Problems to
Quantum Gravity I: Canonical General Relativity WS 20/21

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Sheet 10 — Handout: 18.01.2021 — To present: 25.01.2021

Problem 1: Christoffel symbols

Show that the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ transform as a connection under general coordinate transformations.

Problem 2: Resolution of singularities in spatially flat, homogeneous and isotropic cosmology

a) Solve the equation

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho \tag{1}$$

for the scale factor $a(t)$, where $\rho = \frac{\text{const}}{a^{3(1+\omega)}}$ and ω is a constant determining the so called equation of state of the cosmological model.

b) Compare the case $\omega = 1$ to general relativity coupled to a massless scalar field as discussed in the lecture.

c) Solve the "quantum corrected" equations of motion

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho\left(1 - \frac{\rho}{\rho_{\text{crit}}}\right) \tag{2}$$

where ρ_{crit} is a constant depending on $\hbar$. Discuss the physical properties of these solutions.

Bonus question: Find a "quantum corrected" Hamiltonian that yields (2).

Special bonus question: Find a 'quantum corrected" Lagrangian that yields (2).

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Problem 3: Variation of the Einstein-Hilbert action

Vary the Einstein-Hilbert action (with cosmological constant)

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g}\,(R - 2\Lambda) \tag{3}$$

w.r.t. the inverse metric $g^{\mu\nu}$ to obtain the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{4}$$

Hint: Show first that $\delta R_{\mu\nu}{}^{\rho}{}_{\sigma} = \nabla_\mu \left(\delta\Gamma^{\rho}_{\nu\sigma}\right) - \nabla_\nu \left(\delta\Gamma^{\rho}_{\mu\sigma}\right)$ and neglect boundary terms.

Bonus question: Within the Riemann tensor, substitute the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ by an arbitrary torsion-free connection $A^{\rho}_{\mu\nu}$ that can be varied independently of the metric. Compute and solve the equations of motion obtained from varying $A^{\rho}_{\mu\nu}$.

Bonus question: Add the York-Gibbons-Hawking boundary term

$$S_{\mathrm{YGH}} = \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{|h|} K \tag{5}$$

to the action, where h_{ab} is the induced metric on the boundary $\partial\mathcal{M}$, K_{ab} is the extrinsic curvature (second fundamental form) of $\partial\mathcal{M}$ (see the lecture notes, chapter 5), and $K = K_{ab} h^{ab}$. Show that the variation leads to the Einstein equations without dropping any boundary terms if we fix the boundary condition $\delta h_{ab} = 0$ on the variation.

Special bonus question: What is the value of the contribution of S_{YGH} to the on-shell action at a null crossing of ∂M, i.e. when h_{ab} changes signature?