Calculating a correlation function on the lattice

Calculating a two-point function

We want to calculate a two point correlation function on a particular configuration, for example for a simple meson with momentum $\vec{p}$,

$$C^n_{2pt}(t_f, \vec{p}; t_0) = \pm \sum_{\vec{x}_f} \mathrm{e}^{i\vec{p}\cdot(\vec{x}_f - \vec{x}_0)} \mathrm{M}(\vec{x}_f, t_f) \mathrm{M}^\dagger(\vec{x}_0, t_0).$$

where

$$\mathrm{M}(\vec{x}_f, t_f) = \bar{q}_2(\vec{x}_f, t_f) \Gamma q_1(\vec{x}_f, t_f),$$

and after Wick contractions we have

$$C^n_{2pt}(t_f, \vec{p}; t_0) = \mp \sum_{\vec{x}_f} \mathrm{e}^{i\vec{p}\cdot(\vec{x}_f - \vec{x}_0)} \mathrm{Tr}\left[\Gamma G_1(\vec{x}_f, t_f; \vec{x}_0, t_0) \Gamma^\dagger \gamma_5 G_2^\dagger(\vec{x}_f, t_f; \vec{x}_0, t_0) \gamma_5\right]$$

To calculate $C^n_{2pt}(t_f, \vec{p}; t_0)$ we need to calculate the propagators $G_1(\vec{x}_f, t_f; \vec{x}_0, t_0)$ and $G_2(\vec{x}_f, t_f; \vec{x}_0, t_0)$ and then perform the contractions according to the equation above. Due to translational invariance the expectation value $C_{2pt}(t_f, \vec{p}; t_0)$ is independent of $\vec{x}_0$.

Smearing

Any creation operator excites not only a particle in the ground state (with particular quantum numbers) but also radial excitations.
Our choice of operator,

$$\mathrm{M}^{\dagger}(\vec{x}_0, t_0) = \pm \bar{q}_1(\vec{x}_0, t_0)\Gamma^{\dagger} q_2(\vec{x}_0, t_0)$$

represents a so-called "point source" where q_2 and $\bar{q}_1$ are at the same point. This source normally has large overlaps with radial excitations while you may primarily be interested in the ground state.
The radial excitations decay faster than the ground state with $t_f \gg t_0$, however, the relative statistical errors also increase in this limit.

The choice of operator is not unique. In order to make it easier to extract the ground state a "smeared source" can be used. This is an extended source which is a better approximation to the wavefunction of the ground state. Thus, the operator will have a smaller overlap with excited states.

One possible choice for a smeared source is

$$\mathrm{M}_S^\dagger(\vec{x}_0, t_0) = \sum_{\vec{x}_j, \vec{x}_i} \bar{q}_1(\vec{x}_0, t_0) S(\vec{x}_0, \vec{x}_j) \Gamma^\dagger S(\vec{x}_j, \vec{x}_i) q_2(\vec{x}_i, t_0)$$

where both q_2 and $\bar{q}_1$ are smeared. We assume $S(\vec{x}_i, \vec{x}_j)$ is gauge covariant, local in time and $S = S^\dagger$.

Similarly one can use a "smeared sink" $\mathrm{M}_S(\vec{x}_f, t_f)$.

Calculating the propagator

If we make the spin ($\alpha\beta$) and colour (ab) indices of the propagator explicit,

$$G^{ab}_{\alpha\beta}(x_f; x_0)$$

where x_i is short for $(\vec{x_i}, t_i)$ etc, it is clear a propagator is a 12×12 spin-colour matrix for every path from x_0 to x_f. Due to translational invariance we only need to consider a single starting point and so a propagator can be represented by a 12×12 matrix at every lattice site.

We calculate the propagator by solving the matrix equation,

$$M^{ca}_{\gamma\alpha}(x_i, x_f) G^{ab}_{\alpha\beta}(x_f; x_0) = \delta_{x_i,x_0}\delta_{c,b}\delta_{\beta,\gamma}$$

where M is the Dirac operator as defined for the lattice fermion action you are using. Note that 12 (spin$\times$colour) inversions are needed to construct $G^{ab}_{\alpha\beta}(x_f; x_0)$.

In the case of the Wilson action,

$$M_{Wilson}(x_i, x_f) = \frac{1}{2\kappa}\left(\delta_{x_i,x_f} - \kappa D(x_i, x_f)\right)$$

where

$$D(x_i, x_f) = \sum_{\mu=1}^{4} (1-\gamma_\mu)\, U_\mu(x_i)\delta_{x_f,x_i+\hat{\mu}} + (1+\gamma_\mu)\, U_\mu^\dagger(x_i-\hat{\mu})\delta_{x_f,x_i-\hat{\mu}}$$

For the clover action,

$$D_{clover} = D_{Wilson} - i\frac{c_{SW}}{2}\sum_{\mu\neq\nu}\sigma_{\mu\nu}F_{\mu\nu}\mathbb{1}$$

In the matrix equation the delta function quark source corresponds to a 'point source". For a smeared quark source the matrix equation becomes

$$M^{ca}_{\gamma\alpha}(x_j, x_f)G^{ab}_{\alpha\beta}(x_f; x_0) = S^{cd}(\vec{x_j}, \vec{x_i})\delta_{x_i,x_0}\delta_{d,b}\delta_{\beta,\gamma}$$

If a smeared quark sink is also used then the smearing function is applied to the propagator after the inversion:

$$S^{ca}(\vec{x}_f, \vec{x}_j) G^{ab}_{\alpha\beta}(\vec{x}_j, t_f; \vec{x}_0, t_0).$$

Inversion Algorithms

The matrix equation is a large linear system that can be solved iteratively through some recursive procedure that generates a sequence of increasingly accurate approximate solutions.

Several algorithms are in use. The two algorithms which will be used in the tutorial are,

- the conjugate gradient algorithm (CG),
- the stabilised bi-conjugate gradient algorithm (BiCGStab).

The CG algorithm is slower than BiCGStab but is guaranteed to converge (i.e. find an approximate solution), while BiCGStab does not always converge. Failure to converge is normally only a problem when the quark mass is low.

A measure of the accuracy of an approximate solution ψ is the residue (r)

$$r = \frac{||\eta - M\psi||}{||\eta||}$$

where η is the source in the matrix equation given above. When you invert the propagator you must specify what level of accuracy you wish to have.

Putting it all together

From the above discussion, we know that to calculate a two point function we need to

1. Generate the propagators.
2. Contract the propagators together to form the two point function.

In order to generate the propagators for each propagator we need to

- 1(a) Specify the source (either point source or smeared).
- 1(b) Perform the inversion (specifying the algorithm and the residual).
- 1(c) Possibly smear the propagator at the sink.