**Problems to**
**Quantum Gravity I: Canonical General Relativity** WS 20/21

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*Sheet 9* — Handout: 11.01.2021 — To present: 18.01.2021

## Problem 1: Metric compatibility

Show that for a torsion free connection, i.e. $A^\rho_{\mu\nu} = A^\rho_{\nu\mu}$, the condition

$$D_\mu g_{\rho\sigma} = \partial_\mu g_{\rho\sigma} - A^\nu_{\mu\rho} g_{\nu\sigma} - A^\nu_{\mu\sigma} g_{\rho\nu} = 0 \tag{1}$$

determines $A^\rho_{\mu\nu}$ to be equal to the Christoffel symbols $\Gamma^\rho_{\mu\nu}$.

Hint: Sum various $D_\mu g_{\rho\sigma} = 0$ with suitably interchanged indices.

## Problem 2: Covariant divergence

**a)** Conclude from $\nabla_\mu g_{\rho\sigma} = 0$ that $\nabla_\mu(-g) = 0$, where $g = \det g_{\mu\nu}$. Show furthermore that this implies $\partial_\mu \sqrt{-g} - \sqrt{-g}\Gamma^\nu_{\mu\nu} = 0$.

**b)** Show that $\nabla_\mu(\sqrt{-g} v^\mu) = \partial_\mu(\sqrt{-g} v^\mu)$ for arbitrary vectors $v^\mu$.

**c)** Conclude that covariant derivatives can be partially integrated if the integrand has density weight one, i.e. contains one power of $\sqrt{-g}$, and no uncontracted indices.

## Problem 3: Properties of the Lie derivative

**a)** Show that the Lie derivative is compatible with index contraction

**b)** Compute the Lie derivative of a density $n$ object, such as $\left(\sqrt{\det g}\right)^n$. What happens if the object has additional tensor indices?

**c)** Show that the Lie derivative can be partially integrated if the integrand is a density 1 object with no (uncontracted) tensor indices (as it should be).

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### Problem 4: Parallel transport on a Sphere

Consider a two-dimensional sphere $S^2$ parametrized by the coordinates $\theta \in (0, \pi)$ (vertical angle) and $\phi \in (-\pi, \pi)$ (horizontal angle).

**a)** Calculate the parallel transport of an arbitrary vector along the two curves

$$\gamma_\theta(s) = (\theta_0 + s, \phi_0), \quad s \in (a, b)$$

and

$$\gamma_\phi(s) = (\theta_0, \phi_0 + s), \quad s \in (a, b).$$

**b)** Calculate the parallel transport along a closed curve composed of components of $\gamma_\theta$ and $\gamma_\phi$ along the points: $(\pi/2, 0) \to (\pi/2, \phi^*) \to (\theta^*, \phi^*) \to (\theta^*, 0) \to (\pi/2, 0)$. Compare the parallel transported vector and the initial vector. What happens for an infinitesimal small loop? Compare with the definition of the Riemann tensor.

### Problem 5: BONUS: Geodesic equation

Derive the geodesic equation by varying the length functional

$$d(c) = \int_c ds := \int_c \sqrt{g_{ij} dx^i dx^j} := \int_a^b d\lambda \sqrt{g_{ij}(c(\lambda)) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$$

along a curve $c : [a, b] \to \mathcal{M}_n$. It is easiest to start with the case where the curve parameter $\lambda$ measures proper length, so that

$$\sqrt{g_{ij}(c(\lambda)) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} = \text{const}$$

as a function of $\lambda$ along the desired trajectory. What corrections to the geodesic equation are necessary if $\lambda$ does not measure proper length?