Problems to
Quantum Gravity I: Canonical General Relativity WS 20/21

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Sheet 8 — Handout: 21.12.2020 — To present: 11.01.2021

Problem 1: Covectors

Derive the transformation law for covector components from the definition $dx^i(\partial_j) = \delta^i_j$ for the dual basis. Conclude that the evaluation $w(v) = w_i v^i$ of a vector v using a covector w is independent on the choice of basis.

Problem 2: Push forward and pull back

Compute the coordinate expression for the pull back of a vector field along a diffeomorphism Φ by using its inverse. Confirm that the evaluation of a vector v using a covector w, $w(v)$, is invariant under pull backs, i.e. $(\Phi_* w)(\Phi_* v) = w(v)$. Show also that $\Phi_* \Phi^* = \Phi^* \Phi_* = \mathrm{Id}$.

Hint: $\delta^i_j = \frac{\partial x^i}{\partial x^j} = \frac{\partial (\Phi^{-1})^i(\Phi(x))}{\partial x^j} = \ldots$

Problem 3: Great circles

Show that the shortest path between two points on a sphere is a section of a great circle by solving the geodesic equation in suitable coordinates.

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Problem 4: The classical string

Consider an embedding map $X^\mu(\tau, \sigma)$ from a two-dimensional surface coordinatised by (τ, σ) into four-dimensional Minkowski spacetime with the sign convention $(-, +, +, +)$. The range of this map is called the world sheet of the classical string. We will leave its domain unspecified for this exercise and neglect boundary terms throughout. We define $\dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau}$ and $X'^\mu := \frac{\partial X^\mu}{\partial \sigma}$, as well as $\dot{X}^2 := \dot{X}^\mu \dot{X}_\mu$, $X'^2 := X'^\mu X'_\mu$ and $\dot{X} \cdot X' = \dot{X}^\mu X'_\mu$.

a) Show that the infinitesimal area element of the world sheet in Minkowski spacetime reads

$$dA = \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}\; d\tau d\sigma, \tag{1}$$

where the sign under the square root is chosen to be consistent with $\dot{X}$ being time-like and X' being spacelike.

b) Perform the Legendre transform of the Nambu-Goto action

$$S_{\text{NG}} = -\frac{T}{c} \int dA = -\frac{T}{c} \int d\tau \int d\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}, \tag{2}$$

and show that two primary constraints arise. The constant T is called the string tension and c is the speed of light.

c) How can one see directly in the action that two primary constraints should arise?

d) Compute the total Hamiltonian, the algebra of constraints, and complete the Dirac stability analysis.

Bonus question: Is it possible to arrive at a first order formulation of the classical string analogous to the previous exercise? Perform again the canonical analysis and try to find gauges which simplify the theory.