Problems to
Quantum Gravity I: Canonical General Relativity WS 20/21

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Sheet 2 — Handout: 09.11.2020 — To present: 16.11.2020

Problem 1: Harmonic oscillator in the Hamiltonian formalism

a) The Lagrangian of a harmonic oscillator in suitable coordinates is given by

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2.$$

Compute and solve the Lagrangian equations of motion.

b) Compute the Hamiltonian via a Legendre transform and solve the Hamiltonian equations of motion.

c) Compute explicitly the Hamiltonian flow via the formula

$$f(q(t), p(t)) = e^{t\{\cdot,\mathcal{H}\}} f(q,p)\Big|_{q=q_0,\ p=p_0} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\{f,\mathcal{H}\}_{(n)}\Bigg|_{q=q_0,\ p=p_0}$$

given in the lectures.

d) Plot the integral curves and the Hamiltonian vector field in phase space.

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Problem 2: Legendre Transform

Consider a convex function $f(x)$. Its Legendre transform $g(u)$ is defined as follows:
1. Set $u(x) = f'(x)$ and solve for $x(u)$
2. Set $g(u) = ux(u) - f(x(u))$

a) Show that the Legendre transform is an involution, i.e. that the Legendre transform of $g(u)$ is again $f(x)$.

b) What are the analogues of x, u, f, g in the derivation of the Hamiltonian formalism?

c) Compute the Legendre transform of the function $f(x) = a(x+b)^2$. Check your result by applying the Legendre transform again to recover $f(x)$.

Problem 3: BONUS: Auxiliary fields

Consider the action $S[y^i, \dot{y}^i, z^A]$, where the z^A are auxiliary fields, defined by the property that

$$\frac{\delta S}{\delta z^A} = 0 \quad \Leftrightarrow \quad z^A = z^A(y^i, \dot{y}^i). \tag{1}$$

Show that one can insert the solutions to the auxiliary equations of motion into the action. In other words, consider the modified action principle $S'[y^i] = S[y^i, \dot{y}^i, z^A(y^i, \dot{y}^i)]$ and show that it leads to the same equations of motion for y^i.