

**Problem 1: Constraint algebra**

Show that the Hamiltonian and spatial diffeomorphism constraints

$$\mathcal{H} = \frac{2\kappa}{\sqrt{q}} \left( P^{ab} P_{ab} - \frac{1}{2} P^2 \right) - \frac{\sqrt{q}}{2\kappa} R \quad (1)$$

$$\mathcal{H}_a = -2q_{ac} \nabla_b P^{bc} \quad (2)$$

satisfy the hypersurface deformation algebra

$$\{\mathcal{H}[M], \mathcal{H}[N]\} = \mathcal{H}_a \left[ q^{ab} (M \partial_b N - N \partial_b M) \right] \quad (3)$$

$$\{\mathcal{H}[M], \mathcal{H}_a[N^a]\} = -\mathcal{H}[\mathcal{L}_N M] \quad (4)$$

$$\{\mathcal{H}_a[M^a], \mathcal{H}_a[N^a]\} = -\mathcal{H}_a[\mathcal{L}_N M^a]. \quad (5)$$

Hint: For (3), use that

$$\delta R^{(3)} = -R_{(3)}^{ab} \delta q_{ab} - q^{ab} (\nabla_c \nabla^c \delta q_{ab}) + (\nabla^a \nabla^b \delta q_{ab}). \quad (6)$$

For the other Poisson brackets, use the properties of the Lie derivative generated by  $\mathcal{H}_a$ .

**Problem 2: Gauß law in connection variables**

Show that  $-\int d^3x \Lambda_k \epsilon^{ijk} E_{[i}^a K_{a|j]} = \int d^3x \Lambda^k D_a E_k^a$ .

Hint: This exercise suggests a special form of the covariant divergence of densitized vectors:  $\nabla_a (\sqrt{q} v^a) = ?$

please turn the page

---

**Problem 3: ADM Poisson brackets in triad variables**

Show that

$$\left\{ q_{ab}[E, K](x), P^{cd}[E, K](y) \right\}_{\{K, E\}} = \delta_{(a}^c \delta_{b)}^d \delta^{(3)}(x, y) \quad (7)$$

i.e. that the Poisson brackets of the ADM variables  $q_{ab}$ ,  $P^{ab}$  are reproduced by the Poisson brackets of the triad variables.

Hint:  $0 = \{\delta_a^b, f\} = \{q_{ac}q^{cb}, f\} = q_{ac}\{q^{cb}, f\} + q^{cb}\{q_{ac}, f\}$  and  $\{\det q, f\} = qq^{ab}\{q_{ab}, f\}$  for arbitrary phase space functions  $f$ .

**Bonus question:** Show that  $\{P^{ab}[E, K](x), P^{cd}[E, K](y)\}_{\{K, E\}} = G_{ij}[\dots]$ .

**Problem 4: Canonical connection variables**

To show that  $A_a^i, E_j^b$  is a canonical pair, it was left to show that

$$\left\{ \Gamma_a^i(x), K_b^j(y) \right\} + \left\{ K_a^i(x), \Gamma_b^j(y) \right\} = 0. \quad (8)$$

This can be done either by brute force or by following the hints below. Boundary terms can be neglected throughout.

a) Show that the equation would be satisfied if  $\Gamma_a^i$  has a generating potential, i.e.  $\Gamma_a^i(x) = \frac{\delta F}{\delta E_a^i(x)}$

b) Construct a candidate for  $F$  (The simplest possible will do.)

c) Show that the candidate for  $F$  is indeed a potential for  $\Gamma_a^i$ .

**Bonus question:** What goes wrong if we try to construct similar variables in higher dimensions?

**Bonus question:** What happens to the canonical variables at the boundary of the spatial slice if we keep track of all boundary terms?

**Special Bonus question:** Is there any other useful way to construct canonical connection variables?

please turn the page

---

**Problem 5: BONUS: Variational identities**

For general spatial dimension  $D$ , show that

$$\delta R^{(D)} = -R_{(D)}^{ab} \delta q_{ab} - q^{ab} (\nabla_c \nabla^c \delta q_{ab}) + (\nabla^a \nabla^b \delta q_{ab}) \quad (9)$$

and

$$\delta (\nabla_c \nabla^c \Phi) = -(\nabla^a \nabla^b \Phi) \delta q_{ab} + (\nabla_c \nabla^c \delta \Phi) - (\nabla^b \Phi) (\nabla^a \delta q_{ab}) + \frac{1}{2} q^{ab} (\nabla^c \Phi) (\nabla_c \delta q_{ab}) \quad (10)$$

where  $\Phi$  is a scalar field so that  $\nabla_c \Phi = \partial_c \Phi$  is a co-vector.

Hint: Show that

$$\delta \Gamma_{ab}^c = \frac{1}{2} q^{cd} (\nabla_a \delta q_{db} + \nabla_b \delta q_{da} - \nabla_d \delta q_{ab}) \quad (11)$$

and

$$\delta R_{ab}^{(D)c}{}_d = 2 \nabla_{[a} \delta \Gamma_{b]d}^c. \quad (12)$$