Problems to
Quantum Gravity I: Canonical General Relativity WS 20/21

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Problem 1: Hypersurface deformation algebra

As in the lecture, we consider the canonical theory of a three-dimensional surface σ with coordinates x^a embedded into four-dimensional spacetime with coordinates y^μ. For every point in σ, we have the embedding coordinates $y^\mu(x^a)$. Covectors on spacetime can be pulled back to the hypersurface as $v_a := v_\mu \frac{\partial y^\mu}{\partial x^a}$. We define non-unit time-like conormal $\tilde{n}_\mu = \epsilon_{\mu\nu\rho\sigma}\epsilon^{abc}\frac{\partial y^\nu}{\partial x^a}\frac{\partial y^\rho}{\partial x^b}\frac{\partial y^\sigma}{\partial x^c}$. The unit time-like conormal is given by $n_\mu := \tilde{n}_\mu/\sqrt{-\tilde{n}_\mu\tilde{n}^\mu}$. We choose the metric signature $(-,+,+,+)$ and coordinates such that $n_0 < 0 \Leftrightarrow n^0 > 0$.

The non-vanishing Poisson brackets read $\{y^\mu(x), w_\nu(x')\} = \delta^\mu_\nu \delta^{(3)}(x,x')$.

We define the generators $\mathcal{H} = w_\perp := w_\mu n^\mu, \qquad \mathcal{H}_a = w_\mu \frac{\partial y^\mu}{\partial x^a}$.

Show that the Poisson-algebra of hypersurface deformations reads

$$\{\mathcal{H}[M], \mathcal{H}[N]\} = \mathcal{H}_a \left[q^{ab}\left(M\partial_b N - N\partial_b M\right)\right] \tag{1}$$

$$\{\mathcal{H}[M], \mathcal{H}_a[N^a]\} = -\mathcal{H}\left[\mathcal{L}_N M\right] \tag{2}$$

$$\{\mathcal{H}_a[M^a], \mathcal{H}_a[N^a]\} = -\mathcal{H}_a\left[\mathcal{L}_N M^a\right]. \tag{3}$$

Hints:

- ϵ^{abc} and $\epsilon_{\mu\nu\rho\sigma}$ are both totally antisymmetric. Note that cyclic permutations are only a symmetry of ϵ^{abc}! $\epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\nu\rho\sigma\mu}$.
- $\epsilon^{123} = 1$ and $\epsilon_{0123} = -1$.
- The above Lie derivatives read:
  - Of a scalar M along the vector field N^a: $\mathcal{L}_N M = N^a \partial_a M$
  - Of a vector field M^a along the vector field N^b: $\mathcal{L}_N M^a = N^b\partial_b M^a - M^b \partial_b N^a$
- Show that $\{w_\mu(x), n^\nu(x')\} = \{w_\mu(x), \tilde{n}^\rho(x')\}\frac{1}{\sqrt{-\tilde{n}_\mu\tilde{n}^\mu}} q_\rho{}^\nu$, with $q_\rho{}^\nu := \delta^\nu_\rho + n_\rho n^\nu$
  What role does $q_\rho{}^\nu$ have?
- Show that $3\epsilon_{\mu\nu\rho\sigma}\epsilon^{abc}\frac{\partial y^\rho}{\partial x^b}\frac{\partial y^\sigma}{\partial x^c} = \tilde{n}_\mu \frac{\partial x^a}{\partial y^\nu} - \tilde{n}_\nu \frac{\partial x^a}{\partial y^\mu}$
- Drop boundary terms

Bonus question: What changes if we pick the signature $(+,+,+,+)$?