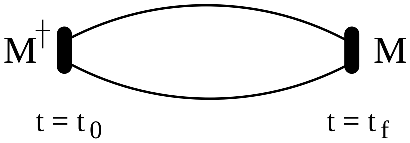
Calculating simple meson and baryon two-point functions

Meson two-point functions

We want to calculate a two-point correlation function for a meson containing a quark q_1 and anti-quark $\bar{q}_2$ (spin and colour indices are supressed).



A meson is created at time t_0 and position $\vec{x}_0$ using the operator,

$$\mathrm{M}^\dagger(\vec{x}_0, t_0) = \pm\bar{q}_1(\vec{x}_0, t_0)\Gamma^\dagger q_2(\vec{x}_0, t_0)$$

and destroyed at time t_f and position $\vec{x}_f$ using

$$\mathrm{M}(\vec{x}_f, t_f) = \bar{q}_2(\vec{x}_f, t_f)\Gamma q_1(\vec{x}_f, t_f),$$

The quantum numbers of the state created depend on Γ. Here we only look at Γ equal to one of the 16 gamma combinations: 1, γ_i, $\gamma_i\gamma_5$, $\gamma_i\gamma_j$ and γ_5. The $\pm$ in $\mathrm{M}^\dagger$ depends on $[\Gamma, \gamma_4]$.

We define

$$C^n_{2pt}(t_f,\vec{p};t_0) = \pm \sum_{\vec{x}_f} \mathrm{e}^{i\vec{p}\cdot(\vec{x}_f-\vec{x}_0)} \mathrm{M}(\vec{x}_f,t_f)\mathrm{M}^\dagger(\vec{x}_0,t_0).$$

where $\vec{p}$ is the momentum and $n = 1 \dots N$ labels the configuration. After Wick contractions this becomes

$$C^n_{2pt}(t_f,\vec{p};t_0) = \mp \sum_{\vec{x}_f} \mathrm{e}^{i\vec{p}\cdot(\vec{x}_f-\vec{x}_0)} \mathrm{Tr}\left[\Gamma G_1(\vec{x}_f,t_f;\vec{x}_0,t_0)\Gamma^\dagger\gamma_5 G_2^\dagger(\vec{x}_f,t_f;\vec{x}_0,t_0)\gamma_5\right]$$

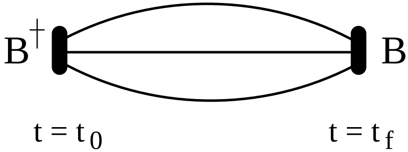
where $G_1(\vec{x}_f,t_f;\vec{x}_0,t_0)$ denotes a quark (q_1) propagating from $(\vec{x}_0,t_0)$ to $(\vec{x}_f,t_f)$ and we have used, for Wilson type fermions, $G_2(\vec{x}_0,t_0;\vec{x}_f,t_f) = \gamma_5 G_2^\dagger(\vec{x}_f,t_f;\vec{x}_0,t_0)\gamma_5$.

To extract physical properties you need to average over configurations:

$$C_{2pt}(t_f,\vec{p};t_0) = \frac{1}{N}\sum_{n=1}^{N} C^n_{2pt}(t_f,\vec{p};t_0)$$

Baryon two-point functions

For baryons the calculations are similar but more complicated as now you have three quarks. We consider the case of the $(\frac{1}{2}^+)$ baryon where the baryon is made up of two quarks (q_1) of the same mass and a quark (q_2) of a different mass. For $q_1 = u$ and $q_2 = d$ this is the proton.



For a nucleon, the baryon operator is given by

$$B_\alpha(\vec{x}_f, t_f) = \epsilon^{abc}(q_2^{Ta}(\vec{x}_f, t_f)C\gamma_5 q_1^b(\vec{x}_f, t_f))q_{1\alpha}^c(\vec{x}_f, t_f)$$

where we introduce color indices, a, b, c and a spin index α for one of the q_1 quarks. $C = \gamma_2\gamma_4$ is the charge conjugation matrix.

The two-point correlation function is given by

$$C^n_{2pt}(t_f, \vec{p}; t_0) = \sum_{\vec{x}_f} \mathrm{e}^{i\vec{p}\cdot(\vec{x}_f - \vec{x}_0)} \mathrm{T}_{\alpha\alpha'} \langle \mathrm{B}_\alpha(\vec{x}_f, t_f) \mathrm{B}_{\alpha'}{}^\dagger(\vec{x}_0, t_0) \rangle.$$

where $\mathrm{T}_{\alpha\alpha'}$ is the spin-projection matrix. At zero momentum, for unpolarised nucleons, $\mathrm{T} = \frac{1}{2}(1 + \gamma_4)$ and for nucleons polarised in the 3 direction, $\mathrm{T} = \frac{1}{2}(1 + \sigma_3)\frac{1}{2}(1 + \gamma_4)$. $\sigma_3 = i\gamma_3\gamma_5$.

After Wick contractions,

$$\begin{aligned} C^n_{2pt}(t_f, \vec{p}; t_0) \;=\; & \sum_{\vec{x}_f} \mathrm{e}^{i\vec{p}\cdot(\vec{x}_f - \vec{x}_0)} \mathrm{T}_{\alpha\alpha'} \epsilon^{abc} \epsilon^{a'b'c'} (C\gamma_5)_{\gamma\beta} (C\gamma_5)^\dagger_{\gamma'\beta'} \cdot \\ & G_2(\vec{x}_f, t_f; \vec{x}_0, t_0)^{aa'}_{\gamma\gamma'} \Big[G_1(\vec{x}_f, t_f; \vec{x}_0, t_0)^{bb'}_{\beta\beta'} G_1(\vec{x}_f, t_f; \vec{x}_0, t_0)^{cc'}_{\alpha\alpha'} \\ & + G_1(\vec{x}_f, t_f; \vec{x}_0, t_0)^{bb'}_{\beta\alpha'} G_1(\vec{x}_f, t_f; \vec{x}_0, t_0)^{cc'}_{\alpha\beta'} \Big] \end{aligned}$$

There are two terms in the expression for $C^n_{2pt}(t_f, \vec{p}; t_0)$ because of the two possibilities for the contractions of the two identical q_1 quarks.

Obtaining the ground state energy of a meson or a baryon

The time behaviour of $C_{2pt}(t_f, \vec{p}; t_0)$ can be obtained by inserting a complete set of states, $|m(\vec{p})\rangle$,

$$1 = \frac{1}{V}\sum_{m,\vec{k}} \frac{|m(\vec{k})\rangle\langle m(\vec{k})|}{2E_m(\vec{k})}$$

where the states correspond to the ground state (gs) and a tower of radial excitations. Now,

$$C_{2pt}(t_f, \vec{p}; t_0) = \sum_{m,\vec{k}}\sum_{\vec{x}_f} \frac{\mathrm{e}^{i\vec{p}\cdot(\vec{x}_f-\vec{x}_0)}}{2VE_m(\vec{k})}\langle \mathrm{M}(\vec{x}_f, t_f)|m(\vec{k})\rangle\langle m(\vec{k})|\mathrm{M}^\dagger(\vec{x}_0, t_0)\rangle$$

$$= \sum_{m} \frac{\mathrm{e}^{-E_m(\vec{p})(t_f-t_0)}}{2E_m(\vec{p})}\langle \mathrm{M}(\vec{x}_0, t_0)|m(\vec{p})\rangle\langle m(\vec{p})|\mathrm{M}^\dagger(\vec{x}_0, t_0)\rangle$$

$$\xrightarrow{t_f\gg t_0} \frac{\mathrm{e}^{-E_{gs}(\vec{p})(t_f-t_0)}}{2E_{gs}(\vec{p})}\langle \mathrm{M}(\vec{x}_0, t_0)|gs(\vec{p})\rangle\langle gs(\vec{p})|\mathrm{M}^\dagger(\vec{x}_0, t_0)\rangle.$$