Problems to
Quantum Gravity I: Canonical General Relativity WS 20/21

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Sheet 6 — Handout: 07.12.2020 — To present: 14.12.2020

Problem 1: Henneaux & Teitelboim: problem 1.18

Let F and G be two gauge invariant functions, i.e. $\{F, \gamma_a\} \approx 0$, $\{G, \gamma_a\} \approx 0$. Prove that $\{F, G\}_* \approx \{F, G\}$, no matter which (good) gauge conditions are adopted.

Problem 2: Dirac bracket and choice of second class constraints

Show that the Dirac bracket is (weakly) unaffected by an alternative choice of second class constraints, $\chi'_\alpha = A_\alpha{}^\beta \chi_\beta$, where $\det A_\alpha{}^\beta \neq 0$.

Problem 3: Henneaux & Teitelboim: problem 1.21

Assume that the second-class constraints $\chi_\alpha = 0$ split as $\chi_\alpha \equiv (\gamma_a, C_a)$ where the subset $\gamma_a = 0$ is first class by itself, $\{\gamma_a, \gamma_b\} = C_{ab}{}^c \gamma_c$. Let F be an arbitrary phase space function. Prove the existence of an equivalent function $\bar{F} = F + \lambda^\alpha \chi_\alpha + \mathcal{O}(\chi^2)$, which is first class w.r.t. the γ_a, $\{\bar{F}, \gamma_a\} = f_a{}^b \gamma_b + \mathcal{O}(\chi)$. Show that the first-class system with constraints $\gamma_a = 0$ and Hamiltonian $\bar{H}$ is equivalent to the original second-class system modulo $\mathcal{O}(\chi)$ terms after evaluating Poisson brackets. Higher powers in χ can be obtained via the gauge unfixing projector, so this exercise checks gauge unfixing to first order in χ.

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Problem 4: BONUS: Recursive Hamiltonians

a) Warmup: Consider the Hamiltonian

$$H_0 = O(q^i, p_i, \mu), \tag{1}$$

with standard Poisson brackets $\{q^i, p_j\} = \delta^i_j$. μ is for now a fixed parameter constant on phase space. Compute the equations of motion.

b) Next, consider the case when $\mu = f(O)$ is a function of the Hamiltonian. The Hamiltonian is then recursively defined as

$$H = O(q^i, p_i, f(O)). \tag{2}$$

Compute the equations of motion and relate the Hamiltonian vector field to that of H_0.

c) Try to rederive the equations of motion along the following lines. First, extend the phase space with the variables μ, p_μ, $\{\mu, p_\mu\} = 1$, then imposing the constraint $\Phi = \mu - f(O) \approx 0$, which Poisson commutes with the Hamiltonian. The new Hamiltonian is the total Hamiltonian $H_T = H_0 + \lambda\Phi$ for λ arbitrary. Choose a gauge fixing $p_\mu - h(q^i, p_j) \approx 0$ for Φ and determine λ by requiring stability of the gauge fixing. For which choices of h do you obtain the wanted equations of motion? Why is this not always the case?

Hint: It may be instructive to start with the simpler case $\mu = f(q^i, p_j)$, where f is independent of O.