Continuous K-theory and Cohomology of Rigid Spaces

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Goal

Compare the continuous K-theory of an affinoid algebra with the cohomology of its associated rigid space.

Setting

Let k be a complete discretely valued field with uniformiser π. An affinoid k-algebra is a quotient

$$A := k\langle t_1, \dots, t_n\rangle / I$$

where $k\langle t_1, \dots, t_n\rangle$ is the k-algebra of power series converging on the unit disk and I is an ideal. Then the elements of norm less or equal to 1 form a subring A_0 of A which is π-adically complete and fulfils $A = A_0[\pi^{-1}]$.

Continuous K-theory

Let A_0 be a complete π-adic ring for some $\pi \in A_0$. Its continuous K-theory is the pro-spectrum

$$\mathrm{K}^{\mathrm{cont}}(A_0) := \text{“}\lim_n\text{”}\, \mathrm{K}(A_0/\pi^n)$$

where K is the nonconnective algebraic K-theory spectrum. Setting $A := A_0[\pi^{-1}]$, we define its continuous K-theory as the pushout

$$\begin{array}{ccc} \mathrm{K}(A_0) & \longrightarrow & \mathrm{K}(A) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{cont}}(A_0) & \longrightarrow & \mathrm{K}^{\mathrm{cont}}(A). \end{array}$$

in the ∞-category of pro-spectra. If $A = A_0'[\lambda^{-1}]$ for another complete λ-adic ring A_0', one obtains a weakly equivalent pro-spectrum, i.e. the pro-homotopy groups are pro-isomorphic. This notion was suggested by Morrow [3] and studied by Kerz-Saito-Tamme [1].

Zariski-Riemann spaces

Geometrically, $\mathrm{Spec}(A)$ is an open subset of $\mathrm{Spec}(A_0)$ whose closed complement is $\mathrm{Spec}(A_0/\pi)$. An admissible model of A is a reduced scheme X together with a projective map $X \to \mathrm{Spec}(A_0)$ which is an isomorphism over $\mathrm{Spec}(A)$.

Assuming resolution of singularities and A to be regular, one can choose a regular model as a good mean to study the geometry of A. Dropping these assumptions, one can alternatively study all models at once.

Thus we define the Zariski-Riemann space $\mathcal{Z}(A)$ as the cofiltered limit of all admissible models of A within the category of locally ringed spaces. Its special fibre $\mathcal{Z}(A)/\pi$ is isomorphic to Huber's adic spectrum $\mathrm{Spa}(A, A_0)$.

Main Result

Theorem 1 (D.). Let A be an affinoid k-algebra of dimension d. Then there is an isomorphism

$$\mathrm{K}^{\mathrm{cont}}_{-d}(A) \xrightarrow{\cong} \mathrm{H}^d(\mathcal{M}(A); \mathbb{Z})$$

where $\mathcal{M}(A)$ is the Berkovich spectrum of A.

Sketch of proof:

- $\mathrm{K}^{\mathrm{cont}}_{-d}(A) \cong \mathrm{K}^{\mathrm{cont}}_{-d}(\mathcal{Z}(A))$ using pro-cdh descent and platification par éclatement.
- $\mathrm{K}^{\mathrm{cont}}_{-d}(\mathcal{Z}(A)) \cong \mathrm{K}_{-d}(\mathcal{Z}(A)/\pi)$ via the Zariski descent spectral sequence. In particular, it is a constant pro-object.
- $K_{-d}(\mathcal{Z}(A)/\pi) \cong \mathrm{H}^d_{\mathrm{cdh}}(\mathcal{Z}(A)/\pi; \mathbb{Z})$ by using [2, Thm. D].
- $\mathrm{H}^*_{\mathrm{cdh}}(\mathcal{Z}(A)/\pi; \mathbb{Z}) \cong \mathrm{H}^*_{\mathrm{Zar}}(\mathcal{Z}(A)/\pi; \mathbb{Z})$ by pulling back covers and using completeness.
- $\mathrm{H}^*_{\mathrm{Zar}}(\mathcal{Z}(A)/\pi; \mathbb{Z}) \cong \mathrm{H}^*(\mathcal{M}(A); \mathbb{Z})$ is classical rigid geometry [4].

Future Work

- Globalise Theorem 1 to arbitrary (nice) rigid spaces.
- Study Zariski-Riemann spaces as almost regular models in order to avoid the assumption of resolution of singularities for other situations.

References

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