark 3.2.3. If $g = 3$, the cone generated by the pink, green and red lines in the figure below intersects the *norm one surface* $\mathbb{R}^3_1$, colored in gray, giving rise to the blue simplex:



In what follows, for any $\sigma \in \Phi_{q+1}$, we denote by $\langle\sigma\rangle$ the class $\mathrm{sgn}(\underline{\alpha})\langle\underline{\alpha}\rangle$ in $C_q(A_\xi)$, where $\underline{\alpha} \in A_\xi^{q+1}$ is a generator of σ. This is well-defined since a generator $\underline{\alpha}$ is uniquely determined up to permutation from σ. We then have the following.

Lemma 3.2.4. *For any integer $q \geq 0$, we let $C_q(\Phi)$ be the $\mathbb{Z}[\Delta_\xi]$-submodule of $C_q(A_\xi)$ generated by $\langle\sigma\rangle$ for all $\sigma \in \Phi_{q+1}$. Then $C_\bullet(\Phi)$ is a subcomplex of $C_\bullet(A_\xi)$ which also gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module. In particular, the natural inclusion induces a quasi-isomorphism of complexes*

$$C_\bullet(\Phi) \xrightarrow{\text{qis}} C_\bullet(A_\xi)$$

compatible with the action of Δ_ξ.

Proof. Note that $C_q(\Phi)$ for any integer $q \geq 0$ is a free $\mathbb{Z}[\Delta_\xi]$-module generated by representatives of the quotient $\Delta_\xi\backslash\Phi_{q+1}$. By construction, $C_\bullet(\Phi)$ can be identified with the chain complex associated to the simplicial decomposition $\{\sigma \cap \mathbb{R}^I_1 \mid \sigma \in \Phi\}$ of the topological space $\mathbb{R}^I_1$. Since $\mathbb{R}^I_1 \cong \mathbb{R}^{g-1}$ is contractible, the chain complex associated to the simplicial decomposition is exact, so we see that the complex $C_\bullet(\Phi)$ is exact and gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module. Since any two free resolutions of $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module are quasi-isomorphic, the claim follows from the fact that $C_\bullet(A_\xi)$ also gives a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}[\Delta_\xi]$-module. □

Since F is totally real, the Dirichlet Unit Theorem (see [4, Theorem 10.7]) shows that the discrete subset $L(\Delta) \subset W$ is a free $\mathbb{Z}$-module of rank $g-1$, and so is $L(\Delta_\xi)$, since Δ_ξ is a subgroup of finite index of Δ by 1.3.1. Hence we have

$$\mathcal{T}_\xi := \Delta_\xi\backslash\mathbb{R}^I_1 \cong \mathbb{R}^{g-1}/\mathbb{Z}^{g-1}.$$

We consider the coinvariant

$$C_q(\Delta_\xi\backslash\Phi) := C_q(\Phi)_{\Delta_\xi}$$

of $C_q(\Phi)$ with respect to the action of Δ_ξ, that is, the $\mathbb{Z}$-module obtained by quotienting $C_q(\Phi)$ by the subgroup generated by $\langle\sigma\rangle - \langle\varepsilon\sigma\rangle$ for $\sigma \in \Phi_{q+1}$ and $\varepsilon \in \Delta_\xi$. For any $\sigma \in \Phi_{q+1}$, we denote by $\overline{\sigma}$ the image of σ in the quotient $\Delta_\xi\backslash\Phi_{q+1}$, and we denote by $\langle\overline{\sigma}\rangle$ the image of $\langle\sigma\rangle$ in $C_q(\Delta_\xi\backslash\Phi)$, which depends only on the class $\overline{\sigma} \in \Delta_\xi\backslash\Phi_{q+1}$. Now, for each such $\overline{\sigma} \in \Delta_\xi\backslash\Phi_{q+1}$, we associate the set $\Delta_\xi\backslash(\Delta_\xi\sigma \cap \mathbb{R}^I_1)$, which gives a q-dimensional simplex of the $(g-1)$-dimensional torus $\mathcal{T}_\xi$. Doing so for every $q \geq 0$, or equivalently for each $\overline{\sigma} \in \Delta_\xi\backslash\Phi$, we obtain a simplicial decomposition of $\mathcal{T}_\xi$ and $C_\bullet(\Delta_\xi\backslash\Phi)$ may be identified with the associated chain complex. Hence we have

$$H_m(C_\bullet(\Delta_\xi\backslash\Phi)) = H_m(\mathcal{T}_\xi, \mathbb{Z}), \qquad H^m\Big(\mathrm{Hom}_{\mathbb{Z}}\big(C_\bullet(\Delta_\xi\backslash\Phi), \mathbb{Z}\big)\Big) = H^m(\mathcal{T}_\xi, \mathbb{Z}),$$

where $H_m(\mathcal{T}_\xi, \mathbb{Z})$ (respectively $H^m(\mathcal{T}_\xi, \mathbb{Z})$) denotes the integral m-th (co)homology group of the $(g-1)$-dimensional torus $\mathcal{T}_\xi$, and therefore is a free abelian group of rank $\binom{g-1}{m}$. Therefore, Kronecker duality implies that the dual pairing

$$H^m(\mathcal{T}_\xi, \mathbb{Z}) \times H_m(\mathcal{T}_\xi, \mathbb{Z}) \to \mathbb{Z} \tag{3.5}$$

is perfect (see for example [12, Theorem 45.8]). Consequently, the bilinear pairing

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}\big(C_m(\Delta_\xi\backslash\Phi),\mathbb{Z}\big)\times C_m(\Delta_\xi\backslash\Phi) & \xrightarrow{\cong} & \mathbb{Z} \\ (\varphi,u) & \longmapsto & \varphi(u) \end{array} \tag{3.6}$$

is also perfect.

We also note that the generator of the homology group

$$H_{g-1}\big(C_\bullet(\Delta_\xi\backslash\Phi)\big) = H_{g-1}(\mathcal{T}_\xi,\mathbb{Z}) \cong \mathbb{Z}$$

is given by the fundamental class

$$\sum_{\overline{\sigma}\in\Delta_\xi\backslash\Phi_g} \langle\overline{\sigma}\rangle \in C_{g-1}(\Delta_\xi\backslash\Phi). \tag{3.7}$$

Now, since $\mathbb{Q}(\xi)$ is a field, the Universal Coefficient Theorem (see [12, Theorem 53.5]) together with the Kronecker pairing (3.6) gives us the following canonical isomorphism

$$\begin{array}{ccc} H^{g-1}\Big(\mathrm{Hom}_{\mathbb{Z}}\big(C_\bullet(\Delta_\xi\backslash\Phi),\mathbb{Q}(\xi)\big)\Big) & \xrightarrow{\cong} & \mathbb{Q}(\xi) \\ \varphi & \longmapsto & \sum_{\overline{\sigma}\in\Delta_\xi\backslash\Phi_g}\varphi(\langle\overline{\sigma}\rangle), \end{array} \tag{3.8}$$

induced by the fundamental class (3.7).

Proposition 3.2.5. *Let $\eta\in H^{g-1}(\xi/\Delta_\xi,\mathscr{O}_\xi)$ be represented by a cocycle*

$$(\eta_{\underline{\alpha}}) \in C^{g-1}(\mathfrak{B}/\Delta_\xi,\mathscr{O}_\xi) = \Bigg(\prod_{\underline{\alpha}\in A_\xi^g}^{\mathrm{alt}} \mathbb{Q}(\xi)\Bigg)^{\Delta_\xi}.$$

For any cone $\sigma\in\Phi_g$, let $\eta_\sigma := \mathrm{sgn}(\underline{\alpha})\eta_{\underline{\alpha}}$ for any $\underline{\alpha}\in A_\xi^g$ such that $\sigma_{\underline{\alpha}}=\sigma$. Then the homomorphism mapping the cocycle $(\eta_{\underline{\alpha}})$ to $\sum_{\overline{\sigma}\in\Delta_\xi\backslash\Phi_g}\eta_\sigma$ induces a canonical isomorphism

$$H^{g-1}(\xi/\Delta_\xi,\mathscr{O}_\xi) \cong \mathbb{Q}(\xi). \tag{3.9}$$

Proof. Since $C_q(\Phi)$ and $C_q(A_\xi)$ are free $\mathbb{Z}[\Delta_\xi]$-modules, the quasi-isomorphism $C_\bullet(\Phi)\xrightarrow{\mathrm{qis}} C_\bullet(A_\xi)$ of Lemma 3.2.4 induces the quasi-isomorphism

$$\mathrm{Hom}_{\Delta_\xi}\big(C_\bullet(A_\xi),\mathbb{Q}(\xi)\big)\xrightarrow{\mathrm{qis}}\mathrm{Hom}_{\Delta_\xi}\big(C_\bullet(\Phi),\mathbb{Q}(\xi)\big).$$

Combining this fact with Lemma 3.2.1 and (3.4), we see that

$$H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi) \cong H^{g-1}\big(\mathrm{Hom}_{\Delta_\xi}\big(C_\bullet(\Phi), \mathbb{Q}(\xi)\big)\big).$$

Since we have $\mathrm{Hom}_{\Delta_\xi}\big(C_\bullet(\Phi), \mathbb{Q}(\xi)\big) = \mathrm{Hom}_{\mathbb{Z}}\big(C_\bullet(\Delta_\xi\backslash\Phi), \mathbb{Q}(\xi)\big)$, our assertion follows from (3.8). □

We will now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. By construction and Lemma 2.2.6, the class $\partial^k\mathcal{G}(\xi)$ is a class defined over $\mathbb{Q}(\xi)$ represented by the cocycle $(\partial^k\mathcal{G}_{\underline{\alpha}}(\xi)) \in C^{g-1}(\mathfrak{B}/\Delta_\xi, \mathscr{O}_\xi)$. By Proposition 3.2.5 and Proposition 2.2.4, the class $\partial^k\mathcal{G}(\xi)$ maps through (3.9) to

$$\sum_{\sigma\in\Delta_\xi\backslash\Phi_g} \partial^k\mathcal{G}_\sigma(\xi) = \sum_{\sigma\in\Delta_\xi\backslash\Phi_g} \zeta_\sigma(\xi, (-k,\dots,-k)).$$

Our assertion now follows from (1.11). □

Corollary 3.2.6. *Assume that the narrow class number of F is one, and let $\chi\colon \mathrm{Cl}^+_F(\mathfrak{f}) \to \mathbb{C}^\times$ be a finite primitive Hecke character of F of conductor $\mathfrak{f} \neq (1)$. If we let $U[\mathfrak{f}] := \mathbb{T}[\mathfrak{f}] \setminus \{1\}$, then we have*

$$L(\chi, -k) = \sum_{\xi\in U[\mathfrak{f}]/\Delta} c_\chi(\xi)\partial^k\mathcal{G}(\xi)$$

for any integer $k \geq 0$.

Proof. The result follows from Theorem 3.1.1 and Proposition 1.4.4. □

Chapter Four

Canonical Decomposition: General Case

In this chapter, we generalize the construction of the Shintani Generating Class given in 2.2.3 to obtain a canonical decomposition of the Hecke L-function at negative integers when the narrow class number of the totally real field is not necessarily one.

4.1 Non-canonical Decomposition: General Case

In this section, we first give a generalization of the Lerch zeta function of F. We then introduce the set $\mathscr{T}_{\mathrm{tors}}$ as a natural parameter space for these generalized Lerch zeta functions, and express the L-function associated to a Hecke character of F canonically in terms of the Lerch zeta functions.

For any fractional ideal $\mathfrak{a}$ of F, we let $\mathbb{T}^{\mathfrak{a}}$ be the algebraic torus

$$\mathbb{T}^{\mathfrak{a}} := \mathrm{Hom}(\mathfrak{a}, \mathbb{G}_m)$$

defined over $\mathbb{Z}$. As before, $\mathbb{T}^{\mathfrak{a}}$ is given as the affine scheme $\mathbb{T}^{\mathfrak{a}} = \operatorname{Spec} \mathbb{Z}[t^{\alpha} \mid \alpha \in \mathfrak{a}]$, where t^{α} are parameters satisfying $t^{\alpha}t^{\alpha'} = t^{\alpha+\alpha'}$ for any $\alpha, \alpha' \in \mathfrak{a}$. The natural left action of Δ on $\mathfrak{a}$ induces a right action of Δ on $\mathbb{T}^{\mathfrak{a}}$. The action of $\varepsilon \in \Delta$, denoted by $\langle \varepsilon \rangle \colon \mathbb{T}^{\mathfrak{a}} \to \mathbb{T}^{\mathfrak{a}}$, maps a character $\xi \in \mathbb{T}^{\mathfrak{a}}$ to the character ξ^{ε} defined by $\xi^{\varepsilon}(\alpha) := \xi(\varepsilon\alpha)$. In terms of the coordinate ring, the isomorphism $\langle \varepsilon \rangle \colon \mathbb{T}^{\mathfrak{a}} \to \mathbb{T}^{\mathfrak{a}}$ is given by $t^{\alpha} \mapsto t^{\varepsilon\alpha}$ for any $\alpha \in \mathfrak{a}$.

For any torsion point $\xi \in \mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$, we define the function $\xi\Delta$ on $\mathfrak{a}$ to be the sum over the Δ-orbit of ξ as before

$$\xi\Delta := \sum_{\varepsilon\in\Delta_\xi\backslash\Delta} \xi^\varepsilon.$$

We (re)define the Lerch zeta function as follows.

Definition 4.1.1. Let $\mathfrak{a}$ be a nonzero fractional ideal of F, and let ξ be a torsion point of $\mathbb{T}^\mathfrak{a}(\overline{\mathbb{Q}})$. We define the *Lerch zeta function* by the series

$$\mathcal{L}(\xi\Delta, s, \mathfrak{a}) := \sum_{\alpha\in\Delta\backslash\mathfrak{a}_+} \xi\Delta(\alpha)N(\mathfrak{a}^{-1}\alpha)^{-s} \tag{4.1}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

This is a natural generalization of the Lerch zeta function defined in (1.8). The sum converges absolutely for $\mathrm{Re}(s) > 1$ and continues meromorphically to the whole complex plane, being entire if $\xi \neq 1$.

We next introduce the set $\mathscr{T}_{\mathrm{tors}}$ as a natural parameter space for the Lerch zeta functions. The action of Δ on $\mathbb{T}^\mathfrak{a}$ generalizes to isomorphisms given by elements of $F_+^\times$ as follows. For any $x \in F_+^\times$, the multiplication by x gives an isomorphism $\mathfrak{a} \cong x\mathfrak{a}$ of O_F-modules, and hence induces an isomorphism of group schemes

$$\langle x\rangle \colon \mathbb{T}^{x\mathfrak{a}} \to \mathbb{T}^\mathfrak{a}. \tag{4.2}$$

Explicitly, this isomorphism maps any character $\xi \in \mathbb{T}^{x\mathfrak{a}}(R)$ to the character $\xi^x \in \mathbb{T}^\mathfrak{a}(R)$ given by $\xi^x(\alpha) := \xi(x\alpha)$ for $\alpha \in \mathfrak{a}$. If $\mathfrak{I}$ denotes the group of nonzero fractional ideals of F, then $\widetilde{\mathbb{T}} := \coprod_{\mathfrak{a}\in\mathfrak{I}} \mathbb{T}^\mathfrak{a}$ has a natural action of $F_+^\times$ given by the isomorphism

$$\langle x\rangle \colon \widetilde{\mathbb{T}} \to \widetilde{\mathbb{T}} \tag{4.3}$$

for any $x \in F_+^\times$ obtained as the collection of isomorphisms $\langle x\rangle \colon \mathbb{T}^{x\mathfrak{a}} \to \mathbb{T}^\mathfrak{a}$ for all $\mathfrak{a} \in \mathfrak{I}$.

Definition 4.1.2. We define $\mathscr{T}(\overline{\mathbb{Q}})$ to be the quotient set

$$\mathscr{T}(\overline{\mathbb{Q}}) := F_+^\times\backslash\widetilde{\mathbb{T}}(\overline{\mathbb{Q}}),$$

and let $\mathscr{T}_{\mathrm{tors}} \subset \mathscr{T}(\overline{\mathbb{Q}})$ be the set of points in $\mathscr{T}(\overline{\mathbb{Q}})$ represented by torsion points of $\mathbb{T}^\mathfrak{a}(\overline{\mathbb{Q}})$ for all $\mathfrak{a} \in \mathfrak{I}$.

Note that if we fix a set of fractional ideals $\mathfrak{C}$ of F representing the narrow ideal class group $\mathrm{Cl}_F^+(1) := \mathfrak{I}/P^+$, then we have $\mathscr{T}(\overline{\mathbb{Q}}) = \coprod_{\mathfrak{a}\in\mathfrak{C}}(\Delta\backslash\mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}}))$. Indeed, each class of $\mathrm{Cl}_F^+(1)$ is represented by an ideal $\mathfrak{a}$ up to multiplication by a principal ideal (x), which in turn is determined by an element $x \in F_+^\times$ up to multiplication by an unit in Δ. This is best shown by the following exact sequence

$$0 \longrightarrow \Delta \longrightarrow F_+^\times \longrightarrow \mathfrak{I} \longrightarrow \mathrm{Cl}_F^+(1) \longrightarrow 0. \tag{4.4}$$

The following Lemma shows that $\mathscr{T}_{\mathrm{tors}}$ is the natural parameter space for the Lerch zeta functions.

Lemma 4.1.3. *Let $\mathfrak{a}$ be a nonzero fractional ideal of F and let $x \in F_+^\times$. Then for any torsion point $\xi \in \mathbb{T}^{x\mathfrak{a}}(\overline{\mathbb{Q}})$, we have*

$$\mathcal{L}(\xi\Delta, s, x\mathfrak{a}) = \mathcal{L}(\xi^x\Delta, s, \mathfrak{a}),$$

where ξ^x is the torsion point of $\mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$ corresponding to ξ through the isomorphism (4.2). In other words, the Lerch zeta function depends only on the class of ξ in $\mathscr{T}_{\mathrm{tors}}$.

Proof. By definitions of the Lerch zeta functions and of ξ^x, we have

$$\mathcal{L}(\xi\Delta, s, x\mathfrak{a}) = \sum_{\beta\in\Delta\backslash x\mathfrak{a}_+} \xi\Delta(\beta)N(\mathfrak{a}^{-1}x^{-1}\beta)^{-s} = \sum_{\alpha\in\Delta\backslash\mathfrak{a}_+} \xi\Delta(x\alpha)N(\mathfrak{a}^{-1}\alpha)^{-s} = \mathcal{L}(\xi^x\Delta, s, \mathfrak{a})$$

as desired. □

We will show in Proposition 4.1.8 that the Lerch zeta functions may be used to express the Hecke L-functions of finite Hecke characters of F. We first use some results concerning the finite Fourier transform as in 1.3.3. For a fractional ideal $\mathfrak{a} \in \mathfrak{I}$ and a nonzero integral ideal $\mathfrak{g}$, we denote by $\mathbb{T}^{\mathfrak{a}}[\mathfrak{g}] := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}/\mathfrak{g}\mathfrak{a}, \overline{\mathbb{Q}}^\times)$ the character group of the finite abelian group $\mathfrak{a}/\mathfrak{g}\mathfrak{a}$. We call it the group of $\mathfrak{g}$-torsion points of $\mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$, since the $\mathcal{O}_F$-action on $\mathfrak{a}$ induces an $\mathcal{O}_F$-module structure on $\mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$, and its submodule of $\mathfrak{g}$-torsion points is identified with $\mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]$ through the natural inclusion $\mathbb{T}^{\mathfrak{a}}[\mathfrak{g}] \hookrightarrow \mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$. Now, for a function $\phi\colon \mathfrak{a}/\mathfrak{g}\mathfrak{a} \to \mathbb{C}$ and $\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]$, let

$$c_\phi(\xi) := N\mathfrak{g}^{-1} \sum_{\beta\in\mathfrak{a}/\mathfrak{g}\mathfrak{a}} \phi(\beta)\xi(-\beta).$$

Then by the Fourier inversion formula for functions on $\mathfrak{a}/\mathfrak{g}\mathfrak{a}$, analogous to 1.3.3, we have

$$\phi(\alpha) = \sum_{\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]} c_\phi(\xi)\xi(\alpha)$$

for any $\alpha \in \mathfrak{a}/\mathfrak{g}\mathfrak{a}$. Moreover, if $\phi^\varepsilon = \phi$ for any $\varepsilon \in \Delta$, where ϕ^ε is the function on $\mathfrak{a}/\mathfrak{g}\mathfrak{a}$ defined by $\phi^\varepsilon(\alpha) = \phi(\varepsilon\alpha)$, then we have $c_\phi(\xi^\varepsilon) = c_\phi(\xi)$, hence we see that

$$\phi(\alpha) = \sum_{\xi \in \Delta \backslash \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]} c_\phi(\xi)\xi_\Delta(\alpha) \tag{4.5}$$

for any $\alpha \in \mathfrak{a}/\mathfrak{g}\mathfrak{a}$ in this case.

For $\mathfrak{a} \in \mathfrak{I}$, we denote by $(\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times$ the subset of $\mathfrak{a}/\mathfrak{g}\mathfrak{a}$ consisting of all elements which individually generate $\mathfrak{a}/\mathfrak{g}\mathfrak{a}$ as an $O_F/\mathfrak{g}$-module. Then we have the following description of $\mathrm{Cl}^+_F(\mathfrak{g})$.

Lemma 4.1.4. *For any $\mathfrak{a} \in \mathfrak{I}$, we have a well-defined map*

$$(\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times \longrightarrow \mathrm{Cl}^+_F(\mathfrak{g})$$

which sends a residue class of $\alpha \in \mathfrak{a}_+$ to the ray class of $\mathfrak{a}^{-1}\alpha$. Moreover, if $\mathfrak{C} \subset \mathfrak{I}$ is a set of representatives of the narrow ideal class group $\mathrm{Cl}^+_F(1)$, the above maps for $\mathfrak{a} \in \mathfrak{C}$ induce a bijection

$$\coprod_{\mathfrak{a} \in \mathfrak{C}} \Delta \backslash (\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times \longrightarrow \mathrm{Cl}^+_F(\mathfrak{g}).$$

Proof. The first claim follows from noting that $\alpha \in \mathfrak{a}_+$ reduces to $\overline{\alpha} \in (\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times$ if and only if $(\alpha) + \mathfrak{g}\mathfrak{a} = \mathfrak{a}$, and then after multiplying by $\mathfrak{a}^{-1}$ we have that $(\alpha)\mathfrak{a}^{-1} + \mathfrak{g} = O_F$, with $(\alpha)\mathfrak{a}^{-1}$ an integral ideal, so the map is well-defined. As for the bijection, we first note that $(\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times \cong (O_F/\mathfrak{g})^\times$ and therefore $\Delta\backslash(\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times \cong (O_F/\mathfrak{g})^\times/\Delta_\mathfrak{g}$. Now the fact that the map is bijective follows from the exactness of the sequence (1.9). □

By composing a finite Hecke character $\chi\colon \mathrm{Cl}^+_F(\mathfrak{g}) \to \mathbb{C}^\times$ with the map $(\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times \to \mathrm{Cl}^+_F(\mathfrak{g})$ given in Lemma 4.1.4, we define a map $\chi_\mathfrak{a}\colon (\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times \to \mathbb{C}^\times$ for each $\mathfrak{a} \in \mathfrak{I}$. Then we have the following.

Lemma 4.1.5. *The map defined above is multiplicative, in the sense that*

$$\chi_{\mathfrak{a}\mathfrak{b}}(\alpha\beta) = \chi_{\mathfrak{a}}(\alpha)\chi_{\mathfrak{b}}(\beta) \tag{4.6}$$

for any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{I}$ *and* $\alpha \in (\mathfrak{a}/\mathfrak{g}\mathfrak{a})^\times$, $\beta \in (\mathfrak{b}/\mathfrak{g}\mathfrak{b})^\times$.

Proof. Given α and β as above, and denoting by (α) and (β) the corresponding fractional ideals of $\mathcal{O}_F$, we clearly have

$$\big((\alpha) + \mathfrak{g}\mathfrak{a} = \mathfrak{a}\big) \wedge \big((\beta) + \mathfrak{g}\mathfrak{b} = \mathfrak{b}\big) \Rightarrow (\alpha\beta) + \mathfrak{g}\mathfrak{a}\mathfrak{b} = \mathfrak{a}\mathfrak{b}.$$

Furthermore,

$$\chi_{\mathfrak{a}\mathfrak{b}}(\alpha\beta) = \chi(\overline{(\alpha\beta)\mathfrak{a}^{-1}\mathfrak{b}^{-1}}) = \chi(\overline{(\alpha)\mathfrak{a}^{-1}})\chi(\overline{(\beta)\mathfrak{b}^{-1}}) = \chi_{\mathfrak{a}}(\alpha)\chi_{\mathfrak{b}}(\beta).$$

□

By extension by *zero*, we will often regard χ as a map $\chi : \mathfrak{I} \to \mathbb{C}$, and $\chi_{\mathfrak{a}}$ as a map $\chi_{\mathfrak{a}} \colon \mathfrak{a}/\mathfrak{g}\mathfrak{a} \to \mathbb{C}$. Then the above formula (4.6) holds for any $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{b}$.

Definition 4.1.6. Let $\chi \colon \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^\times$ be a finite Hecke character. For any $\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]$, we let

$$c_\chi(\xi) \coloneqq c_{\chi_{\mathfrak{a}}}(\xi).$$

Note that by definition, for any $\varepsilon \in \Delta$, we have $\chi_{\mathfrak{a}}^{\varepsilon} = \chi_{\mathfrak{a}}$, hence $c_\chi(\xi^\varepsilon) = c_\chi(\xi)$ for any $\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]$.

Lemma 4.1.7. *Let* $\mathfrak{a} \in \mathfrak{I}$ *and* $x \in F_+^\times$. *Then for any torsion point* $\xi \in \mathbb{T}^{x\mathfrak{a}}[\mathfrak{g}]$, *we have*

$$c_\chi(\xi) = c_\chi(\xi^x),$$

where ξ^x *is the torsion point of* $\mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]$ *corresponding to* ξ *through the isomorphism* (4.2). *In other words, the constant* $c_\chi(\xi)$ *depends only on the class of* ξ *in* $\mathscr{T}_{\mathrm{tors}}$.

Proof. We have

$$c_\chi(\xi) = N\mathfrak{g}^{-1} \sum_{\beta \in x\mathfrak{a}/\mathfrak{g}x\mathfrak{a}} \chi_{x\mathfrak{a}}(\beta)\xi(-\beta) = N\mathfrak{g}^{-1} \sum_{\alpha \in \mathfrak{a}/\mathfrak{g}\mathfrak{a}} \chi_{\mathfrak{a}}(\alpha)\xi(-x\alpha) = c_\chi(\xi^x)$$

as desired. □

Let $\mathscr{T}[\mathfrak{g}] := F_+^\times \backslash \left(\coprod_{\mathfrak{a} \in \mathfrak{I}} \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}] \right)$, which is a finite set. The Hecke L-function may be expressed in terms of the Lerch zeta functions as follows.

Proposition 4.1.8. *Let $\chi \colon \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^\times$ be a finite Hecke character and let $L(\chi, s)$ be the Hecke L-function of χ. Then we have*

$$L(\chi, s) = \sum_{\xi \in \mathscr{T}[\mathfrak{g}]} c_\chi(\xi) \mathcal{L}(\xi\Delta, s, \mathfrak{a}_\xi), \tag{4.7}$$

where $\mathfrak{a}_\xi$ is an ideal such that $\xi \in \mathbb{T}^{\mathfrak{a}_\xi}$. Note that $c_\chi(\xi)\mathcal{L}(\xi\Delta, s, \mathfrak{a}_\xi)$ depends only on the class of ξ in $\mathscr{T}[\mathfrak{g}]$, so this decomposition is canonical.

Proof. In what follows, we let $\mathfrak{C} \subset \mathfrak{I}$ be a set of representatives of the group $\mathrm{Cl}_F^+(1)$. Then $\{\mathfrak{a}^{-1} \mid \mathfrak{a} \in \mathfrak{C}\}$ also represents the group $\mathrm{Cl}_F^+(1)$ and we have

$$\begin{aligned} L(\chi, s) &:= \sum_{\mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a}) N(\mathfrak{a})^{-s} = \sum_{\mathfrak{a} \in \mathfrak{C}} \sum_{\alpha \in \Delta \backslash \mathfrak{a}_+} \chi(\mathfrak{a}^{-1}\alpha) N(\mathfrak{a}^{-1}\alpha)^{-s} = \sum_{\mathfrak{a} \in \mathfrak{C}} \sum_{\alpha \in \Delta \backslash \mathfrak{a}_+} \chi_{\mathfrak{a}}(\alpha) N(\mathfrak{a}^{-1}\alpha)^{-s} \\ &= \sum_{\mathfrak{a} \in \mathfrak{C}} \sum_{\alpha \in \Delta \backslash \mathfrak{a}_+} \sum_{\xi \in \Delta \backslash \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]} c_\chi(\xi) \xi\Delta(\alpha) N(\mathfrak{a}^{-1}\alpha)^{-s} = \sum_{\substack{\mathfrak{a} \in \mathfrak{C} \\ \xi \in \Delta \backslash \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]}} c_\chi(\xi) \mathcal{L}(\xi\Delta, s, \mathfrak{a}), \end{aligned}$$

where the second equality follows from the exactness of (4.4), the third from the definition of $\chi_{\mathfrak{a}}$, the fourth from (4.5) and the last equality from the definition of the Lerch zeta function (4.1). This proves our equality (4.7), since we have $\mathscr{T}[\mathfrak{g}] = \coprod_{\mathfrak{a} \in \mathfrak{C}} (\Delta \backslash \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}])$. □

When the Hecke character is primitive, we may restrict the sum in (4.7) to the subset of primitive additive characters. We denote by $\mathbb{T}_0^{\mathfrak{a}}[\mathfrak{g}]$ the set of primitive elements in $\mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]$, and set $\mathscr{T}_0[\mathfrak{g}] := F_+^\times \backslash \left(\coprod_{\mathfrak{a} \in \mathfrak{I}} \mathbb{T}_0^{\mathfrak{a}}[\mathfrak{g}] \right)$.

Proposition 4.1.9. *If a finite Hecke character $\chi \colon \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^\times$ is primitive, then $c_\chi(\xi) = 0$ holds for any non-primitive $\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}]$. Thus we have*

$$L(\chi, s) = \sum_{\xi \in \mathscr{T}_0[\mathfrak{g}]} c_\chi(\xi) \mathcal{L}(\xi\Delta, s, \mathfrak{a}_\xi). \tag{4.8}$$

Proof. Let $\xi \in \mathbb{T}^{\mathfrak{a}}[\mathfrak{g}']$ for some $\mathfrak{g}' \supsetneq \mathfrak{g}$. Since χ does not factor through $\mathrm{Cl}^+_F(\mathfrak{g}')$, there exists an element $\gamma \in O_{F+}$ prime to $\mathfrak{g}$ such that $\gamma \equiv 1 \pmod{\mathfrak{g}'}$ and $\chi_{O_F}(\gamma) \neq 1$. Then we have $\xi(\alpha) = \xi(\gamma\alpha)$ for any $\alpha \in \mathfrak{a}$, hence

$$\begin{aligned} c_\chi(\xi) &= N\mathfrak{g}^{-1} \sum_{\alpha \in \mathfrak{a}/\mathfrak{g}\mathfrak{a}} \chi_{\mathfrak{a}}(\alpha)\xi(-\alpha) = N\mathfrak{g}^{-1} \sum_{\alpha \in \mathfrak{a}/\mathfrak{g}\mathfrak{a}} \chi_{\mathfrak{a}}(\gamma\alpha)\xi(-\gamma\alpha) \\ &= N\mathfrak{g}^{-1}\chi_{O_F}(\gamma) \sum_{\alpha \in \mathfrak{a}/\mathfrak{g}\mathfrak{a}} \chi_{\mathfrak{a}}(\alpha)\xi(-\alpha) = \chi_{O_F}(\gamma)c_\chi(\xi). \end{aligned}$$

This shows that $c_\chi(\xi) = 0$ as desired, and the identity (4.8) follows immediately from (4.7). □

Now, as in 1.2.2, we set the following definition:

Definition 4.1.10. Let $\mathfrak{a}$ be a nonzero fractional ideal of F. For a finite order additive character $\phi \colon \mathfrak{a} \to \mathbb{C}$ and a cone σ, we define the *geometric Shintani zeta function associated to* $\mathfrak{a}$ by

$$\zeta^{\mathfrak{a}}_\sigma(\phi, \boldsymbol{s}) := \sum_{\alpha \in \breve{\sigma} \cap \mathfrak{a}} \phi(\alpha)\alpha^{-\boldsymbol{s}}, \tag{4.9}$$

where $\boldsymbol{s} = (s_\tau) \in \mathbb{C}^I$ and $\alpha^{-\boldsymbol{s}} := \prod_{\tau \in I} (\alpha^\tau)^{-s_\tau}$. Moreover, for a single variable $s \in \mathbb{C}$, we let

$$\zeta^{\mathfrak{a}}_\sigma(\phi, s) := \zeta^{\mathfrak{a}}_\sigma(\phi, (s, \dots, s)) = \sum_{\alpha \in \breve{\sigma} \cap \mathfrak{a}} \phi(\alpha)N(\alpha)^{-s}.$$

Given a cone generated by $\underline{\alpha} = (\alpha_1, \dots, \alpha_g) \in \mathfrak{a}^g$, after setting

$$\mathscr{G}^{\mathfrak{a}}_\sigma(t) := \frac{\sum_{\alpha \in \breve{P}_{\underline{\alpha}} \cap \mathfrak{a}} e^{2\pi i \mathrm{Tr}(\alpha t)}}{\left(1 - e^{2\pi i \mathrm{Tr}(\alpha_1 t)}\right) \cdots \left(1 - e^{2\pi i \mathrm{Tr}(\alpha_g t)}\right)},$$

it follows mutatis mutandis from 1.2.6 that

$$\partial^{\boldsymbol{k}} \mathscr{G}^{\mathfrak{a}}_\sigma(t)\big|_{t = u \otimes 1} = \zeta^{\mathfrak{a}}_\sigma(\xi_u, -\boldsymbol{k}). \tag{4.10}$$

Furthermore, we have immediately from the definitions and 1.4.3 the following proposition.

Proposition 4.1.11. *Let ξ be a torsion point of $\mathbb{T}^{\mathfrak{a}}(\overline{\mathbb{Q}})$. If Φ is a Shintani decomposition, then we have*

$$\mathcal{L}(\xi\Delta, s, \mathfrak{a}) = N(\mathfrak{a})^s \sum_{\sigma \in \Delta_\xi \backslash \Phi_g} \zeta^{\mathfrak{a}}_\sigma(\xi\Delta, (s, \dots, s)).$$

4.2 Generalized Shintani Generating Class

In this section, we construct the Shintani generating class following Chapter 2, however as a class in the equivariant cohomology of $U := \coprod_{\mathfrak{a}\in\mathfrak{I}} U^{\mathfrak{a}}$ with respect to the action of $F_+^\times$, where $U^{\mathfrak{a}} := \mathbb{T}^{\mathfrak{a}} \setminus \{1\}$ for any $\mathfrak{a} \in \mathfrak{I}$.

We now consider the scheme $\widetilde{\mathbb{T}}$ of §4.1. By the action given in (4.3), the scheme $\widetilde{\mathbb{T}} := \coprod_{\mathfrak{a}\in\mathfrak{I}} \mathbb{T}^{\mathfrak{a}}$ is an $F_+^\times$-scheme. For any $x \in F_+^\times$, the isomorphism $\langle x\rangle\colon \widetilde{\mathbb{T}} \cong \widetilde{\mathbb{T}}$ is the collection of isomorphisms $\langle x\rangle\colon \mathbb{T}^{x\mathfrak{a}} \cong \mathbb{T}^{\mathfrak{a}}$ induced from the isomorphism $\mathfrak{a} \cong x\mathfrak{a}$ given by the multiplication by x for any $\mathfrak{a} \in \mathfrak{I}$. The following proposition allows us to use the theory developed in Chapter 2 in our new context.

Proposition 4.2.1. *Let $\mathfrak{C}$ be a set of fractional ideals of F representing the classes of $\mathrm{Cl}_F^+(1)$. Then the category of $F_+^\times$-equivariant sheaves on $\widetilde{\mathbb{T}}$ is equivalent to the category of Δ-equivariant sheaves on $\coprod_{\mathfrak{a}\in\mathfrak{C}} \mathbb{T}^{\mathfrak{a}}$.*

Proof. Let $\mathfrak{a} \in \mathfrak{C}$ be a representative of some narrow class. Then we denote $\widetilde{\mathbb{T}}^{\mathfrak{a}} := \coprod_{x\in\Delta\setminus F_+^\times} \mathbb{T}^{x\mathfrak{a}}$, so clearly $\widetilde{\mathbb{T}} = \coprod_{\mathfrak{a}\in\mathfrak{C}} \widetilde{\mathbb{T}}^{\mathfrak{a}}$. It is clear that $\widetilde{\mathbb{T}}^{\mathfrak{a}}$ is a $F_+^\times$-scheme, since the $F_+^\times$ action maintains ideals in the same narrow class by definition. We denote by $\left(\widetilde{\mathbb{T}}^{\mathfrak{a}}\right)_x$ the component $\mathbb{T}^{x\mathfrak{a}}$ in $\widetilde{\mathbb{T}}^{\mathfrak{a}}$. We will first show the equivalence of categories between $F_+^\times$-equivariant sheaves on $\widetilde{\mathbb{T}}^{\mathfrak{a}}$ and Δ-equivariant sheaves on $\mathbb{T}^{\mathfrak{a}}$.

Now, let $\theta : \Delta \hookrightarrow F_+^\times$ be the group inclusion and let $\varphi : \mathbb{T}^{\mathfrak{a}} \to \widetilde{\mathbb{T}}^{\mathfrak{a}}$ be the inclusion of schemes which identifies $\mathbb{T}^{\mathfrak{a}}$ with $\left(\widetilde{\mathbb{T}}^{\mathfrak{a}}\right)_1$, and with corresponding ring homomorphisms $\varphi_\sharp : \widetilde{V} \to \varphi^{-1}(\widetilde{V})$ given by the restriction morphisms $\mathscr{O}_{\widetilde{\mathbb{T}}^{\mathfrak{a}}}(\widetilde{V}) \to \mathscr{O}_{\widetilde{\mathbb{T}}^{\mathfrak{a}}}\left(\widetilde{V} \cap \left(\widetilde{\mathbb{T}}^{\mathfrak{a}}\right)_1\right)$.

We claim that φ is θ-equivariant. Indeed, we need to show that $\varphi \circ [\varepsilon] = \langle\varepsilon\rangle \circ \varphi$, but this is clear from the definition of the actions and

$$\varphi \circ [\varepsilon](\mathbb{T}^{\varepsilon\mathfrak{a}}) = \varphi(\mathbb{T}^{\mathfrak{a}}) = \left(\widetilde{\mathbb{T}}^{\mathfrak{a}}\right)_1 \text{ and } \langle\varepsilon\rangle \circ \varphi(\mathbb{T}^{\varepsilon\mathfrak{a}}) = \langle\varepsilon\rangle\left(\left(\widetilde{\mathbb{T}}^{\varepsilon\mathfrak{a}}\right)_\varepsilon\right) = \left(\widetilde{\mathbb{T}}^{\mathfrak{a}}\right)_1.$$

From this, we deduce that the pullback functor φ^* sends equivariant sheaves into equivariant sheaves. To prove that this functor gives an equivalence of categories, we will construct a quasi-inverse. First, we remark that a sheaf $\mathscr{F} \in \mathrm{Sh}_{\widetilde{\mathbb{T}}^{\mathfrak{a}}}$ is given by $(\mathscr{F}_x)_{x \in \Delta \backslash F_+^\times}$, with $\mathscr{F}_x \in \mathrm{Sh}_{(\widetilde{\mathbb{T}}^{\mathfrak{a}})_x}$. Then $(\mathscr{F}_x)_x$ is $F_+^\times$-equivariant if and only if $\langle y \rangle^* \mathscr{F}_x \cong \mathscr{F}_{xy}$ for all $y \in F_+^\times$.

We may now construct the functor ψ which associates to $\widehat{\mathscr{F}} \in \Delta\text{-}\mathrm{Sh}_{\mathbb{T}^{\mathfrak{a}}}$ the sheaf $\mathscr{F} \in \mathrm{Sh}_{\widetilde{\mathbb{T}}^{\mathfrak{a}}}$ given by $(\mathscr{F}_x)_{x \in \Delta \backslash F_+^\times}$ where $\mathscr{F}_x = \langle x \rangle^* \varphi_* \widehat{\mathscr{F}}$. Since $\langle x \rangle^* \mathscr{F} \cong \mathscr{F}$ for any $x \in F_+^\times$, the sheaf $\mathscr{F}$ is $F_+^\times$-equivariant. Therefore we have a functor $\psi : \Delta\text{-}\mathrm{Sh}_{\mathbb{T}^{\mathfrak{a}}} \to \mathrm{F}_+^\times\text{-}\mathrm{Sh}_{\widetilde{\mathbb{T}}^{\mathfrak{a}}}$ sending equivariant sheaves to equivariant sheaves.

We now claim that ψ is a quasi-inverse to φ^*. Indeed, given $\mathscr{F} = (\mathscr{F}_x)_x \in \mathrm{F}_+^\times\text{-}\mathrm{Sh}_{\widetilde{\mathbb{T}}^{\mathfrak{a}}}$, we have

$$((\psi \circ \varphi^*)(\mathscr{F}))_x = (\psi(\varphi^* \mathscr{F}))_x = \langle x \rangle^* \varphi_* \varphi^* \mathscr{F}_1 \cong \langle x \rangle^* \mathscr{F}_1 \cong \mathscr{F}_x,$$

where the last isomorphism is due to the equivariance of the sheaf. Therefore $\psi \circ \varphi^* \cong \mathrm{id}$ in $\mathrm{F}_+^\times\text{-}\mathrm{Sh}_{\widetilde{\mathbb{T}}^{\mathfrak{a}}}$.

Conversely, given $\widehat{\mathscr{F}} \in \Delta\text{-}\mathrm{Sh}_{\mathbb{T}^{\mathfrak{a}}}$, we have

$$(\varphi^* \circ \psi)(\widehat{\mathscr{F}}) = \varphi^* \left((\langle x \rangle^* \varphi_* \widehat{\mathscr{F}})_x \right) = \varphi^* \varphi_* \widehat{\mathscr{F}} \cong \widehat{\mathscr{F}},$$

so $\varphi^* \circ \psi \cong \mathrm{id}$ in $\Delta\text{-}\mathrm{Sh}_{\mathbb{T}^{\mathfrak{a}}}$, which shows that ψ is a quasi-inverse and that φ^* is an equivalence of categories.

Now, since both Δ on $\coprod_{\mathfrak{a} \in \mathfrak{C}} \mathbb{T}^{\mathfrak{a}}$ and $F_+^\times$ on $\widetilde{\mathbb{T}} = \coprod_{\mathfrak{a} \in \mathfrak{C}} \widetilde{\mathbb{T}}^{\mathfrak{a}}$ only act within a class of the narrow class group, the above equivalences of categories for each $\mathfrak{a} \in \mathfrak{C}$ naturally induce the equivalence of categories of the claim. □

Next for $\mathfrak{a} \in \mathfrak{I}$, let $U^{\mathfrak{a}} := \mathbb{T}^{\mathfrak{a}} \setminus \{1\}$. Then any $x \in F_+^\times$ induces an isomorphism $\langle x \rangle \colon U^{x\mathfrak{a}} \to U^{\mathfrak{a}}$, hence $U := \coprod_{\mathfrak{a} \in \mathfrak{I}} U^{\mathfrak{a}}$ is also an $F_+^\times$-scheme. We will now construct the *equivariant Čech complex*, which may be used to express the equivariant cohomology of U with coefficients in an equivariant sheaf $\mathscr{F}$ on U.

Definition 4.2.2. For any fractional ideal $\mathfrak{a}$ in $\mathfrak{I}$, we say that $\alpha \in \mathfrak{a}_+$ is *primitive*, if $\alpha/N \notin \mathfrak{a}_+$ for any integer $N > 1$. We let $A_{\mathfrak{a}} \subset \mathfrak{a}_+$ be the set of primitive elements of $\mathfrak{a}_+$.

If we let $U_\alpha^{\mathfrak{a}} := \mathbb{T}^{\mathfrak{a}} \setminus \{t^\alpha = 1\}$ for any $\alpha \in A_{\mathfrak{a}}$, then $\mathfrak{U}^{\mathfrak{a}} := \{U_\alpha^{\mathfrak{a}}\}_{\alpha \in A_{\mathfrak{a}}}$ gives an affine open covering of $U^{\mathfrak{a}}$ with a natural action of Δ, and $\mathfrak{U} := \{U_\alpha^{\mathfrak{a}}\}_{\mathfrak{a} \in \mathfrak{I}, \alpha \in A_{\mathfrak{a}}}$ gives an affine open covering of $U =$

$\coprod_{\mathfrak{a}\in\mathfrak{I}} U^{\mathfrak{a}}$ with a natural action on $F_+^\times$. Let q be an integer ≥ 0. For any $\underline{\alpha} = (\alpha_0, \ldots, \alpha_q) \in A_{\mathfrak{a}}^{q+1}$, we let $U_{\underline{\alpha}}^{\mathfrak{a}} := U_{\alpha_0}^{\mathfrak{a}} \cap \cdots \cap U_{\alpha_q}^{\mathfrak{a}}$. For an $F_+^\times$-equivariant sheaf $\mathscr{F}$ on U, we write $\mathscr{F}_{\mathfrak{a}}$ for the restriction of $\mathscr{F}$ to $U^{\mathfrak{a}}$, and denote by

$$\prod_{\underline{\alpha}\in A_{\mathfrak{a}}^{q+1}}^{\mathrm{alt}} \Gamma(U_{\underline{\alpha}}^{\mathfrak{a}}, \mathscr{F}_{\mathfrak{a}})$$

the subgroup of $\prod_{\underline{\alpha}\in A_{\mathfrak{a}}^{q+1}} \Gamma(U_{\underline{\alpha}}^{\mathfrak{a}}, \mathscr{F}_{\mathfrak{a}})$ consisting of collections $\boldsymbol{s} = (s_{\underline{\alpha}})$ of sections such that $s_{\rho(\underline{\alpha})} = \mathrm{sgn}(\rho) s_{\underline{\alpha}}$ for any $\rho \in \mathfrak{S}_{q+1}$ and $s_{\underline{\alpha}} = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$.

For any $x \in F_+^\times$ and $\underline{\alpha} = (\alpha_0, \ldots, \alpha_q) \in A_{\mathfrak{a}}^{q+1}$, let $x\underline{\alpha} = (x\alpha_0, \ldots, x\alpha_q)$. Then $\langle x \rangle \colon U^{x\mathfrak{a}} \xrightarrow{\cong} U^{\mathfrak{a}}$ induces $U_{x\underline{\alpha}}^{x\mathfrak{a}} \xrightarrow{\cong} U_{\underline{\alpha}}^{\mathfrak{a}}$, and we have isomorphisms

$$\Gamma(U_{\underline{\alpha}}^{\mathfrak{a}}, \mathscr{F}_{\mathfrak{a}}) \overset{\langle x\rangle^*}{\cong} \Gamma(U_{x\underline{\alpha}}^{x\mathfrak{a}}, \langle x\rangle^* \mathscr{F}_{\mathfrak{a}}) \overset{\iota_x^{\mathfrak{a}}}{\cong} \Gamma(U_{x\underline{\alpha}}^{x\mathfrak{a}}, \mathscr{F}_{x\mathfrak{a}}),$$

which define a natural action of the group $F_+^\times$ on $\prod_{\mathfrak{a}\in\mathfrak{I}} \prod_{\underline{\alpha}\in A_{\mathfrak{a}}^{q+1}}^{\mathrm{alt}} \Gamma(U_{\underline{\alpha}}^{\mathfrak{a}}, \mathscr{F}_{\mathfrak{a}})$. Then we have the following.

Proposition 4.2.3. *Let $\mathscr{F}$ be quasi-coherent and $C^\bullet(\mathfrak{U}/F_+^\times, \mathscr{F})$ be the complex given by*

$$C^q(\mathfrak{U}/F_+^\times, \mathscr{F}) := \left(\prod_{\mathfrak{a}\in\mathfrak{I}} \prod_{\underline{\alpha}\in A_{\mathfrak{a}}^{q+1}}^{\mathrm{alt}} \Gamma(U_{\underline{\alpha}}^{\mathfrak{a}}, \mathscr{F}_{\mathfrak{a}}) \right)^{F_+^\times}$$

for any integer $q \geq 0$, and the differential $d^q \colon C^q(\mathfrak{U}/F_+^\times, \mathscr{F}) \to C^{q+1}(\mathfrak{U}/F_+^\times, \mathscr{F})$ is defined by the alternating sum

$$(d^q f)_{\alpha_0\cdots\alpha_{q+1}} := \sum_{j=0}^{q+1} (-1)^j f_{\alpha_0\cdots\check{\alpha}_j\cdots\alpha_{q+1}}\big|_{U_{(\alpha_0,\cdots,\alpha_{q+1})}}. \tag{4.11}$$

Then for any integer $m \geq 0$, the equivariant cohomology $H^m(U/F_+^\times, \mathscr{F})$ is given as

$$H^m(U/F_+^\times, \mathscr{F}) = H^m(C^\bullet(\mathfrak{U}/F_+^\times, \mathscr{F})).$$

Proof. It follows from Proposition 4.2.1 mutatis mutandis that the category of $F_+^\times$-equivariant sheaves on U is equivalent to the category of Δ-equivariant sheaves on $\coprod_{\mathfrak{a}\in\mathfrak{C}} U^{\mathfrak{a}}$, where $\mathfrak{C}$ is a set of fractional ideals of F representing the classes of $\mathrm{Cl}_F^+(1)$. Furthermore, the pullback functor which gives the equivalence of categories sends quasi-coherent sheaves into quasi-coherent sheaves. Therefore we have

$$C^q(\mathfrak{U}/F_+^\times, \mathscr{F}) \cong \prod_{\mathfrak{a}\in\mathfrak{C}} \Bigg(\prod_{\underline{\alpha}\in A_{\mathfrak{a}}^{q+1}}^{\mathrm{alt}} \Gamma(U_{\underline{\alpha}}^{\mathfrak{a}}, \mathscr{F}_{\mathfrak{a}}) \Bigg)^{\Delta}.$$

Our assertion now follows from 2.1.12. □

We recall that if $\alpha_1, \dots, \alpha_g$ is a $\mathbb{Z}$-basis of O_F, then for each ideal $\mathfrak{a} \in \mathfrak{I}$ there exist rational numbers $q_{\mathfrak{a},1}, \dots, q_{\mathfrak{a},g} \in \mathbb{Q}$ such that $\alpha_1 q_{\mathfrak{a},1}, \dots, \alpha_g q_{\mathfrak{a},g}$ is a $\mathbb{Z}$-basis of $\mathfrak{a}$. Therefore we have, for each $\mathfrak{a} \in \mathfrak{I}$, an isomorphism $(F \otimes \mathbb{C})/(\mathfrak{a}^{-1}\mathfrak{D}^{-1}) \cong \mathbb{T}^{\mathfrak{a}}(\mathbb{C})$ analogous to (2.5).

We define the function $\mathcal{G}_\sigma^{\mathfrak{a}}(t)$ following Corollary 2.2.2 as follows.

Definition 4.2.4. Let $\mathfrak{a} \in \mathfrak{I}$ and $\underline{\alpha} = (\alpha_1, \dots, \alpha_g) \in A_{\mathfrak{a}}^g$, and put $\sigma = \sigma_{\underline{\alpha}}$. Then we define

$$\mathcal{G}_\sigma^{\mathfrak{a}}(t) := \frac{\sum_{\alpha\in\breve{P}_{\underline{\alpha}}\cap\mathfrak{a}} t^\alpha}{(1-t^{\alpha_1})\cdots(1-t^{\alpha_g})} \in \Gamma(U_{\underline{\alpha}}^{\mathfrak{a}}, \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}}).$$

The notation $\mathcal{G}_\sigma^{\mathfrak{a}}(t)$ is justified by the fact that the generator $\underline{\alpha} \in A_{\mathfrak{a}}^g$ is determined up to permutation by $\mathfrak{a}$ and σ if $\dim\sigma = g$, while $\mathcal{G}_\sigma^{\mathfrak{a}}(t)$ is zero if $\dim\sigma < g$ since $\breve{P}_{\underline{\alpha}} = \emptyset$ in this case.

Note that $\mathscr{G}_\sigma^{\mathfrak{a}}(z)$ corresponds to $\mathcal{G}_\sigma^{\mathfrak{a}}(t)$ through the identification $(F \otimes \mathbb{C})/(\mathfrak{a}^{-1}\mathfrak{D}^{-1}) \cong \mathbb{T}^{\mathfrak{a}}(\mathbb{C})$.

As previously, we may also use the formal expansion

$$\mathcal{G}_\sigma^{\mathfrak{a}}(t) = \sum_{\alpha\in\breve{\sigma}\cap\mathfrak{a}} t^\alpha,$$

which depends only on $\mathfrak{a}$ and σ. For any $\underline{\alpha} = (\alpha_1, \dots, \alpha_g) \in A_{\mathfrak{a}}^g$, let $\big(\alpha_j^{\tau_i}\big)$ be the matrix in $M_g(\mathbb{R})$ whose (i,j)-component is $\alpha_j^{\tau_i}$. We let $\mathrm{sgn}(\underline{\alpha}) \in \{0, \pm 1\}$ be the signature of $\det\big(\alpha_j^{\tau_i}\big)$. The Shintani generating class is constructed as in Proposition 2.2.3 as follows.

Proposition 4.2.5. *For any $\mathfrak{a} \in \mathfrak{I}$ and $\underline{\alpha} \in A_{\mathfrak{a}}^g$, let*

$$\mathcal{G}_{\underline{\alpha}}^{\mathfrak{a}} := \mathrm{sgn}(\underline{\alpha})\mathcal{G}_{\sigma_{\underline{\alpha}}}^{\mathfrak{a}}(t) \in \Gamma(U_{\underline{\alpha}}^{\mathfrak{a}}, \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}}).$$

Then $(\mathcal{G}_{\underline{\alpha}}^{\mathfrak{a}})$ is a cocycle in $C^{g-1}(\mathfrak{U}/F_+^\times, \mathscr{O}_{\widetilde{\mathbb{T}}})$, hence defines a cohomology class

$$\mathcal{G} := [\mathcal{G}_{\underline{\alpha}}^{\mathfrak{a}}] \in H^{g-1}(U/F_+^\times, \mathscr{O}_{\widetilde{\mathbb{T}}}),$$

which we call the Shintani generating class.

Proof. The fact that the collection $(\mathscr{G}^{\mathfrak{a}}_{\underline{\alpha}})$ is invariant under the action of $F_+^\times$ follows from the construction. The cocycle condition is proved using the same argument as that of Proposition 2.2.3, again using the formula of [15, Proposition 6.2]. □

In order to account for the $N(\mathfrak{a})^k$ factor in 4.1.11, we define the $F_+^\times$-equivariant twisted sheaf $\mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}}(\boldsymbol{k})$ on $\widetilde{\mathbb{T}}_{\overline{F}}$, analogously to 2.1.6.

Definition 4.2.6. For any $\boldsymbol{k} = (k_\tau) \in \mathbb{Z}^I$, as an $\mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}}$-module we let $\mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}}(\boldsymbol{k}) := \mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}}$, and we define the $F_+^\times$-equivariant structure

$$\iota_x \colon \langle x\rangle^* \mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}}(\boldsymbol{k}) \cong \mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}}(\boldsymbol{k})$$

to be the multiplication by $x^{-\boldsymbol{k}} := \prod_{\tau\in I}(x^\tau)^{-k_\tau}$ for any $x \in F_+^\times$.

Then for any $k \in \mathbb{Z}$, after setting $k^I := (k,\dots,k) \in \mathbb{Z}^I$, we have the following.

Lemma 4.2.7. *For any $k \in \mathbb{Z}$ and $\mathfrak{a} \in \mathfrak{I}$, consider the multiplication by $N(\mathfrak{a})^k$ on $\mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}}$, regarded as a homomorphism $\mathscr{O}_{\mathbb{T}^{\mathfrak{a}}_{\overline{F}}}(k^I) \to \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}_{\overline{F}}}$ of sheaves on $\widetilde{\mathbb{T}}_{\overline{F}}$. Then the collection for all $\mathfrak{a} \in \mathfrak{I}$ gives an isomorphism*

$$\mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}}(k^I) \xrightarrow{\cong} \mathscr{O}_{\widetilde{\mathbb{T}}_{\overline{F}}} \tag{4.12}$$

of $F_+^\times$-equivariant sheaves on $\widetilde{\mathbb{T}}_{\overline{F}} = \coprod_{\mathfrak{a}\in\mathfrak{I}} \mathbb{T}^{\mathfrak{a}}_{\overline{F}}$.

Proof. For any $\mathfrak{a} \in \mathfrak{I}$ and $x \in F_+^\times$, we have a commutative diagram

$$\begin{tikzcd}
\langle x\rangle^* \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}_{\overline{F}}}(k^I) \arrow[r, "N(\mathfrak{a})^k", "\cong"'] \arrow[d, "N(x)^{-k}"', "\cong"] & \langle x\rangle^* \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}_{\overline{F}}} \arrow[d, "\mathrm{id}"', "\cong"] \\
\mathscr{O}_{\mathbb{T}^{x\mathfrak{a}}_{\overline{F}}}(k^I) \arrow[r, "N(x\mathfrak{a})^k", "\cong"'] & \mathscr{O}_{\mathbb{T}^{x\mathfrak{a}}_{\overline{F}}},
\end{tikzcd}$$

of sheaves on $\mathbb{T}^{x\mathfrak{a}}_{\overline{F}}$, which gives our assertion. □

Let $\alpha_1,\dots,\alpha_g$ be a $\mathbb{Z}$-basis of $\mathcal{O}_F$ and $\mathfrak{a} \in \mathfrak{I}$. For any $\tau \in I$, we let $\widetilde{\partial}_{\mathfrak{a},\tau}$ be the differential operator

$$\widetilde{\partial}_{\mathfrak{a},\tau} := \sum_{j=1}^{g} \alpha_j^\tau t^{\alpha_j} \frac{\partial}{\partial t^{\alpha_j}}.$$

As in Lemma 2.2.5, we have a homomorphism of abelian sheaves

$$\widetilde{\partial}_{\mathfrak{a},\tau} \colon \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}_{\overline{F}}}(\boldsymbol{k}) \to \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}_{\overline{F}}}(\boldsymbol{k} - 1_\tau).$$

Now, defining $\widetilde{\partial}_{\mathfrak{a}} \coloneqq \prod_{\tau \in I} \widetilde{\partial}_{\mathfrak{a},\tau}$ as before, we have homomorphisms for each $k \in \mathbb{N}$

$$\widetilde{\partial}^k_{\mathfrak{a}} \colon \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}{}_{\overline{F}}} \to \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}{}_{\overline{F}}}(-k^I),$$

which, after dividing by $N(\mathfrak{a})^k$ and defining $\partial^k_{\mathfrak{a}} \coloneqq N(\mathfrak{a})^{-k}\widetilde{\partial}^k_{\mathfrak{a}}$, induce the homomorphisms

$$\partial^k_{\mathfrak{a}} \colon \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}} \to \mathscr{O}_{\mathbb{T}^{\mathfrak{a}}}$$

from Lemmas 2.2.6 and 4.2.7 above.

From Proposition 2.2.4, we deduce the following.

Proposition 4.2.8. *Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_g) \in A^g_{\mathfrak{a}}$ and $\sigma = \sigma_{\underline{\alpha}}$. For $k \in \mathbb{N}$ and $\partial^k_{\mathfrak{a}}$ as above, we have*

$$\partial^k_{\mathfrak{a}} \mathcal{G}^{\mathfrak{a}}_\sigma(\xi) = N(\mathfrak{a})^k \zeta^{\mathfrak{a}}_\sigma(\xi, -k^I)$$

for any torsion point $\xi \in U^{\mathfrak{a}}_{\underline{\alpha}}$.

If we define ∂ as the collection of $\partial_{\mathfrak{a}}$ over all $\mathfrak{a} \in \mathfrak{I}$, we have a homomorphism of abelian sheaves

$$\partial \colon \mathscr{O}_{\widetilde{\mathbb{T}}} \to \mathscr{O}_{\widetilde{\mathbb{T}}}$$

which is also compatible with the $F^\times_+$-action, by Lemmas 2.2.5 and 4.2.7.

We also have, as in (2.9), an induced homomorphism of groups

$$\partial \colon H^m\big(U/F^\times_+, \mathscr{O}_{\widetilde{\mathbb{T}}}\big) \to H^m\big(U/F^\times_+, \mathscr{O}_{\widetilde{\mathbb{T}}}\big),$$

giving rise to the classes

$$\partial^k \mathcal{G} \in H^{g-1}\big(U/F^\times_+, \mathscr{O}_{\widetilde{\mathbb{T}}}\big). \tag{4.13}$$

4.3 Main Theorem

Exactly as in (3.2), we have a specialization map

$$\xi^* \colon H^m(U/F_+^\times, \mathscr{O}_{\widetilde{\mathbb{T}}}) \to H^m(\xi/\Delta_\xi, \mathscr{O}_\xi), \tag{4.14}$$

since the inclusion of schemes $\xi \to \widetilde{\mathbb{T}}$ is equivariant with respect to the inclusion of groups $\Delta_\xi \hookrightarrow F_+^\times$.

Now, we may use the natural identification from Proposition 3.2.5 to deduce our main theorem:

Theorem 4.3.1 (Main Theorem)**.** *Let F be a totally real number field of degree g, let ξ be a nontrivial torsion point of $\widetilde{\mathbb{T}}$ and let k be an integer ≥ 0. If we let $\mathcal{G}$ be the Shintani generating class defined in Proposition 4.2.5, and if we let $\partial^k \mathcal{G}(\xi) \in H^{g-1}(\xi/\Delta_\xi, \mathscr{O}_\xi)$ be the image by the specialization map (4.14) of the class $\partial^k \mathcal{G}$ defined in (4.13), then we have*

$$\partial^k \mathcal{G}(\xi) = \mathcal{L}(\xi\Delta, -k, \mathfrak{a}_\xi),$$

where $\mathfrak{a}_\xi$ is an ideal such that $\xi \in \mathbb{T}^{\mathfrak{a}}$.

Proof. This follows the same reasoning as Theorem 3.1.1, this time using Propositions 4.2.8 and 4.1.11. □

Corollary 4.3.2. *Let $\chi \colon \mathrm{Cl}_F^+(\mathfrak{g}) \to \mathbb{C}^\times$ be a finite primitive Hecke character and let $L(\chi, s)$ be the Hecke L-function of χ. Then we have*

$$L(\chi, -k) = \sum_{\xi \in \mathscr{T}_0[\mathfrak{g}]} c_\chi(\xi) \partial^k \mathcal{G}(\xi)$$

canonically.

The significance of this result is that the values at nonpositive integers of *any* Hecke L-function associated to a finite Hecke character of F may be expressed as a linear combination of the values of the derivatives of a single canonical cohomology class, the Shintani generating class $\mathcal{G}$ in $H^{g-1}(U/F_+^\times, \mathscr{O}_{\widetilde{\mathbb{T}}})$. We can also see that these values are algebraic numbers.

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