Sheet 12 — Handout: 01.02.2021 — To present: 08.02.2021

Problem 1: Metric determinant

Show that in the ADM decomposition

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N_a & N_a \\ N_a & q_{ab} \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^a/N^2 \\ N^a/N^2 & q^{ab} - N^a N^b/N^2 \end{pmatrix} \qquad (1)$$

of the spacetime metric, we have $\sqrt{-\det g_{\mu\nu}} = |N|\sqrt{\det q_{ab}}$.

Problem 2: Extrinsic curvature

a) Show that $K_{\mu\nu}$ is symmetric.

Hint: $n_\mu \propto \partial_\mu f$, with $f(X_t(x^a)) = t$.

b) Show that $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n q_{\mu\nu}$.

Hint: Show that $2\nabla_{(\mu} n_{\nu)} = \mathcal{L}_n g_{\mu\nu}$.

Problem 3: Spatial diffeomorphism constraint

Show that up to a boundary term

$$-2 \int d^3x \, N^a \nabla_b P^b{}_a = \int d^3x \, P^{ab} \mathcal{L}_N q_{ab} \qquad (2)$$

and compute this boundary term explicitly.

Problem 4: BONUS: Gauß and Codazzi equations

Derive the Gauß equation

$$R^{(3)}_{\mu\nu\rho\sigma} = -2K_{\rho[\mu}K_{\nu]\mu} + q_\mu{}^{\mu'} q_\nu{}^{\nu'} q_\rho{}^{\rho'} q_\sigma{}^{\sigma'} R^{(4)}_{\mu'\nu'\rho'\sigma'} \tag{3}$$

and the Codacci equation

$$R^{(4)} = R^{(3)} + \left(K_{\mu\nu} K^{\mu\nu} - K^2 \right) + 2\nabla_\mu \left(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu \right). \tag{4}$$