DENIS-CHARLES CISINSKI
CHRISTIAN DAHLHAUSEN

SEMINAR
SS 2018

DERIVED CATEGORIES AND SHEAF COHOMOLOGY

Wednesday, 10-12, M 104

"... if I could only understand the beautiful consequence following from the concise property $d^2 = 0$."
Henri Cartan

Derived categories were introduced by Grothendieck as a natural reformulation and generalisation of the main constructions of homological algebra. They lead to the notion of triangulated categories. The aim of the seminar is to study this notion through examples, via the study of cohomology of sheaves on a space.

The main source for this seminar is the book by Gelfand and Manin [GM03].

PRELIMINARIES

Commutative algebra. Some basics in point-set topology. Familiarity with the very basic concepts of category theory, eg. [KS94, I.1.1] or [GM03, II.1-2].

SHEAVES

Talk 1: Presheaves and sheaves.
Motivate sheaves. Define presheaves (on topological spaces with values in sets), first as contravariant functors, and then spell out what that means. Provide plenty examples. Define the notion of a morphism and introduce the categories of presheaves resp. sheaves. Explain that, in the category of sets, filtered colimits commute with finite limits. Define stalks of presheaves as a colimit and make this explicit. State without proof the equivalence between sheaves and étale spaces. [GM03, I.5.1,5.2,5.5], [KS94, II.2.1-2.2.1], [GW10, (2.5)-(2.6)], [MLM94, II.6, Cor. 3].

Talk 2: Sheaves and sheafification.
State the universal property of sheafification, give its construction and proof that it satisfies the desired property. Define the notions of support, direct and inverse images of presheaves and sheaves. Provide the

The sections "Organisatorial remarks", "General remarks", and "Remarks on giving a talk" are freely translated from the hints given in seminar programmes by Clara Löh based on the original by Arthur Bartels. We thank her for her consent to use them.

examples of locally constant sheaves, skyscraper sheaves, and extensions by zero. [GM03, I.5.5], [KS94, II.2.3], [GW10, (2.7)-(2.8)], [Har77, II, Exercises 1.17, 1.19].

Talk 3: Abelian sheaves.
Introduce the category of abelian sheaves, i.e. sheaves with values in abelian groups. Discuss the notions of kernels, images, and cokernels. Be aware that the latter two are defined as sheafifications. Define exact sequences and show that one can check exactness on stalks. Define injective objects, state that there are enough injectives, and sketch its proof. [GM03, I.5.3, 5.4, III.8.1], [KS94, II.2.2].

Abelian categories

Talk 4: Abelian Categories.
Define the notions of an additive and an abelian category and provide examples (in particular, the category of abelian sheaves). Repeat injective objects and define projective objects. State the five lemma and the snake lemma for abelian categories. Maybe sketch their proof. [GM03, II.5.1-11, exc. 5.6,5.7], [KS94, 1.2].

Talk 5: Complexes.
Define the notion of (cochain) complexes (resp. bounded, bounded below, and bounded above ones) and the shift functor. Discuss the equivalence between the two definitions of cohomology. Proof the long exact sequence of cohomology groups associated to a short exact sequence of complexes. Provide examples. Define additive and (left/right) exact functors. Discuss the examples of the tensor product and global sections. If time permits, discuss the truncation functors. [GM03, II.6.1-7], [KS94, 1.3, 1.4].

Derived categories

Talk 6: Homotopy categories as triangulated categories..
Define the notion of homotopy of complexes. Proof that homotopic complexes have isomorphic cohomology. Introduce the homotopy category of complexes (and the variants for bounded, bounded below, and bounded above). Define mapping cones and triangles. Show that a map is a homotopy equivalence if and only if its mapping cone is null-homotopic. Define the notion of a triangulated category and proof that the homotopy category is such a thing (maybe without showing the octahedral axiom). [KS94, I.1.3.3, 1.4, 1.5.1].

Talk 7: Localisation of categories.
Recall the notion of a triangulated category. Define cohomological functors and proof that Hom and cohomology are ones. Recall briefly from commutative algebra the universal property of a localisation of a

commutative ring by a multiplicative subset. State the universal property of a localisation of a category by a multiplicative system. Give the localisation of a ring as a first example. Introduce null systems and discuss localisations of triangulated categories by them. Point to the example of the derived category in the next talk. [KS94, I.1.5, 1.6], cf. [KS94, III.2.2].

Talk 8: Derived categories.
Define quasi-isomorphisms and motivate and introduce the derived category of an abelian category as a localisation of the homotopy category and discuss also the variants for bounded, bounded below, and bounded above complexes. Describe the derived category of bounded below complexes as the homotopy category of bounded below complexes of injective objects and proof this carefully. State without proof that this derived category also can be described as a localisation of the homotopy category of bounded below complexes of adapted objects. [KS94, I.1.7], [GM03, III.5.20-5.25, III.6.4].

Talk 9: Derived functors.
Motivate derived functors. Define and construct right derived functors of additive left exact functors. Describe left derived functors of additive right exact functors via the opposite category. Discuss their uniqueness and their universal property. Show the relation with the existence of enough injective objects. [GM03, III.6.1-6.12], cf. [KS94, 1.8].

Talk 10: Examples of derived functors.
Talk about the derived functors of the internal Hom and the tensor product. Show that their cohomology groups are the classical Ext resp. Tor groups. Discuss also the tensor product of complexes and its derived version. Treat the derived functor of the direct image functor. In particular, the special case where the target is a point and which yields the global sections should be pointed out. Show the relation between adapted classes and acyclic objects. [GM03, III.6.14-16, 7.6, 8.2, 8.3 b),d)]

VERDIER DUALITY

Talk 11: Direct image with compact support.
Define direct images with compact support. Show that soft sheaves are adapted to this functor. Conclude higher direct images with compact support and state that they vanish in degree higher than the dimension. [GM03, III.8.7-8.14].

Talk 12: Inverse image with compact support.
Construct the inverse image with compact support and state that it is right adjoint to the higher direct image (Verdier duality). Introduce

the dualising complex and deduce Poincaré duality. [GM03, III.8.15-8.26].

Talk 13: Base change and projection formula..
Discuss the Mayer-Vietoris theorems, the compatibility of inverse images and derived tensor products, and the interaction between derived direct images and derived Hom. Show the base change formulas and the projection formula. [GM03, III.8 Exercises. 2,3,5,6,7].

References

[GM03] Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR 1950475

[GW10] Ulrich Görtz and Torsten Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010, Schemes with examples and exercises. MR 2675155

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[KS94] Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1994, With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original. MR 1299726

[MLM94] Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994, A first introduction to topos theory, Corrected reprint of the 1992 edition. MR 1300636

Organisatorial remarks

In order to obtain credits for the course, you shall:

- Give a talk of 80 minutes (the remaining 10 minutes are reserved for questions and discussion).
- Be regularly present at the seminar and participate actively (do not hesitate to ask questions if you do not understand something).
- Prepare a handout of one or two pages including the main aspects of your talk as well as some small excercises for the other participants (so that they are encouraged to think about the content of the talk again).
- Submit a written elaboration of your talk of about 5 to 10 pages. This has to be handed in one week before your talk at the latest (as a PDF via email).
- Show up at the latest two weeks before your talk at one of the organisers' office hours with a draft of your elaboration. This serves to clarify questions about your talk.
- Your work will be graded and credited as written in the Prüfungsordnung.

General Remarks

- Begin your preparations in good time! If you have any questions, ask them.
- The basic requirement for giving a talk is to understand the mathematics you talk about. You should know more than you will present in your talk.
- In the beginning, give a short overview. State the main aspects of your talk as early as possible to avoid time pressure at the end of the talk.
- Distinguish clearly between important and not so important parts of your talk. Do not overstrain the audience with too many technical details but explain the essential parts of the proofs.
- Think carefully about the structure of your talk. Keep in mind that the audience has not read as much about the topic as you will hopefully have done.
- Let the talk be independent of the literature used. A structure which is good for a written presentation is not necessarily good for a talk as well!
- Be critical about the literature you use and try to find other sources than those stated in this programme!
- Plan the time schedule of your talk. Think about which part you could skip if time is running out. A trial of your talk helps to estimate the time needed.
- Consider also the content of the talks before and after your talk. If there are any doubts about intersections and inconsistencies, coordinate with the other speakers. Think about what you should recall from the previous talks.
- Please formulate your elaboration in your own words (instead of copying the literature literally) and cite your sources correctly.

Remarks on giving a talk

- Write legibly and give the audience time to read. Avoid erasing or hiding what you just wrote on the board. Plan your board layout.
- Write down all definitions. Be sure about the precise conditions of your statements.
- Speak loudly and clearly!
- Do not be afraid of questions from the audience. Be happy about the interest shown. Mathematics is a social activity and is based on discussions.