Introduction to Quantum Gravity I

Lecture notes, winter term 2018 / 19

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Disclaimer:

This is a set of lecture notes for the lecture "Introduction to Quantum Gravity I". As such, they have not undergone the same level of scrutiny in error checking as published articles and should not be treated as a reference. They are neither necessary nor sufficient substitutes for consulting textbooks or attending the lectures.

Expected course span: 2 semesters.

Duration: 2 hour lecture + 3 hour exercise / week.

First semester: winter term 18/19. 15 lectures.

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Necessary Prerequisites:

- Classical mechanics
- Special relativity

Useful knowledge (basic introductions are provided for what is necessary for this course):

- Classical field theory
- Gauge theory
- Quantum mechanics
- General relativity
- Quantum field theory
- Differential geometry
- Lie groups

About this script:

- *Italic* comments are to be presented only orally, whereas standard font is to be written on the black board. Exceptions are theorems / definitions.

Conventions:

- Einstein summation convention: Repeated indices are summed over their whole range
- Conventions for indices are sometimes changed to facilitate comparison with the most easily available literature

Contents

0 Aim and Literature

0.1 Aim of the lecture

Aim: Basic introduction into canonical quantum gravity, following the canonical loop quantum gravity programme

Content:

- Introduction
- Constrained Hamiltonian systems:
  - Develop a universal classical formalism to describe physical theories with gauge symmetry
  - Understand the geometry of the phase space of gauge systems and learn to manipulate it
- Quantisation of constrained Hamiltonian systems
  - Consistently combine gauge symmetry and quantisation
- Generally covariant systems
  - Understand theories that are invariant under general coordinate transformations
  - Applications to cosmology
- Canonical general relativity
  - Understand the ADM formulation, known as geometrodynamics
  - Formulate general relativity on a Yang-Mills phase space
- Quantum cosmology
  - Test quantisation methods on a simpler system
  - Obtain an understanding of possible quantum gravity effects
- Quantum kinematics
  - Understand how to quantise a basic set of observables
  - Solve the "non-dynamical" quantum constraints
- Geometric operators
  - Quantise the classical expressions for area and volume
  - Understand the physics of spin networks
- Quantum Dynamics
  - Sketch the implementation of the Hamiltonian constraint
  - Overview of existing alternative proposals for the dynamics

0.2 Suggested literature and sources used to assemble these notes

Constrained systems

- Dirac: "Lectures on Quantum Mechanics" (1964, basics, concise and easily accessible)
- Henneaux & Teitelboim: "Quantization of Gauge Systems" (1992, exhaustive, well written)

General relativity

- Carroll: "Spacetime and Geometry", lecture notes available as gr-qc/9712019
- Wald: "General Relativity" (more advanced)

Differential geometry

- Fecko: "Differential Geometry and Lie Groups for Physicists" (very elementary)
- Nakahara: "Geometry, Topology and Physics"
- Frankel: "The Geometry of Physics"

Representation theory of SO(3)

- Sexl, Urbantke: "Relativity, Groups, Particles"

Quantum gravity (general)

- Kiefer "Quantum gravity" (textbook)
- Oriti "Approaches to Quantum Gravity" (broad collection of review articles)

Canonical loop quantum gravity

- Gambini / Pullin: "A First Course in Loop Quantum Gravity" (elementary introduction)
- Rovelli: "Quantum Gravity" (intermediate level)
- Thiemann: "Modern Canonical Quantum General Relativity" (advanced and mathematical presentation)

Covariant path integral formulation

- Rovelli, Vidotto: "Covariant loop quantum gravity" (available at `http://www.cpt.univ-mrs.fr/~rovelli/IntroductionLQG.pdf`)

Online sources

- `wikipedia.org` (for brief introductions to the necessary mathematics)
- Research articles at `arxiv.org`

Other lecture notes on / introductions to the subject:

- Thiemann: "Introduction to Modern Canonical Quantum General Relativity" https://arxiv.org/abs/gr-qc/0110034
- Thiemann: "Lectures on loop quantum gravity" https://arxiv.org/abs/gr-qc/0210094
- Doná, Speziale: "Introductory lectures to loop quantum gravity" https://arxiv.org/abs/1007.0402
- Giesel, Sahlmann: "From Classical To Quantum Gravity: Introduction to Loop Quantum Gravity" https://arxiv.org/abs/1203.2733
- Bilson-Thompson, Vaid: "LQG for the Bewildered" https://arxiv.org/abs/1402.3586
- Bodendorfer: "An elementary introduction to loop quantum gravity" https://arxiv.org/abs/1607.05129

1 Introduction

Shortened version of the introduction of `arXiv:1607.05129` (including references).

1.1 Motivations for studying quantum gravity

Gather some motivations for conducting research in quantum gravity. Choice here represents the personal preferences.

- Geometry is determined by matter, which is quantised
  Einstein equations $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$
  Quantum field theory tells us that matter is quantised
  Two possibilities to reconcile:

  1. Also geometry quantised (considered more likely)
  2. Geometry classical, energy-momentum tensor is an expectation value

  While the second approach seems to be a logical possibility, most researchers consider the first case to be more probable and the second as an approximation to it. Second possibility tricky, e.g. superpositions of particles...

- Singularities in classical general relativity
  "big bang", black hole singularity, ...
  $\rightarrow$ signals breakdown of theoretical description

- Black hole thermodynamics
  Classical black holes exhibit thermodynamic behaviour.
  3 Laws of thermodynamics map to black holes. Thermal Hawking radiation.
  $\rightarrow$ What are the microstates to be counted?

- Cutoff for quantum field theory (QFT)
  Divergences in QFT, need cutoff or regularisation.
  $\rightarrow$ Provided by quantum gravity?

1.2 Possible scenarios for observations

- Modified dispersion relations / deformed symmetries
  Strong bounds from experiments which are sensitive to such effects piling up over a long time or distance, such as observations of particle emission in a supernova.

- Quantum gravity effects at black hole horizons
  While quantum gravity is believed to resolve the singularities inside a black hole, an observation of this fact is a priori impossible due to the horizons shielding the singularity. However, modifications at horizon scale possible in some models / scenarios. On the other hand, it might be possible to observe signatures of evaporating black holes which were formed at colliders, which however generally requires a lowering of the Planck scale in the TeV range, possibly due to extra dimensions.

- Cosmology
  E.g. quantum gravity signature in cosmic microwave background.
  Follows e.g. from singularity resolution of the "big bang"

- Particle spectrum from unification
  Mainly in string theory, often include supersymmetry.

- Gauge / Gravity
  An indirect way of observing quantum gravity effects is via the gauge / gravity correspondence, which relates quantum field theories and quantum gravity.

1.3 Approaches to quantum gravity

List of the largest existing research programmes.

- **Semiclassical gravity**
  - Energy-momentum-tensor is expectation value.
  - Need self-consistent solution

  First step towards quantum gravity, matter fields are treated using full QFT, geometry classical. Beyond QFT on CS: the energy-momentum tensor is QFT expectation value. The state in which this expectation value is evaluated in turn depends on the geometry, need self-consistent solution.

- **Ordinary quantum field theory**
  - Perturbative QFT around given background metric
  - Suffers from non-renormalisability
  - Effective field theory treatment possible

  Quantise the deviation of the metric from a given background. General relativity is non-renormalisable in the standard picture, but possible to use effective field theory up to some energy scale lower than the Planck scale. Does not aim to understand quantum gravity in extreme situations, such as cosmological or black hole singularities.

- **Supergravity**
  - Locally supersymmetric gravity theory
  - Aimed at unification
  - Better UV behaviour, but still non-renormalisable (maybe up to $d = 4, \mathcal{N} = 8$)

  Invented to provide a unified theory of matter and geometry with better UV behaviour. Local supersymmetry relating matter and gravitational degrees of freedom.
  Improved the UV behaviour of the theories, but still non-renormalisable (maybe up to $d = 4, \mathcal{N} = 8$). Nowadays, mostly considered within string theory, where 10-dimensional supergravity appears as a low energy limit.

- **Asymptotic safety**
  - Find non-Gaussian fix point in renormalisation group flow

  Renormalisation group flow assumed to possess a non-trivial fixed point with finite couplings. Solve renormalisation group equations in suitably truncated theory space. Up to now, much evidence in certain truncations.

- **Canonical quantisation: Wheeler-de Witt**
  - No split in background / perturbation
  - Hilbert space hard to define

Canonical quantisation of the Arnowitt-Deser-Misner formulation. Uses spatial metric and its conjugate momentum as canonical variables.
Hamiltonian constraint operator is extremely difficult to define due to its non-linearity, scalar product not known.

- **Euclidean quantum gravity**
  - Wick rotation to Euclidean space
  - Evaluate path integral over all metrics

  Allows to extract thermodynamic properties of black holes. Path integral is often approximated by the exponential of the classical on-shell action. Wick rotation to Euclidean space is well defined only for a certain limited class of spacetimes, dynamical phenomena hard to track.

- **Causal dynamical triangulations**
  - Specific incarnation of asymptotic safety
  - Uses discretisation of action

  Uses certain discretisation, makes it easier to handle on computer. Path integral evaluated using Monte Carlo techniques.

- **String theory**
  - Replace point particle concept by 1-dimensional string
  - Particles as vibration modes of quantum strings

  Initially conceived as a theory of the strong interactions, particle concept replaced by one-dimensional strings. Particle spectrum of string theory includes a massless spin 2 excitation. Consistency demands (in lowest order) the Einstein equations (for supergravity) to be satisfied. Quantisation of gravity is achieved via unification.
  Main problem is wrong spacetime dimension: 26 for bosonic strings, 10 for supersymmetric strings, and 11 in the case of M-theory. Compactify some of the extra dimensions, but large amount of arbitrariness. Limited understanding of non-perturbative string theory.

- **Gauge / gravity**
  - Gravity theory defined via conformal field theory on spacetime boundary
  - Requires dictionary between two descriptions

  Grown out of string theory, but was later recognised to be applicable more widely. Once a complete dictionary known, use the gauge / gravity to define quantum gravity on that class of spacetimes.
  Main problem is the lack of a complete dictionary. Usually very hard to find gauge theory duals of realistic gravity theories, many known examples are very special supersymmetric theories.

- **Loop quantum gravity**
  - Canonical quantisation of GR in connection formulation
  - No unification / particle content added by hand

Spirit of the Wheeler-de Witt approach, but based on connection variables. Main advantage: rigorously define a Hilbert space and techniques to quantise the Hamiltonian constraint. Application to symmetry reduced models: loop quantum cosmology. Main problem: obtain general relativity by coarse graining / renormalisation group flow. Situation roughly the opposite of that in string theory. Regularisation ambiguities present. Path integral approach: spin foams + group field theory approach.

2 Constrained Hamiltonian systems

Hamiltonian formalism is basis for canonical quantisation. We need to incorporate gauge symmetry in this formalism.

2.1 Hamiltonian systems without gauge symmetry

Before moving to constrained systems, we have to recall what happens in the unconstrained case.

2.1.1 Legendre transform and equations of motion

Obtain Hamiltonian system:

1. Define Hamiltonian system from scratch
2. Start with Lagrangian and Legendre transform

The second option usually better:

- Most theories are given in Lagrangian form
- The Lagrangian formalism is simpler to set up
  no Poisson brackets, no interpretation of momenta, ...
- Lagrangians exhibit manifest invariances, such as Lorentz invariance
- No need to guess gauge generators (later)

Consider a time-independent Lagrangian

$$L\left(q^1,\ldots,q^n,\dot{q}^1,\ldots,\dot{q}^n\right) \equiv L\left(q^i,\dot{q}^i\right) \tag{2.1}$$

and the action

$$S = \int dt\, L. \tag{2.2}$$

Time dependent Lagrangians normally don't occur in fundamental physics. The generalisation to field theories is straight forward.

Equations of motion from least action principle $\delta S = 0$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \quad \Leftrightarrow \quad \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\ddot{q}^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j. \tag{2.3}$$

$\Rightarrow$ accelerations $\ddot{q}^j$ are uniquely determined $\Leftrightarrow$ $\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$. We assume this for now.

Canonical momenta:

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \tag{2.4}$$

Idea of Hamiltonian formalism:

- Use the q^i and p_i as <u>independent</u> variables
- Set up first order evolution equations for them

In order to set up equations for q^i and p_i, we could use a function whose variation is the sum of variations in q^i and p_i only:

$$\delta\left(p_i\dot{q}^i - L\right) = \dot{q}^i\delta p_i + p_i\delta\dot{q}^i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i = \dot{q}^i\delta p_i - \frac{\partial L}{\partial q^i}\delta q^i \tag{2.5}$$

so that

$$H := p_i\dot{q}^i(q^j, p_j) - L = H(q^i, p_i) \tag{2.6}$$

H defined uniquely $\Leftrightarrow$ we can express all the $\dot{q}^i$ uniquely as functions of q^j, p_j.

Necessary condition: $\det \frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j} = \det\frac{\partial p_i}{\partial\dot{q}^j} \neq 0$.

Least action principle:

$$\begin{aligned} 0 = \delta\int dt\, L = \delta\int dt\left(p_i\dot{q}^i - H\right) &= \int dt\left(p_i\delta\dot{q}^i + \dot{q}^i\delta p_i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i\right) \\ &= \int dt\left(-\dot{p}_i\delta q^i + \frac{d}{dt}\left(p_i\delta q^i\right) + \dot{q}^i\delta p_i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i\right) \\ &= \int dt\left(\left(-\dot{p}_i - \frac{\partial H}{\partial q^i}\right)\delta q^i + \left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right)\delta p_i\right) \end{aligned} \tag{2.7}$$

$\Rightarrow$ Canonical equation of motion:

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}. \tag{2.8}$$

2.1.2 Phase space and Poisson brackets

The following concepts turn out to be highly useful later.

We will be rather imprecise with the underlying mathematics in this section.

Definition 1. The space coordinatised by $q^1, \ldots, q^n$ is called **configuration space**.

The concept of a manifold etc. will be introduced only later.

Example: The location of a point particle in $\mathbb{R}^n$.

Restrict for simplicity to $q^i \in \mathbb{R}$. ($p_i \in \mathbb{R}$ always).

Definition 2. $\mathbb{R}^{2n}$, coordinatised by all q^i and p_i, is called **phase space** Γ.

Example: The location and momentum of a point particle in $\mathbb{R}^3$.

General case: co-tangential bundle over configuration space.

Definition 3. A **phase space function** f is a "sufficiently smooth" function on phase space, i.e. $f = f(q^i, p_i)$.

All physical observables are phase space functions and vice versa (without gauge symmetry).

The set of phase space function forms an algebra over $\mathbb{R}$ *(roughly: addition + multiplication).*

The algebraic structure of classical mechanics will be discussed in more detail later in section 6.1.2. For now, we do not specify what an algebra is. The mention here is meant for students already familiar with the mathematical concept of an algebra.

Definition 4. The Poisson bracket between two phase space functions f and g is defined as

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \tag{2.9}$$

It satisfies

- Antisymmetry: $\{f, g\} = -\{g, f\}$
- Linearity: for $c_1, c_2 \in \mathbb{R}$: $\{c_1 f_2 + c_2 f_2, g\} = c_1 \{f_1, g\} + c_2 \{f_2, g\}$
- Leibniz property: $\{f_1 f_2, g\} = f_1 \{f_2, g\} + \{f_1, g\} f_2$
- Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

The Poisson bracket adds the structure of a Poisson algebra.

Canonical equation of motion:

$$\dot{q}^i = \left\{q^i, H\right\}, \quad \dot{p}_i = \{p_i, H\}. \tag{2.10}$$

In general

$$\dot{f} = \{f, H\} \tag{2.11}$$

for any phase space function.

$\Rightarrow$ H is the generator of time translations. *Evolution is a flow on phase space.*

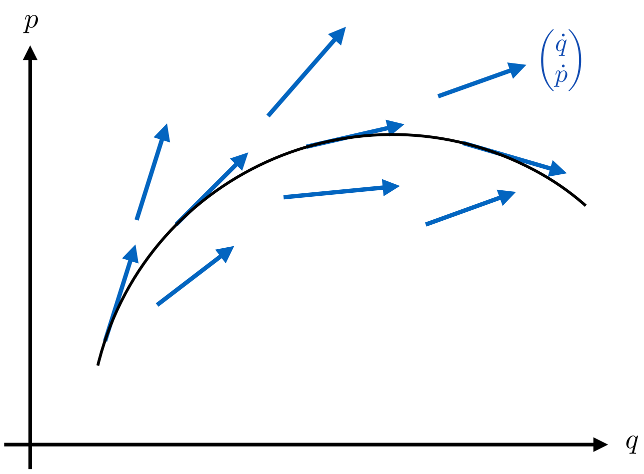


Figure 2.1: An integral curve (black) in phase space with tangents agreeing with the Hamiltonian vector field (blue).

Definition 5. The collection of all the vectors $\left(\dot{q}^1, \ldots, \dot{q}^n, \dot{p}_1, \ldots, \dot{p}_n\right)$ for every point $(p^i, q_i) \in \Gamma$ is called the Hamiltonian vector field $\vec{v}_H$.

$\Rightarrow$ Hamiltonian flow in phase space, can be explicitly exponentiated:

$$\begin{aligned} f(q(t), p(t)) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^n f}{dt^n}\Big|_{t=0} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \{f, H\}_{(n)}\Big|_{q=q_0,\ p=p_0} \\ &=: e^{t\{\cdot, H\}} f(q,p)\Big|_{q=q_0,\ p=p_0} =: e^{t\vec{v}_H} f(q,p)\Big|_{q=q_0,\ p=p_0} \end{aligned} \tag{2.12}$$

$\{f, H\}_{(n+1)} := \{\{f, H\}_{(n)}, H\}, \quad \{f, H\}_{(0)} := f, \quad \vec{v}_H := \frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i}$

Curve $(q^i(t), p_i(t))$ in phase space: integral curve of $\vec{v}_H$.

Hamiltonian systems without gauge symmetry:

- Distinct points in phase space correspond to distinct physical situations
- The Hamiltonian generates a flow on phase space
- The flow is interpreted as physical evolution

Why is this formalism is not sufficient?

- In gauge systems, distinct points in phase space can correspond to the same physical situation
- Therefore, the phase space flow between two physical situations is ambiguous and cannot be generated by a unique Hamiltonian
- For gauge systems, $\det \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} = 0$

2.2 Constrained Hamiltonian systems

2.2.1 Legendre transform

Recall the Lagrangian equations of motion

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \ddot{q}^j = \frac{\partial \mathcal{L}}{\partial q^i} - \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial q^j} \dot{q}^j \tag{2.13}$$

Unique evolution $\Leftrightarrow$ $\ddot{q}_i$ determined as functions of q^i, $\dot{q}^i$ $\Leftrightarrow$ $\det \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$.

For gauge system, the determinant vanishes.

$\Rightarrow$ Canonical momenta $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$ cannot be uniquely expressed as functions of q^i, $\dot{q}^i$, because

$$\det \frac{\partial p^i}{\partial \dot{q}^j} = 0 \tag{2.14}$$

i.e. we can vary the $\dot{q}^i$ without affecting the p_i.

There exists a v^j with $\frac{\partial p^i}{\partial \dot{q}^j} v_j = 0$ by assumption. Therefore, p^i invariant under $\dot{q}^i \mapsto \dot{q}^i + \epsilon v^i$.

We express as many $\dot{q}^i$ through q^i, p_i as possible by using $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$.

Additionally, we obtain relations $\phi_m(q^i, p_i) = 0$, $m = 1, \ldots, M$.

If there were any $\dot{q}^i$ left in the ϕ_m, we could use those equations to express the $\dot{q}_i$ as functions of q^i, p_i.

Definition 6. The $\phi_m(q^i, p_i)$, $m = 1, \ldots, M$ are called **primary constraints**.

The Legendre transform still has the property that

$$\delta \left(p_i \dot{q}^i - \mathcal{L} \right) = \dot{q}^i \delta p_i + p_i \delta \dot{q}^i - \frac{\partial \mathcal{L}}{\partial q^i} \delta q^i - \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \delta \dot{q}^i = \dot{q}^i \delta p_i - \frac{\partial \mathcal{L}}{\partial q^i} \delta q^i \tag{2.15}$$

i.e. q^i, p_i are the dynamical variables of the Hamiltonian formulation.

Hamiltonian H is not unique due to the $\phi_m(q^i, p_i) = 0$.

Any "total" Hamiltonian $H_T = H + u^m \phi_m$ is on the same footing. u^m: arbitrary functions.

2.2.2 Stability algorithm

Strategy: Use H_T as a Hamiltonian and work out consequences.

The u^m are arbitrary functions, sometimes they are velocities which cannot be expressed using only q^i, p_i

Extend Poisson brackets to the u^m (not necessarily phase space functions) in some way consistent with the symmetries of the bracket.

$$\dot{f} = \{f, H + u^m \phi_m\} = \{f, H\} + u^m \{f, \phi_m\} + \{f, u^m\} \phi_m = \{f, H\} + u^m \{f, \phi_m\} \tag{2.16}$$

$\Rightarrow$ Extension is irrelevant, but necessary for the formalism.

Important: Use $\phi_m = 0$ only after evaluating the Poisson brackets.

Definition 7. A **weak equality** is denoted by $\approx$ and means "equality modulo constraints". It may be used only after all Poisson brackets have been evaluated.

Example: $\phi_m \approx 0$, but $\{\phi_m, f(q,p)\} \not\approx 0$ in general.

Consistency of $\phi_m \approx 0$ with the Hamiltonian evolution implies

$$\dot{\phi}_m = \{\phi_m, H_T\} = \{\phi_m, H + u^n \phi_n\} \approx \{\phi_m, H\} + u^n \{\phi_m, \phi_n\} \overset{!}{\approx} 0 \tag{2.17}$$

$\Rightarrow$ Consistency conditions, 4 possibilities:

1. Trivially satisfied, e.g. $0 = 0$
2. Inconsistent theory, e.g. $1 = 0$ (exercise)
3. Condition on the u^n
4. New constraint $\chi_k(q,p) = 0$, independent of the u^n

Definition 8. The set of all $\chi_k(q,p) = 0$ are called **secondary constraints**.

For secondary constraints, one uses the equations of motion, as opposed to primary constraints. Distinction of minor importance.

Secondary constraints $\Rightarrow$ reiterate the consistency algorithm $\Rightarrow$ possibly tertiary constraints, ...

At some point, this algorithm will stop, i.e. give no new conditions, or the theory is inconsistent.

We obtained K new constraints.

Set of all constraints: $\{\phi_1, \ldots, \phi_{M+K}\} := \{\phi_1, \ldots, \phi_M, \chi_1, \ldots, \chi_K\}$

Denote as ϕ_j, $j = 1, \ldots, J = M + K$.

View solving for u^m as solving inhomogeneous linear equation system:

$$\dot{\phi}_j \approx \{\phi_j, H\} + u^m \{\phi_j, \phi_m\} \approx 0 \tag{2.18}$$

J equations for $M \leq J$ unknowns. Assume that solution exists, otherwise theory inconsistent.

Special solution: U^m.

Several homogeneous solutions: $V_a^m \{\phi_j, \phi_m\} \approx 0$, $a = 1, \ldots, A$.

These are vectors V^m in the kernel of $\{\phi_j, \phi_m\}$

General solution: $u^m = U^m + v^a V_a^m$.

Consistent total Hamiltonian: $H_T = H + U^m \phi_m + v^a \phi_a =: H' + v^a \phi_a$, with $\phi_a = V_a^m \phi_m$.

$\Rightarrow$ We are so far left with A arbitrary functions v^a in the Hamiltonian.

2.2.3 Gauge transformations

The following terminology turns out to be very useful and crucial when studying the geometry of the constraint surface later on.

Definition 9. A phase space function f is called **first class** if it has vanishing Poisson bracket with all constraints, i.e. $\{f, \phi_j\} \approx 0$. Otherwise, it is called **second class**.

Linear combination of first class functions are again first class.

Examples:

- All ϕ_a are primary first class constraints by their definition.
- H_T is first class by the consistency algorithm.
  Because all constraints are preserved in time
- $\Rightarrow H'$ is first class by linearity.

Theorem 1. The Poisson bracket of two first class functions is again first class.

Proof: Exercises.

Influence of the v^a on infinitesimal dynamics: (neglect $\mathcal{O}(\delta t^2)$)

$$f(\delta t) = f_0 + \dot{f}\delta t = f_0 + \{f, H_T\}\,\delta t = \underbrace{f_0 + \{f, H'\}\,\delta t}_{\text{unique}} + \underbrace{v^a\,\{f, \phi_a\}\,\delta t}_{\text{arbitrary}} \tag{2.19}$$

Difference in evolution:

$$\Delta f(\delta t) = \underbrace{\delta t\,(v_1^a - v_2^a)}_{\epsilon^a}\,\{f, \phi_a\} = \epsilon^a\,\{f, \phi_a\} \tag{2.20}$$

$\Rightarrow$ Ambiguity is generated by $\epsilon^a\phi_a$, where ϵ^a arbitrary.

$\Rightarrow$ The ϕ_a generate infinitesimal gauge transformations:

- Change the canonical variables q, p
- Do not change the physical state of the system

The consequences of this last statement will be worked out below. It is true for now by the assumption that we have a consistent and predictive theory.

Do the primary first class constraints exhaust the generators of gauge transformations?

Commutator of two infinitesimal gauge transformations: *Exercise*

$$\Delta f = \epsilon_1^a\epsilon_2^b\,\{f, \{\phi_a, \phi_b\}\} \tag{2.21}$$

$\Rightarrow$ Also Poisson brackets of primary first class constraints generate gauge transformations.

These may be secondary constraints.

Similar argument for transformations generated by H' and ϕ_a.

While these arguments extend the list of gauge generators, we cannot proof that they give all generators.

Dirac's conjecture: All first class constraints generate gauge transformations.

Status of this conjecture is disputed.

- Nontrivial to formulate precisely *(what transformations are gauge on the Lagrangian level?)*
- Counterexamples exist, but are pathological
- Proof exists under simplifying regularity conditions that are generically satisfied *(see Henneaux & Teitelboim)*
- True for the main practical examples

Here (and in most literature): Assume the conjecture to be satisfied.

- No natural distinction between primary and secondary constraints at the Hamiltonian level
- Quantisation algorithms treat primary and secondary constraints on the same footing

There is a canonical distinction between first class and second class constraints due to the Poisson bracket, see next section.

Definition 10. The extended Hamiltonian H_E is given by H' plus an arbitrary combination of first class constraints.

We will take H_E as the generator of our dynamics.

2.2.4 Field theory

Generalisations to an infinite number of degrees of freedom:.

- q^n, $n = 1, 2, \dots$ becomes $q(x)$, $x \in \mathbb{R}^3$
- $\sum_n$ becomes $\int d^3x$
- $\frac{\partial L}{\partial \dot{q}^n} = p_n$ becomes $\frac{\delta L}{\delta \dot{q}(x)} = p(x)$
  where $L = \int d^3x \mathcal{L}(x)$ and $p(x)$ is defined as $\delta_{\dot{q}} L = \int d^3x \, p(x) \delta\dot{q}(x)$

  Usually, the variational derivative can be used like a standard derivative of $\mathcal{L}(x)$ w.r.t. $\dot{q}(x)$. This stops working as soon as additional, e.g. spatial, derivatives act inside $\mathcal{L}(x)$.

<u>Example:</u> $\mathcal{L}(x) = \frac{1}{2}\dot{q}(x)^2 - \frac{1}{2}q(x)^2$

- $p(x) = \frac{\delta L}{\delta \dot{q}(x)} = \dot{q}(x)$, because $\delta_{\dot{q}} L = \int d^3x \, \dot{q}(x) \delta\dot{q}(x)$
- $H = \int d^3x \left(p(x)\dot{q}(x) - \mathcal{L}\right) = \int d^3x \left(\frac{1}{2}p(x)^2 + \frac{1}{2}q(x)^2\right)$

2.2.5 Example: Maxwell theory = $U(1)$ gauge theory

- Variables: gauge potential $A_\mu(x)$, $\mu = 0,1,2,3 = t,x,y,z$.
- Field strength: $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$
- Raise and lower indices with $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) = \eta^{\mu\nu}$
- Lagrangian: $L = -\frac{1}{4}\int d^3x\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\int d^3x\, F_{\mu\nu}F_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}$

Observations: two sources of gauge invariance *Which ones?*

- No time derivative of $A_0(x)$
- Lagrangian invariant under $A_\mu(x) \mapsto A_\mu(x) + \partial_\mu\lambda(x)$

Legendre transform:

- Canonical momenta

$$\delta_{\dot{A}}L = -\frac{1}{2}\int d^3x\, F^{\mu\nu}\delta_{\dot{A}}F_{\mu\nu} = -\int d^3x\, F^{0\nu}\delta_{\dot{A}}\partial_0 A_\nu = \int d^3x\, F^{\nu 0}\delta_{\dot{A}}\partial_0 A_\nu \tag{2.22}$$

$$=: \int d^3x E^\mu(x)\delta\dot{A}_\mu(x) \tag{2.23}$$

$\Rightarrow E^\mu(x) = F^{\mu 0}(x)$

$F^{00} = 0$ due to antisymmetry $\Rightarrow E^0(x) \approx 0$ primary constraint.

Other components $E^a = F^{a0}$, $a = 1,2,3$: electric field.

$B^a := \frac{1}{2}\epsilon^{abc}F_{bc}$: magnetic field.

- Poisson brackets $\{A_\mu(x), E^\nu(y)\} = \delta_\mu^\nu\delta^{(3)}(x,y)$
- Hamiltonian (suppressing x-dependence)

$$H = \int d^3x \left(E^\mu\dot{A}_\mu - \mathcal{L}\right) \tag{2.24}$$

$$= \int d^3x \left(F^{a0}\partial_0 A_a + \frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}F_{a0}F^{a0}\right) \tag{2.25}$$

$$= \int d^3x \left(\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}F_{a0}F^{a0} + F^{a0}\partial_a A_0\right) \tag{2.26}$$

$$= \int d^3x \left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}E^a E_a - A_0\partial_a E^a\right) \tag{2.27}$$

Boundary term has been dropped in last step.

Hamiltonian involves only coordinates and momenta, no velocities.

- Stability algorithm: $H_T = H + \int d^3x\, u(x) E^0(x)$

  $\dot{E}^0(x) = \big\{E^0(x), H_T\big\} = \partial_a E^a(x)$

  Does not involve $u(x) \Rightarrow$ new secondary constraint $G(x) := \partial_a E^a(x) \approx 0$ ("Gauß law")

  $\dot{G}(x) = \{G(x), H_T\} = \big\{G(x), \int d^3x\, \frac{1}{4} F_{ab} F^{ab}\big\}$

  $\rightarrow$ Spatial derivatives, need smearing function (but neglect boundary terms):

$$\left\{\int d^3x\, \lambda(x) G(x),\, \frac{1}{4}\int d^3y\, F_{ab}(y) F^{ab}(y)\right\} \tag{2.28}$$

$$= \int d^3x\, d^3y\, \{(\partial_a \lambda(x))\, E^a(x), A_c(y)\}\, \partial_b F^{bc}(y) \tag{2.29}$$

$$= -\int d^3x\, d^3y\, (\partial_a \lambda(x))\, \delta^a_c \delta^{(3)}(x,y) \partial_b F^{bc}(y) \tag{2.30}$$

$$= -\int d^3x\, (\partial_c \lambda(x))\, \partial_b F^{bc}(x) \tag{2.31}$$

$$= \int d^3x\, \lambda(x) \partial_c \partial_b F^{bc}(x) = 0 \tag{2.32}$$

  $\Rightarrow$ Constraint stable, algorithm terminates.

- Extended Hamiltonian $H_E = \int d^3x \left(\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} E^a E_a + \lambda G + \mu E^0\right), \quad \lambda, \mu$ arbitrary
- Infinitesimal gauge transformations:
  - $\big\{E^a(x), \int d^3y\, \lambda(y) G(y)\big\} = 0$
  - $\big\{E^0(x), \int d^3y\, \lambda(y) G(y)\big\} = 0$
  - $\big\{A_a(x), \int d^3y\, \lambda(y) G(y)\big\} = -\partial_a \lambda(x)$
  - $\big\{A_0(x), \int d^3y\, \lambda(y) G(y)\big\} = 0$

  $\Rightarrow A_0, E^0, E^a$ invariant, $A_a \mapsto A_a - \partial_a \lambda$

  - $\big\{A_0(x), \int d^3y\, \mu(y) E^0(y)\big\} = \mu(x)$
  - others are zero

  $\Rightarrow A_a, E^0, E^a$ invariant, $A_0 \mapsto A_0 + \mu$

  Invariant functions (observables): $E^a, F_{ab} = \epsilon_{abc} B^c \Leftrightarrow$ electric + magnetic field

  F_{ab} contains (locally) all gauge invariant information of A_a.
  Counterexamples can be constructed e.g. when the spacetime is not simply connected. Then, consider closed non-contractable field lines.

  E^0 also gauge invariant, but vanishes on constraint surface.

  A_0 takes arbitrary values under gauge transformations.

- Physical degrees of freedom (DOF):

  E^a has to satisfy $\partial_a E^a = 0$. $\Rightarrow$ 2 phase space DOF.

  A_a can be arbitrarily shifted by $\partial_a \lambda$. $\Rightarrow$ 2 phase space DOF.

  In total: 2+2 phase space DOF = 2 configuration space DOF (position + velocity).

- Gauge degrees of freedom:

  A_0 and E^0 do not fulfil any physical purpose.
  A_0 is arbitrary and E^0 is zero.

  Demand that also $A_0 = 0$ throughout the evolution, i.e. impose constraint $A_0 \approx 0$.

  Stability algorithm: $\Rightarrow \mu = 0$.

  Discard A_0 and E^0 from theory, as they don't appear in H_E and are consistently zero.

  More complicated for Gauß law, but similarly possible in principle.

  This process is known as **gauge fixing**.

  We note that

  1. $\{A_0(x), E^0(y)\} = \delta(x, y)$
     $\rightarrow$ Gauge generator and gauge fixing condition are second class pairs.
     the generator sets one variable to zero, while it generates arbitrary changes in the other one.

  2. Original theory with gauge freedom and gauge fixed theory are equivalent.
     Given a gauge fixed theory, it has to be possible to construct a "gauge-unfixed" theory with additional gauge invariance.
     Reverse process must be possible: gauge unfixing

  These concepts will now be formalised by studying the geometry of the constraint surface.

Legendre transform for Constrained systems

- Not all velocities can be solved for the momenta, leading to constraints
- Stability of constraints under evolution may lead to further constraints
- First class constraints generate gauge transformations

2.3 The geometry of the constraint surface

2.3.1 Regularity conditions

Many equivalent ways to define a constraint, e.g. $p_1 = 0 \Leftrightarrow p_1^2 = 0 \Leftrightarrow \sqrt{|p_1|} = 0$.

Why are two of the above constraints ill-suited for the Hamiltonian formalism?

Some regularity assumptions needed:

- For simplicity: assume constraints to be linearly independent
  Otherwise, one can usually pick locally an independent subset
- The constraints can be taken as the first J coordinates in a regular coordinate system in the vicinity of the constraint surface
- The variations $\delta\phi_j = \frac{\partial\phi_j}{\partial q^i}\delta q^i + \frac{\partial\phi_j}{\partial p_i}\delta p_i$ are non-vanishing, well defined, and locally linearly independent on the constraint surface (excludes $p_1^2 = 0$ and $\sqrt{|p_1|} = 0$)
  (here, locally = everywhere, e.g. with arbitrary smearing functions)
- We assume these conditions to be valid globally

With these restrictions in mind, we continue our investigation.

2.3.2 First and second class split

Recall:

- first class constraints $\leftrightarrow$ gauge transformations
- second class pairs $\leftrightarrow$ transformation generator + gauge fixing

$\rightarrow$ Need to separate the constraints in first and second class.

Is this always possible? Is this unique in some sense?

Define the matrix $C_{ij} = \{\phi_i, \phi_j\}$.

Assume $\mathrm{rank}(C_{ij})$ constant on the constraint surface *as another regularity condition.*

Theorem 2. If $\det C_{ij} \approx 0$, then there exists at least one first class constraint among the ϕ_i.

Proof: If $\det C_{ij} \approx 0$, then there exits $\lambda^i \neq 0$ such that $\lambda^i C_{ij} \approx 0\ \forall j$.
Then, $\lambda^i\phi_i$ is first class. $\square$

Now, redefine the constraints as $\phi'_i = A_i{}^j\phi_j$ so that $\phi'_1 = \lambda^i\phi_i$.

$\Rightarrow C'_{1i} = -C'_{i1} \approx 0$.

$$C'_{ij} \approx \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C'_{22} & \dots & C'_{2J} \\ 0 & \dots & \dots & \dots \\ 0 & C'_{J2} & \dots & C'_{JJ} \end{pmatrix}_{ij} \tag{2.33}$$

Rename C' by C and reiterate this procedure until $\det C_{ij} \not\approx 0$.

$\Rightarrow$ Split into first class constraints γ_a and second class constraints χ_α.

$$C_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & C_{\alpha\beta} \end{pmatrix} \tag{2.34}$$

$C_{\alpha\beta}$ antisymmetric $\Rightarrow$ Number of second class constraints is even.

Determinant of antisymmetric matrix of odd dimension vanishes.

Because: $\det C = \det C^T = \det -C = (-1)^n \det C$

Note that this is not true any more for fermions due to Graßmann numbers.

The above split is not unique. Invariant under

$$\gamma_a \mapsto A_a{}^b \gamma_b, \qquad \chi_\alpha \mapsto A_\alpha{}^\beta \chi_\beta + A_\alpha{}^a \gamma_a \tag{2.35}$$

for $\det A_a{}^b \neq 0$ and $\det A_\alpha{}^\beta \neq 0$

Also, one can add squares of second class constraints to first class constraints.

In the following, we assume $\det C_{\alpha\beta} \neq 0$ everywhere on $\chi_\alpha = 0$ (without necessarily having $\gamma_a = 0$) as a technical condition.

2.3.3 Small excursion: quantisation

The following will be made more precise later in section 6.

Quantisation: maps phase space functions to linear operators on a Hilbert space, so that

- $[\hat{f}, \hat{g}] := \hat{f}\hat{g} - \hat{g}\hat{f} = i\hbar\widehat{\{f, g\}}$

Works only for a limited set up phase space functions. (Groenewold-van Hove theorem)

In general: ordering ambiguities.

Elements of the Hilbert space are "kets": $|\psi\rangle$

Constraints: $\phi_i \approx 0 \mapsto \hat{\phi}_i |\psi\rangle \stackrel{!}{=} 0$

Physical states: $\hat{\phi}_i |\psi\rangle_{\text{phys}} = 0 \quad \Leftrightarrow \quad e^{i\lambda^j \hat{\phi}_j} |\psi\rangle_{\text{phys}} = |\psi\rangle_{\text{phys}}$

Is this consistent?

- Assume $\hat{\phi}_i |\psi\rangle_{\text{phys}} = 0$
- $\Rightarrow \hat{\phi}_i \hat{\phi}_j |\psi\rangle_{\text{phys}} = 0$
- $\Rightarrow \left(\hat{\phi}_i \hat{\phi}_j - \hat{\phi}_j \hat{\phi}_i\right) |\psi\rangle_{\text{phys}} = 0$

- $\Rightarrow \widehat{\{\phi_i, \phi_j\}} \left|\psi\right\rangle_{\text{phys}} = 0$ *(up to ordering problems)*

Two cases:

- Only first class constraints: $\{\phi_i, \phi_j\} = {c_{ij}}^k \phi_k$, $\Rightarrow \widehat{{c_{ij}}^k \phi_k} \left|\psi\right\rangle_{\text{phys}} = 0$
  Consistent (up to ordering)
- Second class constraints present: $\exists\, i, j : \{\phi_i, \phi_j\} \neq {c_{ij}}^k \phi_k \quad \Rightarrow$ "$\mathbb{1}$" $\left|\psi\right\rangle_{\text{phys}} = 0$
  Inconsistent

Two options to proceed:

- Change quantisation prescription for second class constraints (later)
- Get rid of second class constraints classically (now)

There is no general rule of path is best to follow. Solving constraints classically can be very hard in practise. Quantising constraints is ambiguous. Therefore, both options should be explored.

2.3.4 The Dirac bracket

The action of second class constraints doesn't preserve the constraint surface.

Simply because they don't Poisson-commute with some of the constraints. E.g. choose the constraints as local coordinates off the constraint surface.

$\Rightarrow$ they cannot be treated as gauge generators.

$\Rightarrow$ develop a strategy for solving them classically.

Consider the following example: *from Dirac's book, similar to the Maxwell example*

- Configuration space is $\mathbb{R}^n$, coordinates $q^1, \dots q^n$.
- Canonical momenta $p_1, \dots, p_n$.
- Second class constraints $\chi_1 = q^1 \approx 0, \quad \chi_2 = p_1 \approx 0$

q^1 and p_1 are not of importance, we would like to simply set them to zero and thus solve the constraints.

However, $\{q^1, p_1\} = 1 \neq 0$, $\rightarrow$ inconsistent with Poisson bracket.

Need to modify the Poisson bracket after solving constraints:

$$\{f, g\}_* = \sum_{i=2}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right) \tag{2.36}$$

New bracket $\{\cdot, \cdot\}_*$ is consistent with strongly setting $\chi_1 = \chi_2 = 0$ and still satisfies all properties of $\{\cdot, \cdot\}$.

$\rightarrow$ Need to generalise this idea!

Guiding principles:

- $\{\chi_\alpha, \cdot\}_* = 0$ strongly.
- Preserve all properties of the Poisson bracket (bi-linearity, ...)
  This is particularly important, because those properties are reflected by commutators upon quantisation
- Modification should depend only on the bracket arguments and the second class constraints

Solution: **Dirac bracket**

$$\{f, g\}_* = \{f, g\} - \{f, \chi_\alpha\} C^{\alpha\beta} \{\chi_\beta, g\} \tag{2.37}$$

where $C^{\alpha\beta} C_{\beta\gamma} = \delta^\alpha_\gamma$.

Properties of the Dirac bracket:

- Antisymmetry: $\{f, g\}_* = -\{g, f\}_*$
- Linearity: for $c_1, c_2 \in \mathbb{R}$: $\{c_1 f_2 + c_2 f_2, g\}_* = c_1 \{f_1, g\}_* + c_2 \{f_2, g\}_*$
- Leibniz property: $\{f_1 f_2, g\}_* = f_1 \{f_2, g\}_* + \{f_1, g\}_* f_2$
- Jacobi identity: $\{f, \{g, h\}_*\}_* + \{g, \{h, f\}_*\}_* + \{h, \{f, g\}_*\}_* = 0$
- Second class compatibility: $\{\chi_\alpha, \cdot\}_* = 0$ strongly
- First class compatibility: $\{f, \cdot\}_* \approx \{f, \cdot\}$ for any first class f

Proof: Exercises.

No changes in formalism:

- H_E still generates the dynamics, as it is first class
- First class constraints still generate gauge transformations
- Solving second class constraints is consistent with the Dirac bracket
  E.g. solving constraints by using reduced set of phase space coordinates so that $\chi_\alpha = 0$.
- First class constraints cannot be set to zero even with Dirac bracket

The Dirac bracket is weakly unaffected by choosing a different (but equivalent) set of second class constraints. (Exercises)

2.3.5 Gauge fixing

We may want to get rid of the gauge degrees of freedom and work only with second class constraints / Dirac bracket.

Given gauge generators γ_a:

$\rightarrow$ introduce gauge conditions $C_b(q, p) \approx 0$

$C_b(q, p) \approx 0$ restricts the allowed part of phase space

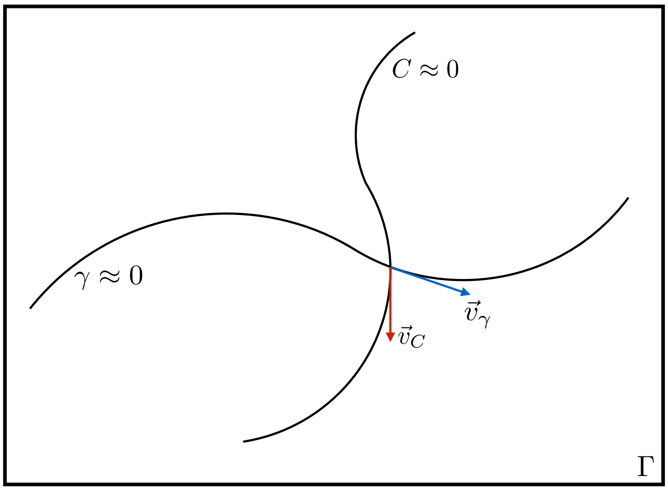


Figure 2.2: The two constraint surfaces $\gamma \approx 0$ and $C \approx 0$ intersect non-tangentially. The vector fields $\vec{v}_\gamma$ and $\vec{v}_C$ prescribe a flow along their respective constraint surfaces. $\vec{v}_\gamma \nparallel \vec{v}_C$ at the intersection, which is equivalent to $\{\gamma, C\} \not\approx 0$. Gauge fixing $C \approx 0$ thus selects a representative of the equivalence class of points on $\gamma \approx 0$ under the flow generated by $\vec{v}_\gamma$.

Necessary properties:

- Accessibility:
  For any given (q, p), there must exist a gauge transformation $q \mapsto q',\ p \mapsto p'$, such that $C_b(q', p') \approx 0$.

- Completeness:
  The gauge is fixed completely, i.e. no more gauge transformations are possible.
  Infinitesimally, $\delta u^a \{C_b, \gamma_a\} \approx 0 \ \Rightarrow\ \delta u^a = 0$, or $\det\{C_b, \gamma_a\} \not\approx 0$.

After a complete gauge fixing, no first class constraints are left.

Geometric interpretation: figure 2.2

Completeness is globally non-trivial in general: Gribov copies, figure 2.3.

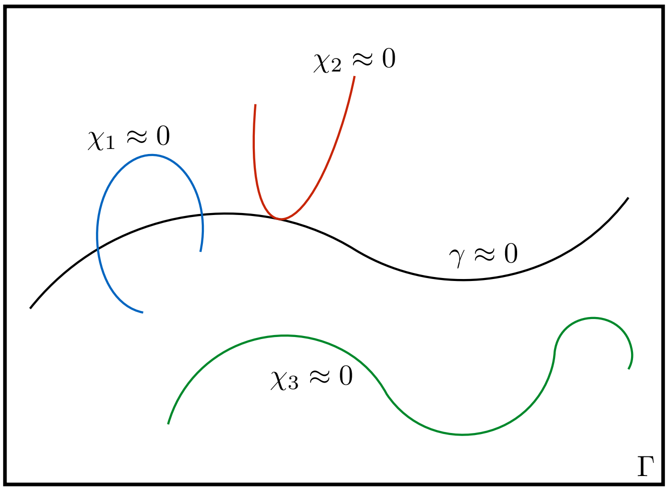


Figure 2.3: Three obstacles to a good gauge fixing for the constraint $\gamma \approx 0$ are shown. χ_1 intersects the constraint surface twice, the gauge fixing is not unique (there is a Gribov copy). χ_2 intersects the constraint surface degenerately, $\{\gamma, \chi_2\} = 0$. $\chi_3 \approx 0$ does not intersect $\gamma \approx 0$, the gauge is not accessible.

Accessibility not always given in all of phase space: Gribov obstruction

E.g. no good global gauge conditions are known in general relativity.

Geometry of the constraint surface:

- Second class constraints correspond to fixed gauges
- Solving second class constraints requires the Dirac bracket for consistency

2.3.6 Degrees of freedom

Before discussing first class constraints, we look at the degrees of freedom for guidance.

Physical DOF = All DOF - Gauge DOF

Only second class constraints:

- Solving one constraint eliminates 1 DOF

$\Rightarrow$ 1 DOF per second class constraint.

Only first class constraints:

- Solving one constraint eliminates 1 DOF
- Additionally, physical observables Poisson-commute with first class constraints

$\Rightarrow$ 2 DOF per first class constraint.

Consistent with gauge fixing: 1 first class constraint = 2 second class constraints (gauge generator and fixing)

DOF counting so simple only for finite dimensional systems.

For field theories, need to discuss the functional spaces of the Lagrange multipliers. Case by case study with physical input.

2.3.7 Gauge invariant functions

Recall phase space functions $C^\infty(\Gamma)$:

- Algebra $\mathcal{A}$ with addition "$+ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$" and multiplication "$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$"
- Lie Algebra with Lie bracket (= Poisson bracket) $\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$
- Operations related via $\{fg, h\}_* = \{f, h\}_* \, g + f \, \{g, h\}_*$

System constrained to be on constraint surface Σ.

$\Rightarrow$ phase space functions that agree on Σ cannot be distinguished.

This means that the relevant functions are only those on Σ and we should develop a formalism that refers only to such functions.

We want to study $C^\infty(\Sigma)$.

$\mathcal{N}$ = functions vanishing on Σ.

- $\mathcal{N}$ is an ideal in $C^\infty(\Gamma)$: $f \cdot g \in \mathcal{N} \; \forall \; f \in \mathcal{N}, \; g \in C^\infty(\Gamma)$
- $\mathcal{N} = \lambda^a \gamma_a + \lambda^\alpha \chi_\alpha$

Define quotient algebra $C^\infty(\Gamma)/\mathcal{N}$
= equivalence class of phase space functions differing by an element of $\mathcal{N}$

$C^\infty(\Gamma)/\mathcal{N} = C^\infty(\Sigma)$ with addition "+" and multiplication "$\cdot$"

Any function on Σ defines an equivalence class. Conversely, every equivalence class defines a function on Σ.

Ideal property of $\mathcal{N}$ is needed:

$$(f_1 + \lambda_1^a \gamma_a + \lambda_1^\alpha \chi_\alpha) \cdot (f_2 + \lambda_2^a \gamma_a + \lambda_2^\alpha \chi_\alpha) = \left(f_1 f_2 + \underbrace{(\lambda_1^a \gamma_a + \lambda_1^\alpha \chi_\alpha) \, f_2 + f_1 \, (\lambda_2^a \gamma_a + \lambda_2^\alpha \chi_\alpha) + \ldots}_{\stackrel{!}{=} \lambda_3^a \gamma_a + \lambda_3^\alpha \chi_\alpha} \right) \tag{2.38}$$

Otherwise, the product would depend on the choice of representative of the equivalence class.

Note that we didn't show so far that the Lie bracket extends to $C^\infty(\Sigma)$!

Definition 11. An **observable** F is a function on the constraint surface $C^\infty(\Sigma)$ that Poisson-commutes weakly with all the first class constraints:

$$\{F, \gamma_a\}_* \approx 0. \tag{2.39}$$

F does not depend on the representative, as $\{\gamma_a + \chi_\alpha, \gamma_b\}_* \approx 0$.

Two steps:

1. Restrict to constraint surface Σ
2. Gauge invariance condition w.r.t. Dirac bracket

We do not explain how the measurement process is supposed to take place. While this is classically often clear, it becomes a problem at the quantum level. Determining observables of the theory in our formalism was possible purely starting from the action principle.

Alternative characterisation: Well defined bracket structure in $C^\infty(\Sigma)$

- Addition and multiplication well defined
- Bracket well defined if only second class constraints (Dirac Bracket)
- With first class constraints:

$$\{f + \lambda^a \gamma_a, g\}_* = \{f, g\}_* + \{\lambda^a \gamma_a, g\}_* \overset{!}{\approx} \{f, g\}_* \tag{2.40}$$

  $\Rightarrow$ g has to Poisson commute weakly with the first class constraints. *Similar for f*

$\Rightarrow$ Bracket on $C^\infty(\Sigma)$ well defined only for observables!

$\Rightarrow$ Necessary to consider all first class constraints (primary and secondary) as gauge generators!

The well defined bracket structure is mandatory for quantisation.

2.3.8 Gauge unfixing

Gauge fixing suggests that second class systems can also be viewed as first class systems.

How to construct a first class system from a second class one?

Example:

- Phase space coordinates: $(q^1, q^2, p_1, p_2) \in \mathbb{R}^4$
- $\{q^i, p_j\} = \delta^i_j$
- Second class constraints $\chi_1 = q^2 \approx 0$, $\chi_2 = p_2 \approx 0$
- First class Hamiltonian $H = H(q^1, q^2, p_1, p_2) = \sum_{i,j,k,l=0}^{\infty} c_{ijkl} (q^1)^i (q^2)^j (p_1)^k (p_2)^l$
  Taylor expansion in all four variables.
  Note that $c_{i1k0} = 0$ and $c_{i0k1} = 0$ due to first class property.

Physical content: Only q^1, p_1 interesting.

Solve with Dirac bracket:

- Dirac bracket: $\left\{q^1, p_1\right\}_* = 1$
- Solve constraints explicitly: $q^2 = 0 = p_2$
- Hamiltonian: $H = \sum_{i,k=0}^{\infty} c_{i0k0}(q^1)^i (p_1)^k$

Transform to first class system now.

- Drop one constraint
- Call the other one the gauge generator
- Use original Poisson bracket

Observation: Not unique. Keep e.g. $q^2 \approx 0,\ p_2 \approx 0,\ q^2 + p_2 \approx 0, \ldots$

As an example, keep p_2 and drop q^2.

$\Rightarrow H$ not gauge invariant in general

$$\{H, p_2\} = \sum_{i,j,k,l=0}^{\infty} c_{ijkl}\,(q^1)^i j\,(q^2)^{j-1}(p_1)^k(p_2)^l \tag{2.41}$$

Need first class Hamiltonian $\tilde{H}$ that agrees with H upon setting $q^2 = 0$

$\Rightarrow \tilde{H} = \sum_{i,k,l=0}^{\infty} c_{i0kl}(q^1)^i(p_1)^k(p_2)^l$

We can also add arbitrary powers of $p_2 \approx 0$.

We removed all powers of q^2 from H. This was trivial here, because we could simply Taylor expand H in the constraints. For more complicated constraints, this is more involved.

Physical observables: q^1, p_1 both Poisson-commute with p_2.

Evolution:

$$\begin{aligned}
&\left\{f(q^1,p_1), \tilde{H}\right\} \\
= \quad &\left\{f(q^1,p_1), \sum_{i,k=0}^{\infty} c_{i0k0}(q^1)^i(p_1)^k\right\} + \sum_{i,k=0}^{\infty}\sum_{l=1}^{\infty} c_{i0kl}(q^1)^i(p_1)^k \underbrace{\left\{f(q^1,p_1),(p_2)^l\right\}}_{0} + \mathcal{O}(p_2) \\
\approx \quad &\left\{f(q^1,p_1), H\right\}_*
\end{aligned} \tag{2.42}$$

Physical evolution of observables invariant.

We can redefine the Hamiltonian by adding arbitrary powers of p_2, in particular remove all powers of p_2 from it. Gives H on surface $q^2 = 0 = p_2$.

Conclusion from example:

- Dropping gauge fixing conditions generally leads to second class Hamiltonians
- Second class property comes from powers of the gauge fixing condition inside H
- Need to remove these powers by adding powers of the gauge fixing conditions
- By going back to the gauge $q^2 = 0$, we recover the original Hamiltonian theory
  We do not necessarily recover the exact first class Hamiltonian that we started from, since we are free to add arbitrary powers of first class constraints to it, which in this case means we can add powers of at least 2 *of second class constraints. But we recover the same physics.*
- This is simple if we have a Taylor expansion of H in terms of the gauge fixing condition, but this is usually not the case.

$\rightarrow$ Formalise this idea by only using the available structure (Poisson bracket)

Simplify notation, $q^2 \mapsto q$, $p_2 \mapsto p$.

Heuristic idea:

- The remaining first class constraint p generates changes in the gauge fixing q, here because $\{q, p\} = 1$
- We do not want H to depend on q
- Flow evaluation point along the gauge orbit of p to $q = 0$

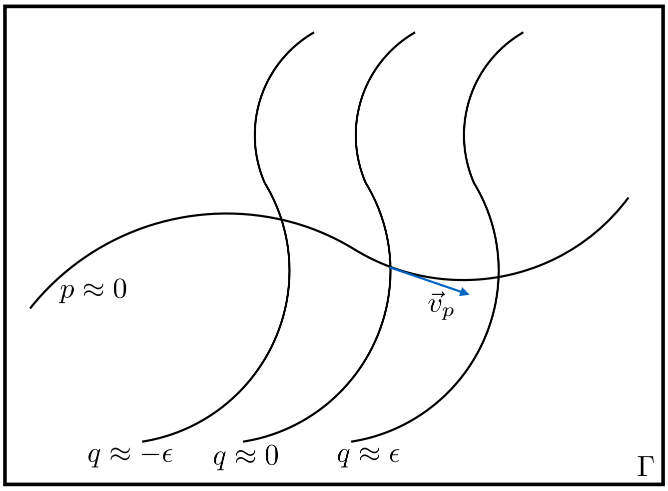


Figure 2.4: The gauge unfixing projector will evaluate a function at $q = 0$ by moving the evaluation point along $\vec{v}_p$ until it satisfies $q = 0$. The reason why $\mathbb{P}f$ Poisson-commutes with p is then simply that changes in q don't matter for the evaluation of the phase space function, as we always flow to $q = 0$.

$$\mathbb{P}H = e^{-\{\cdot,p\}q} H = \sum_{n=0}^{\infty} \frac{(-q)^n}{n!} \{H, p\}_{(n)} \tag{2.43}$$

For any phase space function: **gauge unfixing projector**

$$\mathbb{P} = e^{-\{\cdot,p\}q} = \sum_{n=0}^{\infty} \frac{(-q)^n}{n!} \{\cdot, p\}_{(n)} \tag{2.44}$$

$\{\cdot,\gamma\}_{(n+1)} := \big\{\{\cdot,\gamma\}_{(n)},\gamma\big\}, \quad \{\cdot,\gamma\}_{(0)} := \cdot.$

Applying $\mathbb{P}$ takes us to $q = 0$ along the gauge orbit of p.

Therefore, $\mathbb{P}f$ has to Poisson-commute with p. If we would have flown not along the Hamiltonian vector field of p, this would not have been true.

$\mathbb{P}$ computes a gauge invariant extension. Two constraints needed to define $\mathbb{P}$.

Example: if one thinks of p being the Hamiltonian and q the time (here then also a phase space variable, see later in parametrised systems), then we always evaluate a given phase space function at a given time $t = 0$. In this case, $\mathbb{P}$ would map any phase space function to its initial values at $t = 0$.

Check that $\mathbb{P}$ deletes powers of the gauge condition:

- Gauge condition q, gauge generator p, c_n phase space functions independent of q.
- $\{q, p\}_{(1)} = 1$
- $\{q^k, p\}_{(1)} = kq^{k-1}$
- $\{q^k, p\}_{(n)} = k(k-1)\dots(k-n+1)q^{k-n} = \frac{k!}{(k-n)!}q^{k-n}$

$$\begin{aligned}
\mathbb{P}\sum_{k=0}^{\infty} c_k q^k &= \sum_{k=0}^{\infty} c_k q^k - \sum_{k=0}^{\infty} \frac{1}{1!} k c_k q^{k-1} q + \sum_{k=0}^{\infty} \frac{1}{2!} k(k-1) c_k q^{k-2} q^2 \pm \dots & (2.45)\\
&= \sum_{k=0}^{\infty} c_k q^k \sum_{n=0}^{k} \frac{(-1)^n k!}{(k-n)! n!} & (2.46)\\
&= \sum_{k=0}^{\infty} c_k q^k \sum_{n=0}^{k} (-1)^n \binom{k}{n} = \sum_{k=0}^{\infty} c_k q^k \delta_{k,0} & (2.47)\\
&= c_0\, q^0 = c_0 & (2.48)
\end{aligned}$$

For general second class constraints: $C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}$

Pick first class subset γ_a (half number) + other half χ_b

$$\begin{aligned}
\mathbb{P} = e^{-\{\cdot, C^{ab}\gamma_b\}\chi_a} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Big\{\dots\Big\{\cdot, C^{a_1 b_1}\gamma_{b_1}\Big\}, \dots, C^{a_n b_n}\gamma_{b_n}\Big\} \chi_{a_1}\dots\chi_{a_n} & (2.49)\\
&\overset{\gamma_a}{\approx} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \{\dots\{\cdot, \gamma_{b_1}\}, \dots, \gamma_{b_n}\} C^{a_1 b_1}\chi_{a_1}\dots C^{a_n b_n}\chi_{a_n} & (2.50)
\end{aligned}$$

In practise, gauge unfixing is useful only if the series terminates or can be summed.

This happens e.g. in the connection formulation of general relativity, useful for quantum gravity.

Question: when does this not happen? How do the gauge constraints have to look like?

E.g. when the gauge generator includes several powers of the canonical variables.

Some properties of $\mathbb{P}$ (locally):

- $(\mathbb{P}f) + (\mathbb{P}g) = \mathbb{P}(f+g)$
- $\{\mathbb{P}f, \mathbb{P}g\} \overset{\gamma_a}{\approx} \{f, g\}_{*(\gamma_a,\chi_b)}$ *Formalism equivalent to Dirac bracket*
- $(\mathbb{P}f) \cdot (\mathbb{P}g) \overset{\gamma_a}{\approx} \mathbb{P}(f \cdot g)$
  One can see that $\mathbb{P}$ is consistent with multiplication only up to constraints only when taking the projector w.r.t. to several first class constraints. Then, the order in which gauge transformations are applied is important, but one can show that the Hamiltonian vector fields of the constraints $C^{ab}\gamma_b$ <u>weakly</u> commute. See theorem 2.2.1 in Thiemann's book.
- $\mathbb{P}(\lambda^a\gamma_a + \mu^b\chi_b) \overset{\gamma_a}{\approx} 0$
- $\mathbb{P}$ generates all γ_a-observables *(Take a γ_a-observable $\mathcal{O}_\gamma$, then $\mathbb{P}\mathcal{O}_\gamma = \mathcal{O}_\gamma$)*

These properties are important for reduced phase space quantisations. In particular, the first identity allows one to compute the Poisson bracket of observables, which one needs to find representations of classical observables.

One may choose the clarifying notation $\mathbb{P}^\chi_\gamma$.

Remark: Batalin-Fradkin-Tyutin-formalism is an alternative to gauge unfixing, but introduces new DOF.

Gauge fixing / unfixing:

- One can rewrite first class systems (partially) as second class systems, and the other way around
- Physically, this corresponds to fixing gauge conditions, or lifting them
- While these descriptions are equivalent classically, they may have different values as starting points for a quantisation

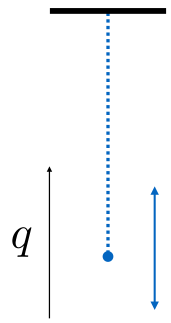
3 Time Reparametrisation Invariant Systems

Generally covariant systems are a fundamental technical tool to account for the fact that choices of coordinates, which are merely a tool for convenient descriptions of a phenomenon, should be of no physical relevance.

Goal of this section: Study time-reparametrisation invariant systems.

Next two sections: Generally covariant systems (e.g. general relativity)

Gedankenexperiment: Measure position of a harmonic oscillator.



Question: How is this done? Which input is needed? Is the above question well defined by itself?

- Question not well defined. We need to specify a measurement time
- What is time?
- Some physical significance of time coordinate, e.g. clock.
- Any clock is a physical object
  Already Newton was unhappy with his definition of absolute time and aware that one would need some relational notion to supersede it.

Main idea:

- Clock should be modelled in our theoretical description: $t(\tau)$
- τ arbitrary temporal parameter
- Describe "correlations" clock $t(\tau)$ - position $q(\tau)$
- Physics invariant under (monotonic) relabelings $\tau \mapsto f(\tau)$

3.1 Parametrised systems

Consider example system:

- canonical variables q^i, p_i
- Hamiltonian H_0

Action:

$$S[q^i(t), p_i(t)] = \int_{t_1}^{t_2} dt \left(p_i \frac{dq^i}{dt} - H_0 \right) \tag{3.1}$$

Introduce time variable $q^0 := t$, conjugate momentum p_0.

Search for equivalent action where time is a variable:

$$S[q^0(\tau), p_0(\tau), q^i(\tau), p_i(\tau)] = \int_{\tau_1}^{\tau_2} d\tau \left(p_0 \frac{dq^0}{d\tau} + p_i \frac{dq^i}{d\tau} - u^0 \left(p_0 + H_0\right) \right) \tag{3.2}$$

Equations of motion for new variables:
Variations of p and q independently:

$$\begin{aligned} \frac{\delta S}{\delta p_0} &= \quad \frac{dq^0}{d\tau} - u^0 = 0 && (3.3) \\ \frac{\delta S}{\delta q^0} &= \quad -\frac{d}{d\tau} p_0 = 0 && (3.4) \\ \frac{\delta S}{\delta u^0} &= - \left(p_0 + H_0\right) = 0 && (3.5) \end{aligned}$$

(2): p_0 constant of motion
(3): $p_0 = -H_0$ *The only constant of motion (without solving the EOM) in general time-independent systems*
(1): u^0 measures change of clock q^0 w.r.t. τ

Insert into action: *Possible due to exercise, consider p_0, u^0 as auxiliary fields, but not q^0*

$$\begin{aligned} S &= \int_{\tau_1}^{\tau_2} d\tau \left(p_0 \frac{dq^0}{d\tau} + p_i \frac{dq^i}{d\tau} - u^0 \left(p_0 + H_0 \right) \right) && (3.6) \\ &= \int_{\tau_1}^{\tau_2} d\tau \left(p_i \frac{dq^i}{d\tau} - \frac{dq^0}{d\tau} H_0 \right) && (3.7) \\ &= \int_{t_1}^{t_2} dt \left(p_i \frac{dq^i}{dt} - H_0 \right) && (3.8) \end{aligned}$$

The new action is therefore equivalent to the original one.

$u^0 \left(p_0 + H_0\right)$ in action leads to constraint: $\gamma = p_0 + H_0 \approx 0$:

- No $\frac{du^0}{d\tau}$ in action $\Rightarrow p_u \approx 0$ primary constraint
- Hamiltonian: $u^0 \left(p_0 + H_0\right) =: u^0 \gamma$
- Stability: $\frac{dp_u}{d\tau} = -\left(p_0 + H_0\right) \overset{!}{\approx} 0 \Rightarrow \gamma \approx 0$

$\Rightarrow$ The Hamiltonian vanishes on the constraint surface

- H is sum of constraints *Here only 1*
- H is not strongly zero! $\{\cdot, H\} \not\approx 0$ in general

Interpretation:

- Time evolution = gauge transformation
- Flow generated by $p_0 + H_0(q^i, p_i)$:
  - H_0 evolves q^i, p_i in the usual way
  - p_0 evolves the time q^0
  - $p_0 = -H_0$ stays constant

  $\Rightarrow q^i, p_i$ and q^0 evolve (advance in coordinate time τ) simultaneously!

  This means that the clock "ticks" when the other canonical variable evolve.

- Changing u^0 changes evolution speed of $q^i(\tau), p_i(\tau), q^0(\tau)$ similarly.
  $\Rightarrow$ Correlations $q^i(q^0), p_i(q^0)$ independent of u^0

  This means that it does not matter how fast we proceed in the evolution: Changing speed (u^0) just changes how fast we sample all correlations, but eventually we will sample all of them.

How to recover the usual observables, e.g. $q(t_0)$?

Simple model: free particle, $H_0 = \frac{p^2}{2m}, \quad \gamma = p_0 + H_0 \approx 0$

- Need constant of motion equal to $q(t_0)$ *How could this be done?*
- $q_{t_0}(\tau) := q(\tau) - \frac{p(\tau)}{m}\left(q^0(\tau) - t_0\right)$
- $\left\{q_{t_0}, p_0 + \frac{p^2}{2m}\right\} = -\frac{p(\tau)}{m} + \frac{p(\tau)}{m} = 0$

$\Rightarrow q_{t_0}$ constant of motion that agrees with $q(t_0)$

Remarks on constructing observables:

- Construction principle:
  Evolve q^0, q^i, p_i in "time" τ until $q^0(\tau) = t_0$
- Requires solving the equations of motion (EOM)
  This succeeded here because we chose a very simple system
- Solutions to EOM $\Leftrightarrow$ initial data at some time t_0 $\Leftrightarrow$ Constants of motion in the (q^0, p_0, q^i, p_i)-system
  Note that "constants of motion" here are not only those that one considers in classical mechanics, e.g. the energy or angular momentum, which can be found by looking at the symmetries of the system. Here, we need to explicitly solve the EOM to obtain these constants of motion.

It is always possible to parametrise a given Hamiltonian system:

1. Add canonical pair q^0, p_0 and Lagrange multiplier u^0

2. Replace extended Hamiltonian: $H_E \mapsto u^0\,(p_0 + H_E)$
3. Add constraint $p_0 + H_E \approx 0$
4. Keep all other constraints
   The first class constraints appearing in H_E are still first class constraints here, their multipliers are just multiplied by u^0, which doesn't change anything.

This can bring explicitly time dependent systems in time-independent form.

However, "deparametrising" is not straight forward and may even be impossible to achieve globally (e.g. general relativity).

$\Rightarrow$ Important to develop a formalism with gauge invariance

Once one restricts to a part of phase space where a certain gauge condition used for deparametrisation is accessible, it is possible to compute the Poisson brackets of physical observables (the constants of motion) by using properties of the gauge unfixing projector, without solving the equations of motion. This allows one to construct reduced phase space quantisations of these subsectors of the theory.

3.2 General examples

Example: Free relativistic particle:

- Action: $S[X^\mu(\tau)] = -m\int_w ds = -m\int_{\tau_1}^{\tau_2} d\tau\,\sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}$
- w world line of particle, Embedding map $X^\mu(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))^\mu$,
  $\eta = \mathrm{diag}(-1,1,1,1)$, τ arbitrary temporal parameter
  The action is the proper length of the world line.
  Lower and raise indices with η
- Canonical momenta: $p_\mu := \frac{dL}{d\left(\frac{dX^\mu}{d\tau}\right)} = \frac{m}{\sqrt{\cdots}}\frac{dX_\mu}{d\tau}$
- Proper time s: $ds = \sqrt{\ldots}d\tau \quad\Rightarrow\quad \frac{d}{ds} = \frac{1}{\sqrt{\cdots}}\frac{d}{d\tau} \quad\Rightarrow\quad p_\mu = m\frac{dX_\mu}{ds} = mu_\mu$ (u_μ = four velocity)
- Constraint: $p_\mu p^\mu = -m^2 \;\Rightarrow\; \gamma := p_\mu p^\mu + m^2 \approx 0$ (mass shell condition)
  Movement of the particle is in the temporal direction, follows from sign choice in the action.
- Hamiltonian: $H = p_\mu\frac{dX^\mu}{d\tau} - L = \frac{\sqrt{\cdots}}{m}\left(p_\mu p^\mu + m^2\right)$
- Extended Hamiltonian: $H_E = \lambda\gamma \approx 0$, λ Lagrange multiplier
- Canonical equations of motion:
  - $\frac{d}{d\tau}p_\mu = 0$ (four velocity is constant)

- $\frac{d}{d\tau}X^\mu = 2\lambda\, p^\mu$ (for $\lambda = \frac{1}{2m}$: $\frac{dX_\mu}{ds} = \frac{1}{m}p_\mu$)

- Independent Dirac observables:
  - p_i (p_0 from $\gamma \approx 0$)
  - $X^i - p^i \frac{X^0 - t_0}{p^0}$ (X^0 gauge DOF, shifted by γ)

  Note: Other parametrisation, e.g. w.r.t. X^1 instead of X^0 possible.

  $\Rightarrow$ *Many physically equivalent choices in formulating Dirac observables.*

This example (worldline) can be generalised to higher dimensions:

- World-surface: classical strings (exercises)
- World-volumes: branes
- Vary metric: general relativity (with different action)

Example: Homogeneous Lagrangian: $\mathcal{L}(q^i, c\,\dot{q}^i) = c\,\mathcal{L}(q^i, \dot{q}^i)$ (exercises)

Reparametrisation invariant systems:

- Any Hamiltonian system can be written as a reparametrisation invariant system
- The Hamiltonian is a sum of constraints
  (if no time-dependent canonical transformations)
- Time evolution = gauge transformation
- Physical statements are correlations between evolving objects

4 Crash course in General Relativity

How do we measure distance?

- Experiment: Compare to a given ruler (mètre des archives, Urmeter)
- Theoretical description: assign length to coordinate units
  E.g. distance $= (x_2 - x_1)$ implicitly refers to regular units of x

Generalisation: units of x may be irregular and subject to change over time.

Physical picture: the spacetime on which physics takes place is dynamical

To describe this, we need a few concepts:

- Manifold (arena where physics takes place)
- Metric (assignment of distance between points)
- Geodesics (what are straight lines, c.f. Newton's axions)
- Curvature (tensors derived from metric)
- Integration theory (for well-defined actions, i.e. coordinate independent)

In the following: Crash course on those subjects, only relevant details, no mathematical rigour.

4.1 Manifolds

The main point of defining manifolds for us is to allow for more general spaces than that of $\mathbb{R}^n$*, possibly with globally non-trivial topologies.*

We do not discuss issues like local topology here, in the sense of defining continuity. Usually, one would start from a topological manifold and work ones way up.

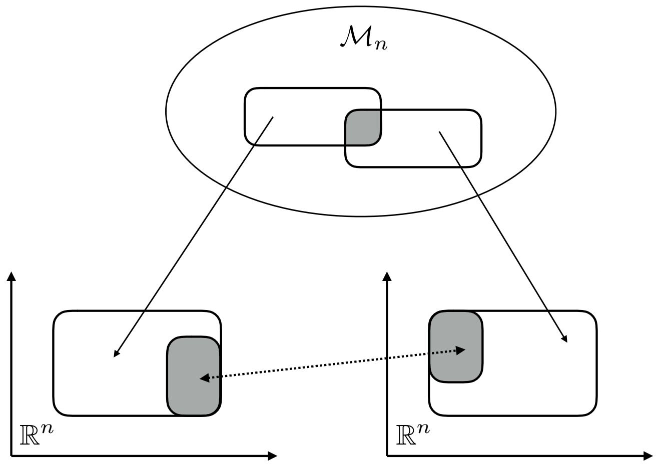
Properties of an n-dimensional (differentiable) manifold $\mathcal{M}_n$

- A space that locally looks like $\mathbb{R}^n$
- There exists a collection of invertible maps (charts), from subsets of $\mathcal{M}_n$ to subsets of $\mathbb{R}^n$. A collection of maps covering all of $\mathcal{M}_n$ is called atlas.
  All points in the manifold need to be included in the charts.
  $\rightarrow$ provides local coordinates
- Maps are consistent with each other on overlaps and sufficiently smooth
  $\rightarrow$ change of coordinates well-defined
  $\rightarrow$ transfers differential calculus from $\mathbb{R}^n$ to $\mathcal{M}_n$

Example: 2-Sphere

- We need at least 2 charts *(e.g. northern / southern hemisphere)*

Further examples: 2-Torus, handle-body, ...



Definition 12. A diffeomorphism is a bijective map from one manifold to another (or itself), where both the map and its inverse are sufficiently often differentiable.

- A diffeomorphism induces a change of coordinates
  Moves coordinates from one point to another along its inverse
- Later: diffeomorphism invariance = invariance under general coordinate transformations

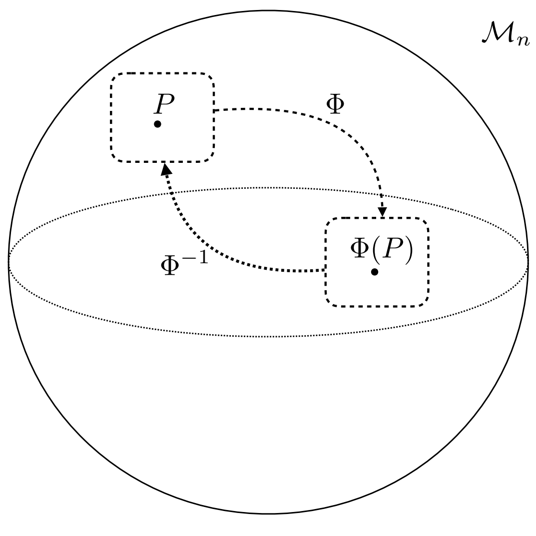


Figure 4.1: A diffeomorphism induces a change of coordinates: we can assign to P the coordinates of $\Phi(P)$ or equivalently move the coordinates in a neighbourhood of $\Phi(P)$ to a neighbourhood of P using Φ^{-1}. In local coordinates: $\Phi^\alpha(x^i) = y^\alpha(x^i)$.

Bottom line:

- We can consistently use coordinates in spaces with general topologies
  These will usually be simple spaces like $\mathbb{R}^n$ or spheres.

- We can transfer differential calculus from $\mathbb{R}^n$ to those spaces

4.2 Vectors and covectors

4.2.1 Vectors

Define what we mean by a vector on a manifold. Important for some later concepts.

Natural objects on manifold: curves

Idea:

1. Define vectors as tangent vectors to curves.
   $\rightarrow$ Need to differentiate the curve

2. Evaluate change of functions along the curves

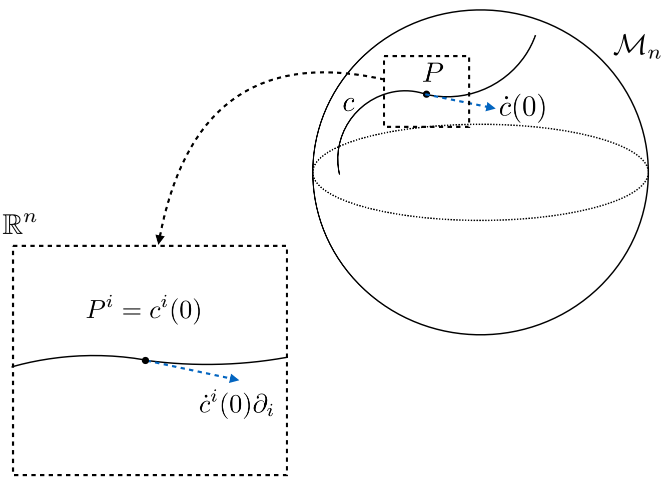
Consider on our manifold: *drop the n-index of the manifold*

- function $f : \mathcal{M} \rightarrow \mathbb{R}$.

- curve $c : [-\epsilon, \epsilon] \rightarrow \mathcal{M}, \quad c(0) = P$

Define

$$\frac{d}{dt} \underbrace{f(c(t))}_{f \text{ on } \mathcal{M}} \Big|_{t=0} = \frac{d}{dt} \underbrace{f(c^i(t))}_{f \text{ in local coordinates}} \Big|_{t=0} = \underbrace{\dot{c}^i(t=0)}_{\frac{d}{dt} c^i(t)\big|_{t=0}} (\frac{\partial}{\partial x^i} f)(c(t=0)) =: \dot{c}^i \, \partial_i f(t=0) =: \dot{c}(f)(t=0) \tag{4.1}$$



Remarks:

- Definition independent of coordinates *Refers only to the curve*

- Change of coordinates $y^\alpha = y^\alpha(x^i)$

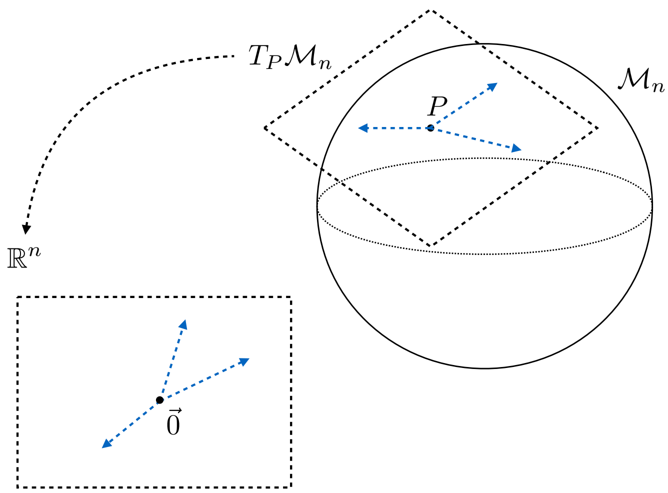
$$\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha} \quad \Leftrightarrow \quad \frac{\partial}{\partial y^\alpha} = \frac{\partial x^i}{\partial y^\alpha}\frac{\partial}{\partial x^i}$$

$$\Rightarrow \quad \dot{c}^\alpha = \frac{\partial y^\alpha}{\partial x^i}\dot{c}^i \quad \Leftrightarrow \quad \dot{c}^i = \frac{\partial x^i}{\partial y^\alpha}\dot{c}^\alpha$$

  Transformation behaviour of a vector (upper index)

- $\dot{c}^i$: components of a vector, $\quad \partial_i$: basis vectors

Space of all vectors at a point P in $\mathcal{M}$: tangent space $T_P\mathcal{M}$



Collection of all tangent spaces: tangential bundle $T\mathcal{M}$

Vector field: maps every point $P \in \mathcal{M}$ to an element of $T_P\mathcal{M}$.
$\rightarrow$ assigns vector to every point

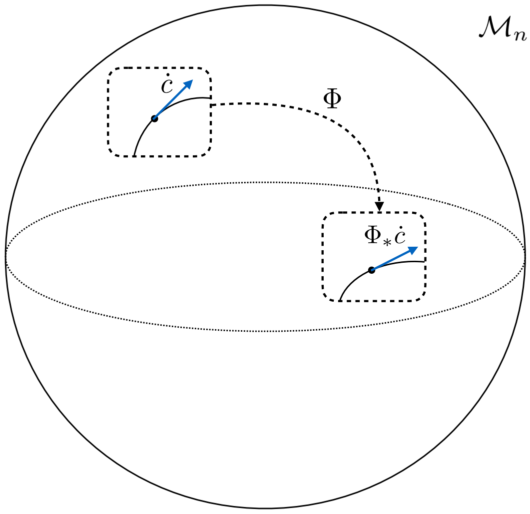
Example: Hamiltonian vector field

Manifold = phase space $\mathbb{R}^{2n}$

$v_H(f) := \{f, H\} = \left(\frac{\partial H}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p^i}\right) f$

Attention: no canonical identification of neighbouring tangential spaces!
Important for derivatives.

We can "push forward" a vector along a diffeomorphism:



We move the curve c with Φ to a new curve $\Phi(c)$ and use this curve to define a vector at $\Phi(c(0))$.

$$(\Phi_* \dot{c})\,(f)(\Phi(P)) := \tfrac{d}{dt} f(\Phi(c(t)))\big|_{t=0} = \underbrace{\frac{\partial \Phi(x)^\alpha}{\partial x^i}(P)\dot{c}^i(t=0)}_{(\Phi_*\dot{c})^\alpha(\Phi(P))}(\tfrac{\partial}{\partial y^\alpha} f)(\Phi(P))$$

We can push forward along general maps, not only diffeomorphisms.

4.2.2 Covectors

Idea: Covector = linear map from vectors to $\mathbb{R}$.

Cotangent space at P: $T_P^*\mathcal{M}$

- Dual basis: dx^i
- $dx^i(\partial_j) := \partial_j x^i = \delta^i_j$
- Expansion in basis: $w = w_i dx^i$
- $w(v) = w_i dx^i(v^j \partial_j) = v^j w_i \delta^i_j = v^i w_i$
- Coordinate change: $w_\alpha = w_i \frac{\partial x^i}{\partial y^\alpha}$ (exercise)

Collection of all cotangent spaces: cotangent bundle $T^*\mathcal{M}$

Covectorfield analogous.

Evaluation $w(v)$ independent of the choice of coordinates. (Exercise)

Attention: No canonical identification of vectors and covectors! Needs additional structure.

We can "pull back" a covector along a diffeomorphism: *Also along general maps*

We simply push the vector it takes as an argument forward. This means that the co-vector is defined at $\Phi(P)$ and the vector at P, thus we pull back the co-vector from $\Phi(P)$ to P.

$(\Phi^* w)(v)(P) := w(\Phi_* v)(\Phi(P))$

$(\Phi^* w)(v)(P) = (\Phi^* w)_i(P)dx^i(v^i(P)\partial_i) = w_\alpha(\Phi(P))dy^\alpha \left((\Phi_* v)^\beta (\Phi(P))\partial_\beta\right) = w_\alpha(\Phi(P))v^i(P)\frac{\partial\Phi^\alpha}{\partial x^i}(P)$

$\Rightarrow w_i(P) = w_\alpha(\Phi(P))\frac{\partial\Phi^\alpha}{\partial x^i}(P)$

Diffeomorphisms are bijective:

- Push forward covectors along Φ = Pull back covectors along Φ^{-1}
  e.g. $(\Phi_* w)_\alpha(\Phi(P)) = w_i(P)\frac{\partial(\Phi^{-1})^i}{\partial y^\alpha}(\Phi(P))$ (exercise)
- Pull back vectors along Φ = Push forward vectors along Φ^{-1} (exercise)

In general: involved maps not bijective *Then no such thing as pushing forward a form if the map used is not invertible.*

Pullbacks and pushforwards are compatible with index contraction (exercise).

4.3 Metrics and tensors

We need an assignment of distance to a curve in a manifold.

Infinitesimal line element:

- Euclidean space = $\mathbb{R}^3$ with standard metric:

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = \sum_{i,j=1}^{3} \delta_{ij} dx^i dx^j \tag{4.2}$$

- Generalisation: x^i are local coordinates on $\mathcal{M}_n$

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j, \quad g_{ij}(x) = \text{metric tensor (symmetric)} \tag{4.3}$$

Tensor: something that transforms like a tensor (see the following).

The indices of a tensor transform like the indices of vectors / co-vectors.

The infinitesimal distance ds^2 should be coordinate independent:

- Consider change of coordinates $x^i = x^i(y^\alpha)$
- $dx^i = dx^i(y) = \frac{dx^i}{dy^\alpha}dy^\alpha$
- $\Rightarrow\ g_{ij}(x)dx^i dx^j = \underbrace{g_{ij}(x(y))\frac{dx^i}{dy^\alpha}\frac{dx^j}{dy^\beta}}_{g_{\alpha\beta}(y)} dy^\alpha dy^\beta =: g_{\alpha\beta}(y)dy^\alpha dy^\beta$

Rules for tensors:

- Tensorial objects with a lower (covariant) index transform as $T_\alpha(y) = \frac{dx^i}{dy^\alpha} T_i(x(y))$
- Tensorial objects with an upper (contravariant) index transform as $T^\alpha(y) = \frac{dy^\alpha}{dx^i} T^i(x(y))$

Note:

- Same index structure on both sides of the equality
  The index structure automatically fixes how a tensor index transforms.
- Summed always over upper / lower indices
- $\partial_i = \frac{\partial}{\partial x^i}$ behaves as with a lower index
  Note however that an object with the components $1/T^i$ *is not a tensor, in particular not a covariant one.*
- Transformations from coordinate changes cancel in summations:
  $T^\alpha{}_\alpha(y(x)) = \frac{dy^\alpha}{dx^i} \frac{dx^j}{dy^\alpha} T^i{}_j(x) = \delta_i^j T^i{}_j(x) = T^i{}_i(x)$
- Multiple indices transform as $T^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n} = \frac{dy^{\alpha_1}}{dx^{i_1}} \cdots \frac{dy^{\alpha_m}}{dx^{i_m}} \frac{dx^{j_1}}{dy^{\beta_1}} \cdots \frac{dx^{j_n}}{dy^{\beta_n}} T^{i_1 \dots i_m}{}_{j_1 \dots j_n}$
- Important: coordinates $x^i, y^\alpha, \dots$ are <u>not</u> tensors. Here, $i, \alpha, \dots$ label the different coordinates.
- But the differentials dx^i, dy^α transform as tensors with upper index

Notation:

- The metric tensor transforms as a rank 2 covariant tensor.
- Upper indices: contravariant
- $T^{i_1 \dots i_m}{}_{j_1 \dots j_n}$ is a tensor of rank (m, n)
- A scalar, e.g. $T^i{}_i$, does not change under coordinate transformations

Inverse metric: g^{ij} such that $g_{ij} g^{jk} = \delta_i^k$ *rank 2 contravariant tensor*

Raise and lower indices with the metric:

- $T^i g_{ij} =: T_j$ transforms with a lower index
- $U_i g^{ij} =: U^j$ transforms with an upper index

How to obtain metrics?

1. Prescribe them from scratch
2. Induce a metric by embedding $\mathcal{M}_n$ into $\mathbb{R}^{n+m}$ with Euclidean metric

Embedding example: 2-sphere.

- Define subset S^2 as the subset of $\mathbb{R}^3$ satisfying $(x^1)^2 + (x^2)^2 + (x^3)^2 = R^2$

- Examples of local coordinates:
  - Cartesian coordinates (x^1, x^2, x^3) on subset $\mathbb{S}^2 \subset \mathbb{R}^3$: metric known
  - Spherical coordinates θ, ϕ: metric via change of coordinates
    $x^1 = R\sin\theta\cos\phi$
    $x^2 = R\sin\theta\sin\phi$
    $x^3 = R\cos\theta$
    Compute metric: $g_{\theta\theta} = \frac{dx^i}{dy^\theta}\frac{dx^i}{dy^\theta}\delta_{ij} = R^2, \quad g_{\phi\phi} = R^2\sin^2\theta, \quad g_{\theta\phi} = 0$

Define distance $d(c)$ along a curve $c : [a, b] \to \mathcal{M}_n$:

$$d(c) = \int_c ds := \int_c \sqrt{g_{ij}dx^i dx^j} := \int_a^b d\lambda \sqrt{g_{ij}(c(\lambda))\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}} \tag{4.4}$$

λ parametrises the curve.

$d(c)$ is called the "proper distance". *As opposed to coordinate distance*

Two invariances:

- Reparametrisation of λ
- Changes of coordinates $x^i \to y^\alpha$

Example: Length of great circle on a sphere with radius R.

- Great circle: $\phi \in [0, 2\pi), \theta = \pi/2$. Take $\phi = \lambda$ to parametrise the great circle.

$$\int_0^{2\pi} d\lambda \sqrt{g_{ij}(c(\lambda))\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}} = \int_0^{2\pi} d\phi \sqrt{g_{\phi\phi}} = \int_0^{2\pi} d\phi\, R\sin(\pi/2) = 2\pi R \tag{4.5}$$

- Other parameterisations possible, e.g. take twice $\phi = \text{const}$, $\theta \in [0, \pi)$ as exercise

Pull backs and push forwards along diffeomorphisms can be generalised to tensors:

$$(\Phi_* T)^{\alpha_1 \ldots \alpha_m}(\Phi(P)) = \frac{\partial \Phi(x)^{\alpha_1}}{\partial x^{i_1}}(P) \ldots \frac{\partial \Phi(x)^{\alpha_m}}{\partial x^{i_m}}(P)\, T^{i_1 \ldots i_m}(P) \tag{4.6}$$

Similar for covariant indices.

So far: Riemannian metric, positive definite.

For general relativity: Pseudo-Riemannian metric: only non-degenerate.

E.g. Minkowski metric $\eta = \text{diag}(-1, 1, 1, 1)$

Use greek indices $\mu, \nu, \ldots$ instead of $i, j, \ldots$ to remember this.

4.4 Geodesics

How do test particles move on a Riemannian manifold without exterior forces?

In flat space: straight line, constant distance per time

- Straight w.r.t. cartesian coordinates
- Not "straight" w.r.t. general coordinates

Invariant generalisation: shortest path = "geodesic"
$\rightarrow$ Need equations to compute shortest path

Extremise length functional $d(c)$ w.r.t. c for given endpoints of the curve c.

Assume that the curve parameter λ is "affine", i.e. measures proper distance.

- $\sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} = \text{const}$
- Curved space analogue of constant velocity magnitude = speed
- Call curve parameter now s to remember this!

$\delta d(c) = 0$, after a long calculation: *(see wikipedia: "Geodesics in general relativity")*

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \qquad \Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk} \right) \tag{4.7}$$

Γ^i_{jk}: Christoffel symbols. They are not tensors! *See next section*

In flat space and cartesian coordinates, this reduces to vanishing acceleration.

Newtonian mechanics without external forces on curved manifolds:

- Particles move along geodesics in space
- Constant speed $\frac{ds}{dt}$ w.r.t. absolute time $t \sim \lambda$

General relativity without external forces (later)

- Particles more along geodesics in spacetime
  There then is no question of how fast one traverses the geodesic. If one traverses it faster, one is at a later time, i.e. also the observer is.

NB: The geodesic equation is more complicated in a non-affine parametrisation.
For example w.r.t. a coordinate, e.g. the time.

4.5 Integration

We need to define integration over Riemannian manifolds in order to define an action principle for the metric and in order to construct invariant quantities.

Substitution rule for multidimensional integration over region R in the x-coordinate space:

$$\int_R d^n x = \int_{y(R)} \left| \det \left(\frac{\partial x^i}{\partial y^\alpha} \right) \right| d^n y \tag{4.8}$$

$\left| \det \left(\frac{\partial x^i}{\partial y^\alpha} \right) \right|$ *is called the (absolute value of the) Jacobian (determinant).*

Multi-dimensional generalisation of $dx = \frac{\partial x}{\partial y} dy$.

Infinitesimal shifts in the new variables y^α *define a parallelepiped in the* x^i*-coordinate space whose volume is accounted for by the Jacobi determinant. In other words, it follows from* $d^n x \propto \epsilon_{i_1 \dots i_n} dx^{i_1} \dots dx^{i_n}$.

We want a coordinate-invariant integral:

$$\int_R \dots (x) d^n x \stackrel{!}{=} \int_{y(R)} \dots (y) d^n y \tag{4.9}$$

Otherwise, one would always have to specify the coordinate system in which the integral is to be performed.

This integral should somehow also include the metric, as we would like an integral over the whole space to give its proper volume.

Requirements:

- Should only depend on a region in $\mathcal{M}$
- Should reproduce the proper volume of $\mathcal{M}$ if the unit function is integrated

An object built from the metric with no indices is its determinant.

Transformation of the metric determinant:

$$\det(g_{ij}) \to \det \left(\frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} g_{ij} \right) = \det \left(\frac{\partial x^i}{\partial y^\alpha} \right)^2 \det(g_{ij}) \tag{4.10}$$

Invariant integral for some scalar function f: *change of coordinates with positive determinant*

$$\int f(x) \sqrt{\det(g_{ij})(x)} d^n x \tag{4.11}$$

$$= \int f(x(y)) \sqrt{\det(g_{ij})(x(y))} \left| \det \left(\frac{\partial x^i}{\partial y^\alpha} \right) \right| d^n y \tag{4.12}$$

$$= \int f(y) \sqrt{\det(g_{\alpha\beta})(y)} d^n y \tag{4.13}$$

Integration is over the same regions in the manifold.

The integral does not depend on the choice of coordinates.

Note that the absolute value here is necessary, since we don't specify the orientation of the region over which we integrate. In 1 dimension, one can keep track of this orientation e.g. by flipping the integration boundaries in the substitution $x \mapsto y = -x$, in which case we should not use the absolute value, i.e. $dx = \frac{dx}{dy} dy$.

4.6 Covariant derivatives

For action principle, need derivatives $\partial_\mu := \frac{\partial}{\partial x^\mu}$ of the metric and other tensors.

Coordinate change:

$$\partial_\mu T^\nu \to \frac{dx^{\mu'}}{dy^\mu} \partial_{\mu'} \frac{dy^\nu}{dx^{\nu'}} T^{\nu'} \neq \frac{dx^{\mu'}}{dy^\mu} \frac{dy^\nu}{dx^{\nu'}} \partial_{\mu'} T^{\nu'} \tag{4.14}$$

Need to compensate the $\partial_{\mu'} \frac{dy^\nu}{dx^{\nu'}}$ term.

Usual way in physics: Covariant derivative $D_\mu T^\nu := \partial_\mu T^\nu + A^\nu_{\mu\rho} T^\rho$

Example: Electromagnetism with charged scalar particles

- Scalar particle wave functions $\Psi(x)$ transform as $\Psi(x) \to e^{i\phi(x)}\Psi(x)$ under U(1) gauge transformation
- $\partial_\mu \Psi(x) \to e^{i\phi(x)} \partial_\mu \Psi(x) + e^{i\phi(x)} \Psi(x) i \partial_\mu \phi(x)$
- Introduce gauge potential A_μ, so that $A_\mu(x) \to A_\mu(x) - \partial_\mu \phi(x)$
  Those are the gauge transformations generated by the Gauß law in the previous chapter.
- $D_\mu \Psi(x) := (\partial_\mu + i A_\mu) \Psi(x) \to e^{i\phi(x)} D_\mu \Psi(x)$

Now back to connections on tangential bundle:

Connection $A^\nu_{\mu\rho}$ transforms as $A^\nu_{\mu\rho} \to \underbrace{\frac{dx^{\mu'}}{dy^\mu} \frac{dx^{\rho'}}{dy^\rho} \frac{dy^\nu}{dx^{\nu'}} A^{\nu'}_{\mu'\rho'}}_{\text{tensorial piece}} - \underbrace{\frac{dx^{\mu'}}{dy^\mu} \frac{dx^{\nu'}}{dy^\rho} \partial_{\mu'} \frac{dy^\nu}{dx^{\nu'}}}_{\text{non-tensorial piece}}$

Neither $\partial_\mu T^\nu$ nor $A^\nu_{\mu\rho} T^\nu$ transform as a tensor, but $D_\mu T^\nu$ does.

Generalise: Require that D_μ reduces to ∂_μ scalars and Leibniz property:
that is the usual rules for derivatives

- $D_\mu T_\nu = \partial_\mu T_\nu - A^\rho_{\mu\nu} T_\rho$
- $D_\mu T = \partial_\mu T$ (T is a scalar, no indices)
- $D_\mu T^\rho{}_\nu = \partial_\mu T^\rho{}_\nu + A^\rho_{\mu\sigma} T^\sigma{}_\nu - A^\sigma_{\mu\nu} T^\rho{}_\sigma$
- $D_\mu \left(S^{\cdots}{}_{\ldots} T^{\cdots}{}_{\ldots} \right) = S^{\cdots}{}_{\ldots} D_\mu \left(T^{\cdots}{}_{\ldots} \right) + \left(D_\mu S^{\cdots}{}_{\ldots} \right) T^{\cdots}{}_{\ldots}$ Leibniz property

General rules for D_μ:

- Act once with partial derivative

- Act on each index separately with connection as above
- Sum all pieces

A vector is parallel transported along a curve $c(\lambda) \Leftrightarrow \dot{c}^\mu(\lambda) D_\mu(c(\lambda)) v^\nu(\lambda) = 0$

- Connects neighbouring tangent spaces
- Provides a notion of constancy of a vector field

A priori, we can define any connection on our manifold.

For GR: $A^\nu_{\mu\rho} = \Gamma^\nu_{\mu\rho}$.

$\nabla_\mu := D_\mu$ with $A^\nu_{\mu\rho} = \Gamma^\nu_{\mu\rho}$.

Properties:

- $\nabla_\mu g_{\nu\sigma} = 0$ (metric compatibility)
  Length of parallely transported vectors remains constant.
- $\Gamma^\nu_{\mu\rho} = \Gamma^\nu_{\rho\mu}$ (torsion freeness)
  Assumption in the standard formulation of GR.
  Equivalent alternative: Teleparallel gravity has torsion, but no curvature. Connection is still metric compatible.
  Einstein-Cartan-Theory encodes torsion as additional DOFs. Connection is still metric compatible.
- An affinely parametrised geodesic curve c with $\dot{c}^\mu = \frac{dc^\mu}{ds}$ satisfies

$$\dot{c}^\mu \nabla_\mu \dot{c}^\nu = 0 \tag{4.15}$$

- A non-affinely parametrised geodesic with $\dot{c}^\mu = \frac{dc^\mu}{d\lambda}$ satisfies

$$\dot{c}^\mu \nabla_\mu \dot{c}^\nu = \alpha(\lambda) \dot{c}^\nu \tag{4.16}$$

4.7 Lie derivatives

Is there another natural derivative for tensors which does not require any new structure?

Covariant derivative:

- Requires connection, may be metric compatible
- Derivative w.r.t. vector at the evaluation point

Lie derivative:

- Does not require any additional structure on top of the differentiable manifold
- Derivative w.r.t. vector field in a neighbourhood of the evaluation point

→ computes the change of a tensor w.r.t. the flow along a vector field

→ infinitesimal diffeomorphism

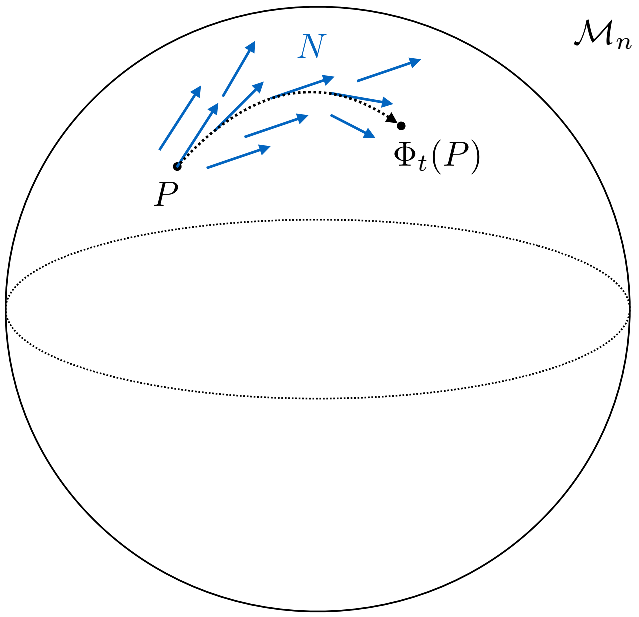


Figure 4.2: An 1-parameter family Φ_t of diffeomorphisms satisfies $\frac{\partial}{\partial t} f(\Phi_t(P))\big|_{t=t_0} = N(\Phi_{t_0}(P))f$, where $N = N^i \partial_i$ is a vector field and f a function, in a neighbourhood of P. $\Phi_0 = \mathrm{id}$. In other words, we take $\Phi_t(P)$ to be the curve $c(t)$ in defining vectors.

Lie derivative of a tensor field T along a vector field N:

$$\left(\mathcal{L}_N T\right)(P) = \frac{d}{dt}\Big((\Phi_{-t})_* T(\Phi_t(P))\Big)\Big|_{t=0} = \frac{d}{dt}\Big(\left((\Phi_t)^* T\right)(P)\Big)\Big|_{t=0} \tag{4.17}$$

Indices of T are suppressed, i.e. contracted with basis vectors.

Note the brackets in the definition. The push-forward is of a tensor at $\Phi_t(P)$, whereas the pullback is evaluated at P.

The pullback of a tensor T is the push-forward along the inverse diffeomorphism, which implies equality of the two definitions follows.

The push forward / pull back allows us to identify the tangent spaces at different points, but depends on chosen diffeomorphism!

Infinitesimally around P: $\left(\Phi_t(x)\right)^i = x^i + tN^i$

$\Rightarrow \frac{\partial(\Phi_t)^i}{\partial x^j} = \delta^i_j + t\frac{\partial N^i}{\partial x^j}$

Here we see that knowledge of the vector field is needed in a neighbourhood of P.

Example:

$$
\begin{aligned}
(\mathcal{L}_N v) &= \frac{d}{dt}\Big((\Phi_{-t})_* v(\Phi_t(P))\Big)\Big|_{t=0} && (4.18)\\
&= \lim_{t\to 0}\frac{\frac{\partial(\Phi_{-t})^i}{\partial x^j}(\Phi_t(P))v^j(P+tN)\partial_i - v^i(P)\partial_i}{t} && (4.19)\\
&= \lim_{t\to 0}\frac{v^i(P+tN)-v^i(P)-tv^j\frac{\partial N^i}{\partial x^j}(\Phi_t(P))}{t}\partial_i && (4.20)\\
&= \left(N^j\partial_j v^i - v^j\partial_j N^i\right)(P)\partial_i && (4.21)
\end{aligned}
$$

Other properties:

- Reduces to partial derivative on scalar functions f: $\mathcal{L}_N f = N^i\partial_i f$
- On covectors: $(\mathcal{L}_N w)_i = N^j\partial_j w_i + w_j\partial_i N^j, \quad \mathcal{L}_N w = \left(N^j\partial_j w_i + w_j\partial_i N^j\right)dx^i$
- Compatible with index contraction
- Leibniz rule: $\mathcal{L}_N(T_1T_2) = T_1\mathcal{L}_N T_2 + (\mathcal{L}_N T_1)\,T_2$
- General tensors: $(\mathcal{L}_N T)^i{}_j = N^k\partial_k T^i{}_j - T^k{}_j\partial_k N^i + T^i{}_k\partial_j N^k$
  - One partial derivative
  - For each index: add one transformation as above
- Lie bracket: $[v_1,v_2](f) := v_1(v_2(f)) - v_2(v_1(f)) = (\mathcal{L}_{v_1}v_2)(f)$

4.8 Riemann tensor

What is curvature? Examine $\mathbb{R}^2$ vs $\mathbb{S}^2$.

Move vector around closed curve:

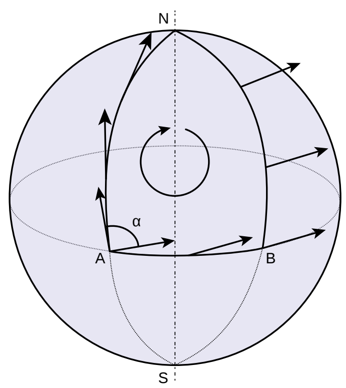


Figure 4.3: Parallel transport of a vector around a closed loop on a sphere reveals a deficit angle α. *Picture by Fred the Oyster, CC-BY-SA 4.0,* `https://commons.wikimedia.org/w/index.php?curid=35124171`.

In case of parallel transport along geodesics, the scalar product (angle) of the transported vector and the curve tangent is constant.

Idea: Transport of vectors around closed curves reveals curvature.

How to compute curvature infinitesimally?

Infinitesimal generator of translation:

- Euclidean space: $\partial_i = \frac{\partial}{\partial x^i}$. $e^{t^i \partial_i} f(x^1, \ldots, x^n) = f(x^1 + t^1, \ldots, x^n + t^n)$
- Curved space: D_μ, ∇_μ

We cannot use the Lie derivative because the concept of curvature has to require more structure than merely a differential manifold. The part on S^2 which we parallel transport around looks like $\mathbb{R}^2$ from the point of view of the manifold.

We compute: $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) T^\rho = R_{\mu\nu}{}^\rho{}_\sigma T^\sigma$

$$R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\gamma_{\sigma\nu}\Gamma^\rho_{\mu\gamma} - \Gamma^\gamma_{\sigma\mu}\Gamma^\rho_{\nu\gamma} \tag{4.22}$$

is called Riemann curvature tensor.
By construction, it transforms as a tensor.

The matrix $M^\rho{}_\sigma := \delta^\rho{}_\sigma + \delta v^\mu \delta w^\nu R_{\mu\nu}{}^\rho{}_\sigma$ computes the parallel transport around an infinitesimal parallelogram spanned by $\delta v^\mu, \delta w^\nu$.

$R_{\mu\nu}{}^\rho{}_\sigma$ depends only on the metric. *More generally, on a connection $A^\rho_{\mu\nu}$.*

Further definitions:

- Ricci curvature tensor: $R_{\mu\nu} = R_\mu{}^\sigma{}_{\nu\sigma} = R_{\mu\rho}{}^{\nu'}{}_\sigma \, g_{\nu\nu'} \, g^{\rho\sigma}$
- Ricci scalar: $R = R^\mu{}_\mu = R_{\mu\nu} g^{\mu\nu}$ *Gauß curvature in 2 dimensions*

4.9 Action and field equations

Aim: field equations form-invariant under arbitrary changes of coordinates
→ tensor equations
→ action principle invariant under the choice of coordinates
We then just vary the action principle w.r.t. tensors to obtain tensor equations

Simplicity assumptions:

- Metric is the only field *(No torsion and non-metricity)*
- At most second time derivatives of the metric *Otherwise phase space enlarged, more initial data*

In 3+1 dimensions ⇒ Einstein-Hilbert action

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-\det g}\,(R + 2\Lambda) \tag{4.23}$$

Plus boundary terms depending on boundary conditions.
Free parameters:

- Newton constant G. Often: $8\pi G = \kappa$ or similar conventions.
- Cosmological constant Λ

NB: In higher dimensions: additional non-trivial Lovelock terms allowed.
Powers of the Riemann tensor, contracted such that only second derivatives appear.

In four dimensions: Variation yields Einstein equations: (exercises)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{4.24}$$

All involved quantities are tensors:

- Equations transform as tensors
- Equations are form-invariant under coordinate changes
  This means that in any coordinate system, they are given by the above tensors
- No coordinates are preferred or have physical meaning

Addition of matter:

$$S = S_{\mathrm{EH}} + S_{\mathrm{matter}} \tag{4.25}$$

Example: Scalar field with potential:

$$S_{\mathrm{scalar}} = -\frac{1}{2} \int d^4x \sqrt{-\det g}\, \left((\nabla_\mu \phi)(\nabla^\mu \phi) + V(\phi) \right) \tag{4.26}$$

Note that the covariant derivative reduces to a partial derivative when action on a scalar such as ϕ.

$\rightarrow$ Need proper definition of matter action on curved spacetime.

Field equations with matter:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{4.27}$$

$$T_{\mu\nu} = \frac{-2}{\sqrt{-\det g}} \frac{\delta S_{\mathrm{matter}}}{\delta g^{\mu\nu}} \quad \text{Energy-Momentum-Tensor} \tag{4.28}$$

Note that $\delta_g S_{matter} = \int d^4x \underbrace{(\ldots)_{\mu\nu}}_{\frac{\delta S_{matter}}{\delta g^{\mu\nu}}} \delta g^{\mu\nu}$

4.10 Physical effects

Very short qualitative sketch about effects appearing in General Relativity. Field equations non-linear and hard to analyse.

Solve field equations and analyse:

- Gravitational attraction can overwhelm all other repulsive forces
  $\rightarrow$ Collapse to a black hole

- Event horizons prohibit information from exiting certain regions
  Attention: Very non-trivial to define "region", because one should not refer to coordinates. In the case of black holes, the geometry is distorted in such a way that saying that no particle can escape the black hole just means that particles cannot escape their own light-cone.

- Generic occurrence of singularities, but typically shielded by horizons
  Cosmic censorship hypothesis (Penrose)
  Note that the cosmic censorship hypothesis excludes the big bang explicitly.

- Cosmological solutions determine the evolution of the universe

- Gravitational waves in the weak field limit

Generic difficulties:

- Non-linearity of the equations, hard to solve

- Avoid coordinate related artefacts

  ◦ coordinate singularities
  ◦ Gauge invariant notions of energy, ...

4.11 Cosmology

In this section, we study what general relativity tells us about the evolution of the universe as a whole. Symmetry assumptions lead to solvable equations.

Observation: universe looks homogeneous and isotropic on large scales.

Idea: attempt to solve Einstein equations under symmetry assumptions.

Choose metric

$$g = \begin{pmatrix} -N^2(t) & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t) & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix} \tag{4.29}$$

Spatially homogeneous, isotropic, and flat, but non-trivial time evolution.

We choose the coordinate volume $\int d^3x$ of σ to be 1. Either:

- σ is compact, e.g. Torus.

- Work in fiducial cell for non-compact σ.

Action, $\Lambda = 0$, up to boundary terms: *(remember that $\kappa = 8\pi G$)*

$$S_{\text{EH}} = \frac{1}{2\kappa} \int d^4x \sqrt{-\det g} R = -\frac{3}{\kappa} \int dt\, \frac{1}{N} \dot{a}^2 a \tag{4.30}$$

Equations of motion w.r.t. N:

$$\frac{\delta S_{\text{EH}}}{\delta N} = \frac{3}{\kappa N^2} \dot{a}^2 a = 0 \tag{4.31}$$

For non-degenerate metric ($a, N \neq 0, \infty$): $\dot{a} = 0$

$\Rightarrow$ Trivial evolution without matter

Add matter content: massless scalar field.

$$\begin{aligned} S = S_{\text{EH}} + S_{m=0} &= \frac{1}{2\kappa}\int d^4x\sqrt{-\det g}R - \frac{1}{2}\int d^4x\sqrt{-\det g}\left((\nabla_\mu\phi)(\nabla^\mu\phi)\right) \\ &= -\frac{3}{\kappa}\int dt\,\frac{1}{N}\dot{a}^2 a + \int dt\,\frac{a^3}{2N}\dot{\phi}^2 \end{aligned} \tag{4.32}$$

Variation w.r.t. N:

$$\frac{\delta S}{\delta N} = \frac{1}{N^2}\left(\frac{3}{\kappa}\dot{a}^2 a - \frac{1}{2}\dot{\phi}^2 a^3\right) = 0 \tag{4.33}$$

Relates change in geometry and change in matter.

EOM for a:

$$\frac{\delta S}{\delta a} = -\frac{3\dot{a}^2}{\kappa N} + \frac{3a^2\dot{\phi}^2}{2N} \tag{4.34}$$

$$= \frac{d}{dt}\frac{\delta S}{\delta \dot{a}} = \frac{d}{dt}\left(-\frac{6\dot{a}a}{\kappa N}\right) = \frac{d}{dt}\,p_a \tag{4.35}$$

EOM for ϕ:

$$\frac{\delta S}{\delta \phi} = 0 = \frac{d}{dt}\frac{\delta S}{\delta \dot{\phi}} = \frac{d}{dt}\left(\frac{a^3\dot{\phi}}{N}\right) := \frac{d}{dt}\,p_\phi \tag{4.36}$$

Hamiltonian formulation:

- $\dot{\phi} = N p_\phi / a^3$
- $\dot{a} = -\frac{N\kappa p_a}{6a}$
- S independent of $\dot{N} \Rightarrow$ primary constraint $p_N \approx 0$
- Hamiltonian:

$$H_T = p_a\dot{a} + p_\phi\dot{\phi} + p_N\dot{N} - \mathcal{L} + \lambda p_N \tag{4.37}$$

$$= -\frac{N\kappa p_a^2}{6a} + \frac{Np_\phi^2}{a^3} + \frac{N\kappa p_a^2}{12a} - \frac{Np_\phi^2}{2a^3} + \tilde{\lambda}p_N \tag{4.38}$$

$$= N\left(-\frac{\kappa p_a^2}{12a} + \frac{p_\phi^2}{2a^3}\right) + \tilde{\lambda}p_N \tag{4.39}$$

- Stability of p_N: $\mathcal{H} := -\frac{\kappa p_a^2}{12a} + \frac{p_\phi^2}{2a^3} \approx 0$
- N functions as Lagrange multiplier, drop p_N
- Poisson brackets: $\{a, p_a\} = 1, \quad \{\phi, p_\phi\} = 1$

For quantum theory (later): change variables

- $v = a^3$

- $b = -\frac{3\dot{a}}{aN} = \kappa \frac{p_a}{2a^2}$
- Set $12\pi G = 1 \Rightarrow \kappa = 8\pi G = 2/3$
- $\{v, b\} = \{a^3, \kappa \frac{p_a}{2a^2}\} = \frac{2}{3}\frac{3}{2} = 1$

Then, $H = N\left(\frac{p_\phi^2}{2v} - \frac{b^2 v}{2}\right) = N\mathcal{H} \approx 0$

Time evolution = gauge transformation

Coordinate time t has no physical significance

Hamiltonian equations of motion:

$$\dot{v} = \{v, H\} = -Nbv \qquad \dot{b} = \{b, H\} = N\frac{p_\phi^2}{2v^2} + \frac{1}{2}Nb^2 \tag{4.40}$$

$$\dot{\phi} = \{\phi, H\} = N\frac{p_\phi}{v} \qquad \dot{p}_\phi = \{p_\phi, H\} = 0. \tag{4.41}$$

Furthermore: $\mathcal{H} = 0 \quad \Leftrightarrow \quad p_\phi^2 = b^2 v^2$

Options:

- Set $N = 1$, then t measures proper time
- Use scalar field as a clock

Here, take second route, but also set $N = 1$
The correlations between ϕ and the geometry are the same, no matter how fast we traverse the spacetime in the temporal direction by choosing N

p_ϕ: constant of motion.

Insert $p_\phi^2 = b^2 v^2$ into $\dot{b}$: $\dot{b} = b^2$, $\Rightarrow b = \frac{-1}{t - t_0}$.

Insert into $\dot{v}$: $\dot{v} = \frac{v}{t - t_0} \quad \Rightarrow \quad v = \pm |p_\phi|(t - t_0)$

Choose $v > 0$ *Corresponds to an arbitrary choice of orientation of the manifold.*

We reach $v = 0$, $b = \infty$ within finite proper time

$v = 0$: "Big Bang" / "Big Crunch" singularity

$R \propto \frac{\dot{a}^2}{a^2} \propto \frac{\dot{v}^2}{v^2} \propto b^2 \to \infty$: curvature singularity

$R \propto \frac{P_\phi^2}{v^2} \propto$ matter energy density

Scalar field. Inserting $v(t)$ into $\dot{\phi}$

$$\dot{\phi} = \pm \mathrm{Sign}(p_\phi)\frac{1}{t - t_0} \quad \Leftrightarrow \quad \phi - \phi_0 = \pm \mathrm{Sign}(p_\phi)\,\mathrm{Log}\,(t - t_0)\,. \tag{4.42}$$

v as a function of ϕ, p_ϕ

$$v(\phi) = |p_\phi| \exp\left(\pm \mathrm{Sign}(p_\phi)(\phi - \phi_0)\right). \tag{4.43}$$

Dirac observable: v at a given "time" $\tilde{\phi}$

$$v|_{\phi=\tilde{\phi}} := v \exp\left(\mp \mathrm{Sign}(p_\phi)(\phi - \tilde{\phi})\right). \tag{4.44}$$

$v|_{\phi=\tilde{\phi}}$: volume v of the universe at scalar field time $\tilde{\phi}$.

Show as exercise that $v|_{\phi=\tilde{\phi}}$ Poisson-commutes with the Hamiltonian constraint.

Other independent Dirac observable: p_ϕ

Physics:

- at some point $\tilde{\phi}$ in scalar field time: fix v and p_ϕ
- b determined via $\mathcal{H} = 0$

Deparametrisation:

Consider ϕ as time.

P_ϕ generates shifts in ϕ.

$\mathcal{H} = 0 \quad \Leftrightarrow \quad p_\phi = \pm\sqrt{b^2 v^2}$

True Hamiltonian: $H_{\text{true}} = \pm bv$ = Generator of ϕ-time translations

$\frac{d}{d\phi} v(\phi) = \{v, H_{\text{true}}\} = \pm v(\phi)$

$\Rightarrow v \propto \exp(\pm\phi)$

$\frac{d}{d\phi} b(\phi) = \{b, H_{\text{true}}\} = \mp b(\phi)$

$\Rightarrow b \propto \exp(\mp\phi)$

$\Rightarrow bv \propto P_\phi = \text{const}$

Spatially flat, homogeneous, isotropic cosmology:

- Hamiltonian is constrained to vanish
- Time parameter has no physical meaning
- Dirac observables are correlations between physical fields

5 Canonical General Relativity

Spacetime-covariant formulation of GR:

- Compute spacetime metric for the spacetime as a whole from manifestly spacetime-covariant equations
  "Manifestly" here means "obviously"
- Extract fields on a three-dimensional "equal time" spatial slice
  by pulling back the relevant quantities to the slice

Canonical formulation of GR:

- Set up a Hamiltonian system on the spatial slice
- Spatial metric + its conjugate momentum as variables
- Evolve to a neighbouring spatial slice using the Hamiltonian

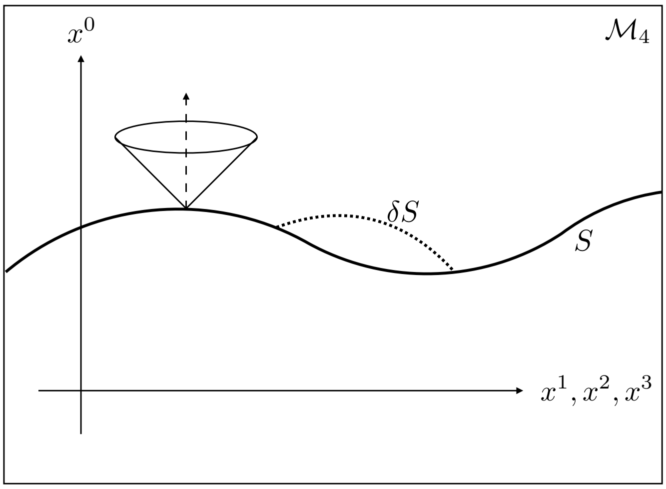


Figure 5.1: An everywhere spacelike surface S within the four-dimensional spacetime is shown. We may make small deformations δS of the surface, resulting in four independent functions in the solutions to the Hamiltonian equations of motion. An example light-cone is shown, indicating the spacelike nature of S.

Causal structure:

- Providing the values of all canonical variables on a spatial slice should determine the future evolution (up to gauge)
- We restrict ourselves to spacetimes which can be foliated by spatial surfaces ("global hyperbolicity")
  Important for initial value problem, no closed timelike curves, ...

Expectations:

- 4 first class constraints per point on S encoding arbitrariness in moving S
- A weakly vanishing Hamiltonian *(the choice of S has to be pure gauge)*

$\Rightarrow$ 3 constraints per point generating spatial diffeomorphisms (no dynamical information)
$\Rightarrow$ 1 constraint per point generating time-like evolution (contains dynamics, not simple diffeomorphism "off-shell")

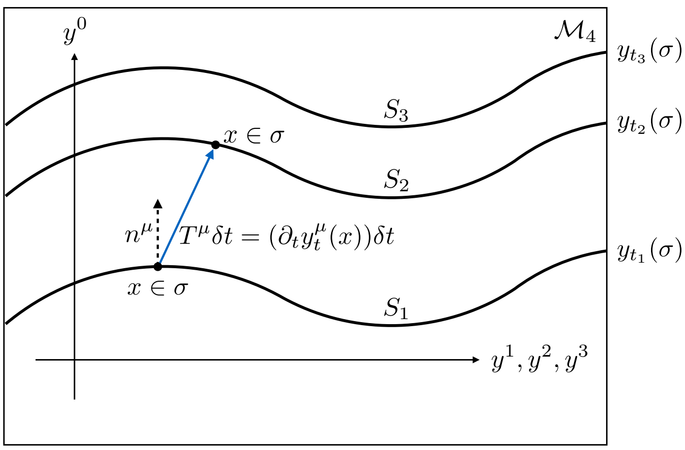
One cannot speak a priori about time-like diffeomorphisms of both canonical variables unless one uses the Einstein equations, as we will see later.

A rough analogy is to consider the harmonic oscillator. There, $\{q, H\} = p$, which can be identified as a time-like diffeomorphism by appealing to the definition of p as $\dot{q}$. However, $\dot{p} = -x$ cannot be identified as diffeomorphism, because there is no time derivative of p within the canonical variables. Rather, one needs to solve the equations of motion and one can then ask whether on shell, $-x$ agrees with $\dot{p}$, which would be the case. Therefore, on shell we can have that $\{p, H\} = -x = \dot{p}+$ "equation of motion" is a time-like diffeomorphism.

5.1 Hypersurface deformations

The following formalism was set up by Dirac and is described in his book (see literature list)

Goal: Evolve 3-dimensional surfaces through spacetime (with fixed metric)



Dirac has set up a simple formalism:

- 3-surface σ with coordinates x^a, $a = 1, 2, 3$
- Spacetime with coordinates y^μ, $\mu = 0, 1, 2, 3$
- Configuration data: Embedding maps $y^\mu(x^a)$
  4 distinct fields in 3-dimensional space, c.f. string embedding map

$y^\mu_t(x^a)$: 1-parameter family of configurations = embeddings of σ
t should not be confused with the "time" y^0 in the target space

- Canonical momenta $w_\mu(x)$ *generate shifts in embedding coordinates y^μ.*
- Poisson bracket $\{y^\mu(x), w_\nu(x')\} = \delta^\mu_\nu \delta^{(3)}(x, x')$

The histories above are generated by the Hamiltonian $H = \int_\sigma d^3x\, \omega_\mu T^\mu$, where T^μ are arbitrary (t-dependent) parameters.

Infinitesimally, $y^\mu \mapsto y^\mu + T^\mu \delta t$

We want a formalism purely on σ.

Decompose $T^\mu = Nn^\mu + N^\mu$, with $N^\mu = \frac{\partial y^\mu}{\partial x^a} N^a$ *Push-forward of a vector on σ to S.*

Pull back covectors to σ: $v_a := v_\mu \frac{\partial y^\mu}{\partial x^a}$

Need also perpendicular part to S:
→ non-unit time-like conormal $\tilde{n}_\mu = \epsilon_{\mu\nu\rho\sigma} \epsilon^{abc} \frac{\partial y^\nu}{\partial x^a} \frac{\partial y^\rho}{\partial x^b} \frac{\partial y^\sigma}{\partial x^c}$.

→ Unit time-like conormal $n_\mu := \tilde{n}_\mu / \sqrt{-\tilde{n}_\mu \tilde{n}^\mu}$.

Define generators relative to S:

- $\mathcal{H} = w_\perp := w_\mu n^\mu$ moves surface normal to S
- $\mathcal{H}_a = w_\mu \frac{\partial y^\mu}{\partial x^a}$ changes coordinates on S, but leaves S invariant.

$\{y^\mu(x), \int_\sigma d^3x\, N^a H_a\} = N^a \partial_a y^\mu(x)$

→ one constraint per point for dynamics
→ three constraints per point for spatial diffeomorphisms, no dynamical information

Compute algebra: (exercises)

$$\{\mathcal{H}[M], \mathcal{H}[N]\} = \mathcal{H}_a \left[q^{ab} \left(M\partial_b N - N\partial_b M\right)\right] \tag{5.1}$$
$$\{\mathcal{H}[M], \mathcal{H}_a[N^a]\} = -\mathcal{H}\left[\mathcal{L}_N M\right] \tag{5.2}$$
$$\{\mathcal{H}_a[M^a], \mathcal{H}_a[N^a]\} = -\mathcal{H}_a\left[\mathcal{L}_N M^a\right]. \tag{5.3}$$

Dirac algebra / hypersurface deformation algebra

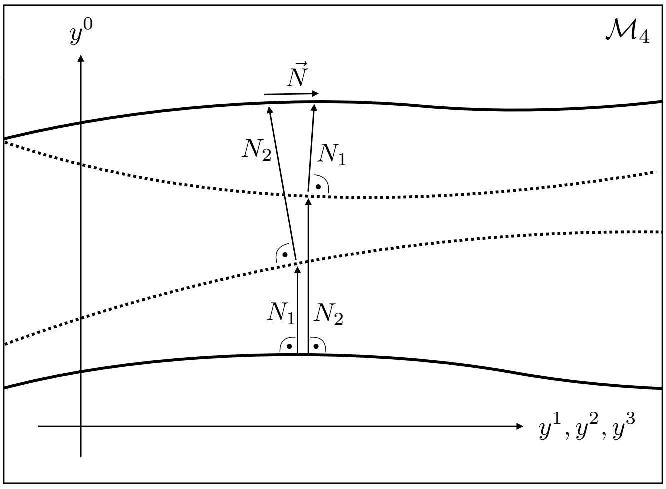


Figure 5.2: Executing two normal deformations in reverse order leads to a tangential shift with $N^a = q^{ab} (N_1 \partial_b N_2 - N_2 \partial_b N_1)$.

Hypersurface deformations:

- A simple model $\Rightarrow$ algebra of surface deformations
- Any generally covariant canonical theory needs to satisfy this algebra
  Next section: derive it from GR by brute force
- Represent this algebra in quantum theory
  This provides a notion of quantum general covariance

5.2 The ADM formulation

Start with the EH action and perform the Dirac analysis by brute force.

5.2.1 Strategy

Goal: Hamiltonian formulation of Einstein-Hilbert action ($\kappa = 8\pi G$)
Note that there are different conventions for κ in the literature!

$$S_{\text{EH}} = \frac{1}{2\kappa} \int_{\mathcal{M}} d^4x \sqrt{-g} R \tag{5.4}$$

Strategy:

- Formalism includes split into spacetime-slices
- But slicing arbitrary

$\Rightarrow$ Hamiltonian evolution along some time-like vector field T^μ

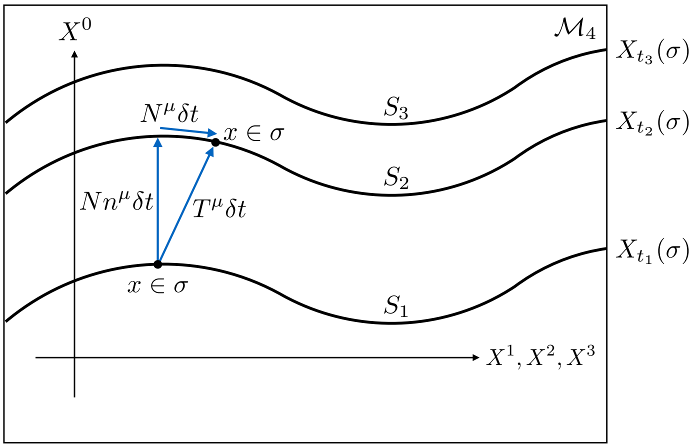


Figure 5.3: The class of spacetimes we consider is topologically equivalent to $\mathbb{R} \times \sigma$, where σ is a three-dimensional manifold. $\mathcal{M}_4$ is foliated by three-dimensional spatial slices $X_t(\sigma)$, $t \in \mathbb{R}$. We distinguish between the slices S, subsets of $\mathcal{M}_4$, and σ, which is mapped to S_t via the embedding map X_t. Time evolution is along the vector field $T^\mu = \partial_t X_t^\mu$. The decomposition into normal and tangential components is $T^\mu = Nn^\mu + N^\mu$.

We slightly change notation here from $y_t \mapsto X_t$ to conform with literature.

- T^μ is arbitrarily chosen, i.e. gauge parameter
- Decomposition $T^\mu = Nn^\mu + N^\mu$
  $N =$ Lapse function, $N^\mu =$ Shift vector. $(N^\mu n_\mu = 0)$
- T^μ determines a slicing $t \mapsto X_t(\sigma)$ given a slice S_0 at t_0
- all slicings are equivalent (gauge related)
- space-like foliation: $T^\mu T_\mu = -N^2 + N^\mu N_\mu < 0 \quad \Rightarrow N \neq 0$, choose $N > 0$.

Idea: $\int d^4x = \int dt \int d^3x$, write integrands in coordinates t, x^a adapted to the slicing.

- Metric not specified yet, can be still chosen arbitrarily in those coordiantes!
- The physical location of a neighbouring slice (i.e. how the slices are embedded in a given spacetime) is determined by N and N^μ

5.2.2 Fundamental forms

Needs some geometrical concepts to be introduced first: "time"-derivative of the metric for momenta

Goal: metric and its time derivative as tensors on σ.

1. Define suitable tensors on S as a subset of $\mathcal{M}$

2. Pull back to σ along embedding map.
   The second step is trivial if all tensors are purely spatial

Definition 13. The first fundamental form is defined as $q_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$.

- $q_{\mu\nu}$ is the spatial projection of $g_{\mu\nu}$
- $q_{\mu\nu} n^\nu = 0$
- $q_{\mu\nu}$ invertible only in spatial directions

$\rightarrow$ spatial metric $q_{ab} = (\partial_a X_t^\mu)(\partial_b X_t^\nu) q_{\mu\nu}$ after pull back to σ.

Definition 14. The second fundamental form is defined as $K_{\mu\nu} := q_\mu{}^\rho q_\nu{}^\sigma \nabla_\rho n_\sigma$.

- $K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n q_{\mu\nu}$ (exercise)

  $K_{\mu\nu}$ is 1/2 the Lie derivative of the spatial metric along the hypersurface normal
- $K_{\mu\nu} = K_{(\mu\nu)} := \frac{1}{2}\left(K_{\mu\nu} + K_{\nu\mu}\right)$ *symmetric*
- $K_{\mu\nu} n^\nu = 0$

With $T^\mu = N n^\mu + N^\mu$:

- $2K_{\mu\nu} = \frac{1}{N}\left(\mathcal{L}_T q_{\mu\nu} - \mathcal{L}_{\vec{N}} q_{\mu\nu}\right)$

We choose to write $\vec{N}$ in the second Lie derivative to point out the difference between N and N^a.

Need to understand geometry of the 3-surface σ.

Covariant derivative associated to $q_{\mu\nu}$: D_μ, defined by

- $D_\mu q_{\nu\rho} = 0$
- $D_{[\mu} D_{\nu]} f := \frac{1}{2}\left(D_\mu D_\nu - D_\nu D_\mu\right) f = 0$ on scalars f
- Preserves spatial tensors

Unique by theorem in differential geometry.

Can be constructed as

- $D_\mu f = q_\mu{}^\nu \nabla_\nu \tilde{f}$ for scalars f
- $D_\mu v_\nu = q_\mu{}^\rho q_\nu{}^\sigma \nabla_\rho \tilde{v}_\sigma$
- Obeys Leibniz rule

$\tilde{f}, \tilde{v}_\nu$: arbitrary smooth extensions of f, v_ν in a neighbourhood of S.

D_μ is independent of the extensions.

Pull back to σ: $D_a v_b = \partial_a v_b - \Gamma^c_{ab} v_c, \qquad \Gamma^c_{ab} = \frac{1}{2} q^{cd} \left(\partial_b q_{da} + \partial_a q_{db} - \partial_d q_{ab} \right)$

Construct Riemann tensor from D_μ: $\left(D_\mu D_\nu - D_\nu D_\mu \right) v_\rho =: R^{(3)}_{\mu\nu\rho}{}^{\sigma} v_\sigma \qquad$ with $v_\rho n^\rho = 0$

Fundamental identities: (exercises)

- Gauß equation: $R^{(3)}_{\mu\nu\rho\sigma} = -2 K_{\rho[\mu} K_{\sigma]\nu} + q_\mu{}^{\mu'} q_\nu{}^{\nu'} q_\rho{}^{\rho'} q_\sigma{}^{\sigma'} R^{(4)}_{\mu'\nu'\rho'\sigma'}$
- Codacci equation: $R^{(4)} = R^{(3)} + \left(K_{\mu\nu} K^{\mu\nu} - K^2 \right) + 2 \nabla_\mu \left(n^\nu \nabla_\nu n^\mu - n^\mu \nabla_\nu n^\nu \right)$
  with $K := K_\mu{}^\mu$

Need metric in coordinates t, x^a

The idea is now to obtain the form of the metric by expressing the infinitesimal line element via the changes that varying t and x produces via the embedding map X

$dX^\mu = (\partial_t X^\mu) dt + (\partial_a X^\mu) dx^a$

Suppress t-subscript: $X(t, x^a) := X_t(x^a)$

$T^\mu = \partial_t X^\mu = N n^\mu + N^\mu$

$$\begin{aligned} ds^2 &= g_{\mu\nu} dX^\mu dX^\nu && (5.5) \\ &= g_{\mu\nu} \left((\partial_t X^\mu) dt + (\partial_a X^\mu) dx^a \right) \Big((\partial_t X^\nu) dt + (\partial_b X^\nu) dx^b \Big) && (5.6) \\ &= (-N^2 + N^\mu N_\mu) dt^2 + 2 N^\mu g_{\mu\nu} (\partial_a X^\nu) dt dx^a + (\partial_a X^\mu)(\partial_b X^\nu) g_{\mu\nu} && (5.7) \end{aligned}$$

ADM (Arnowitt-Deser-Misner)-form of the metric.

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N_a & N_a \\ N_a & q_{ab} \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^a/N^2 \\ N^a/N^2 & q^{ab} - N^a N^b / N^2 \end{pmatrix} \qquad (5.8)$$

Coordinate expressions:

$n_\mu \propto \epsilon_{\mu\nu\rho\sigma} \epsilon^{abc} (\partial_a X^\nu)(\partial_b X^\rho)(\partial_c X^\sigma) \propto (1, \vec{0})_\mu$

Spatial vectors have a vanishing t-component

$\Rightarrow \quad n_\mu = (-N, \vec{0})$ and $n^\mu = (1/N, -N^a/N)$

$T^\mu = N n^\mu + N^\mu = (1, (N^a - N^a)) = (1, \vec{0})^\mu \qquad$ *not in conflict with arbitrary lapse and shift!*

$\Rightarrow \quad T^\mu \partial_\mu$ has only a ∂_t component

Coordinates are adapted to the slicing, not the metric!

In these coordinates, $\sqrt{-g} = |N| \sqrt{q}$ (exercise).

For Einstein-Hilbert action:

- By Codacci equation: $\sqrt{-g}R^{(4)}$ expressed in terms of $q_{\mu\nu}$, $K_{\mu\nu}$, N, + total derivative

$\rightarrow$ pull everything back to σ.

ADM-form of the action:

$$S_{\mathrm{ADM}} = \frac{1}{2\kappa}\int_{\mathbb{R}} dt \int_\sigma d^3x\,\sqrt{\det q}\,N\left(R + K_{ab}K^{ab} - K^2\right) \tag{5.9}$$

From now on: $R := R^{(3)}$.

5.2.3 Legendre transform

Perform Legendre transform & canonical analysis of

$$S_{\mathrm{ADM}} = \frac{1}{2\kappa}\int_{\mathbb{R}} dt \int_\sigma d^3x\,\sqrt{\det q}\,N\left(R + K_{ab}K^{ab} - K^2\right) \tag{5.10}$$

Neglect boundary terms throughout.

Note that while we chose adapted coordinates to a specific slicing, we are still completely general in that we can assign any metric to those coordinates. Only by specifying lapse, shift, etc. we actually specify how this foliation lies inside a given spacetime equipped with a metric. Changing these assignments is what constraints are doing, so the gauge transformations of our theory will take care of this freedom.

Time derivative $\dot{q}_{ab} = \mathcal{L}_T q_{ab}$ hidden in K_{ab} *indices pulled back to σ.* *(due to $T = \partial_t$)*

More precisely, $\dot{q}_{ab} = (\partial_a X_t^\mu)(\partial_b X_t^\nu)\ \mathcal{L}_T q_{\mu\nu}$

Due to $T^\mu = (1, \vec{0})$, $\mathcal{L}_T q_{\mu\nu} = \partial_t q_{\mu\nu}$.

$K_{ab} = \frac{1}{2N}\left(\mathcal{L}_T q_{ab} - \mathcal{L}_{\vec{N}} q_{ab}\right)$

Canonical momenta:

- $P^{ab} := \frac{\delta S}{\delta \dot{q}_{ab}} = \frac{1}{2\kappa}\sqrt{q}\left(K^{ab} - q^{ab}K\right)$, $\quad \{q_{ab}(x), P^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta^{(3)}(x,y)$
- $P_N := \frac{\delta S}{\delta \dot{N}} = 0$
- $P_{N^a} := \frac{\delta S}{\delta \dot{N}^a} = 0$

$\dot{q}_{ab} = 2NK_{ab} + \mathcal{L}_{\vec{N}} q_{ab}$

3+1 primary constraints per point

$$C := P_N \approx 0, \qquad C_a := P_{N^a} \approx 0. \tag{5.11}$$

Useful identities:

- $P = -\frac{\sqrt{q}}{\kappa}K$
- $K_{ab} = \frac{\kappa}{\sqrt{q}}\left(2P_{ab} - q_{ab}P\right)$
- $K_{ab}K^{ab} = \frac{\kappa^2}{q}\left(4P_{ab}P^{ab} - P^2\right)$

Total Hamiltonian:

$$H_T = \int d^3x \left(P^{ab}\dot{q}_{ab} + P_N \dot{N} + P_{N^a}\dot{N}^a - \mathcal{L}_{\text{EH}} + \lambda C + \lambda^a C_a \right) \tag{5.12}$$

$$= \int d^3x \left(N \left(\frac{2\kappa}{\sqrt{q}} \left(P^{ab}P_{ab} - \frac{1}{2}P^2 \right) - \frac{\sqrt{q}}{2\kappa}R \right) + P^{ab}\mathcal{L}_{\vec{N}} q_{ab} + \tilde{\lambda}C + \tilde{\lambda}^a C_a \right) \tag{5.13}$$

Stability of constraints:

- $\dot{C} = \frac{2\kappa}{\sqrt{q}} \left(P^{ab}P_{ab} - \frac{1}{2}P^2 \right) - \frac{\sqrt{q}}{2\kappa}R =: \mathcal{H}$
- $\dot{C}_a = -2q_{ac}\nabla_b P^{bc} = -2\nabla_b P^b{}_a =: \mathcal{H}_a$

We used $-2\int d^3x\, N^a \nabla_b P^b{}_a = \int d^3x\, P^{ab}\mathcal{L}_{\vec{N}} q_{ab}$ up to boundary terms (exercises).

The form $\int d^3x\, P^{ab}\mathcal{L}_{\vec{N}} q_{ab}$ is convenient in that it manifestly generates Lie derivatives. However, the expression does not vanish in general depending on the boundary conditions chosen, as it differs from $-2\int d^3x\, N^a \nabla_b P^b{}_a$ by a boundary term. For the distinction to be relevant, N^a has to be non-vanishing on the boundary.

Two new constraints:

- $\mathcal{H}$: scalar constraint, Hamiltonian constraint *also super-Hamiltonian constraint*
- $\mathcal{H}_a$ vector constraint, (spatial) diffeomorphism constraint

Compute algebra: (exercise)

$$\{\mathcal{H}[M], \mathcal{H}[N]\} = \mathcal{H}_a \left[q^{ab} \left(M\partial_b N - N\partial_b M \right) \right] \tag{5.14}$$

$$\{\mathcal{H}[M], \mathcal{H}_a[N^a]\} = -\mathcal{H}\left[\mathcal{L}_N M \right] \tag{5.15}$$

$$\{\mathcal{H}_a[M^a], \mathcal{H}_a[N^a]\} = -\mathcal{H}_a\left[\mathcal{L}_N M^a \right]. \tag{5.16}$$

$\Rightarrow$ All constraints are stable and first class.

Interpretation of $\mathcal{H}_a$: (again dropping boundary terms)

- $\{q_{ab}(x), \mathcal{H}_a[N^a]\} = \{q_{ab}(x), \int d^3y\, P^{ab}\mathcal{L}_N q_{ab}\} = \mathcal{L}_N q_{ab}(x)$
- $\{P^{ab}(x), \mathcal{H}_a[N^a]\} = \{P^{ab}(x), -\int d^3y\, q_{ab}\mathcal{L}_N P^{ab}\} = \mathcal{L}_N P^{ab}(x)$

$\mathcal{H}_a$ generates spatial diffeomorphisms

By Leibniz property of the Lie derivative:

- $\{f(P^{ab}, q_{ab}), \mathcal{H}_a[N^a]\} = \mathcal{L}_N f(P^{ab}, q_{ab})$

This also explains 2 of the above algebra relations.

No direct physical information in $\mathcal{H}_a$, changes only coordinates on initial value slice.

Interpretation of $\mathcal{H}$:

- $\{q_{ab}(x), \mathcal{H}[N]\} = 2N\kappa K_{ab} = \mathcal{L}_{Nn} q_{ab}$

- $\{P^{ab}(x), \mathcal{H}[N]\} = \frac{q^{ab}N\mathcal{H}}{2} - N\sqrt{q}\left(q^{a\rho}q^{b\sigma} - q^{ab}q^{\rho\sigma}\right)R^{(4)}_{\rho\sigma} + \mathcal{L}_{Nn}P^{ab}$

Remarks:

- $\mathcal{H}$ generates diffeomorphism normal to hypersurface S $\underline{\text{on } q_{ab}}$
- Second computation very hard
- Normal diffeomorphism only if subset of vacuum Einstein equations ($R^{(4)}_{\mu\nu} = 0$) and $\mathcal{H} = 0$ are satisfied
- Einstein equations needed to turn second time derivative of q_{ab} into canonical data
  We need an equivalent formulation, the covariant EOM, where we have a manifest identification of $L_{Nn}P^{ab}$ with the canonical data.

- Einstein equations logically needed here to specify theory
  Otherwise, it could have been any theory / Lagrangian that we are dealing with here.
- Other vacuum Einstein equations from
  - $\mathcal{H} \propto G_{\mu\nu}n^\mu n^\nu$
  - $\mathcal{H}_a \propto G_{\mu\nu}n^\mu {q^\nu}_a$
- Group generated by $\mathcal{H}, \mathcal{H}_a$: Bergmann-Komar group BK($\mathcal{M}$)
- BK($\mathcal{M}$) agrees with Diff($\mathcal{M}$) only on-shell
  We also say that BK($\mathcal{M}$) is the dynamical symmetry group, while Diff($\mathcal{M}$) is a kinematical one

5.3 Phase space extension

It is possible to start canonical quantisation at this point, leading to the Wheeler-de Witt theory, or Geometrodynamics. However, mathematical problems occur along the way, in particular it is not known how to construct a spatially diffeomorphism-invariant integration measure on the space of all metrics. It is therefore helpful to change the canonical variables to some "more compact" objects, e.g. a connection taking values in the Lie algebra of a compact Lie group, and then to use holonomies of that connection.

Idea from coupling fermions.

Minkowski space: γ-matrices $\gamma^I\gamma^J + \gamma^J\gamma^I = 2\eta^{IJ}\mathbb{1}_4$, $I = 0, 1, 2, 3$

Curved spacetime: $\eta^{IJ} \to g^{\mu\nu}$.

If we have e^μ_I so that $g^{\mu\nu} = e^\mu_I e^\mu_J \eta^{IJ}$, we can use $\gamma^\mu = e^\mu_I \gamma^I$

e^μ_I is called a vier-bein, or an orthonormal frame

At each point in σ, three linearly independent tangent vectors.

Construct orthonormal frame:

- e^a_i, $i = 1, 2, 3$, so that $e^a_i q_{ab} e^b_j = \delta_{ij}$
- e^a_1, e^a_2, e^a_3: three orthonormal vectors
- Encodes metric: $q_{ab} = e^i_a e^j_b \delta_{ij}$, where $e^i_a e^b_i = \delta^b_a$
- Also, $q^{ab} = e^a_i e^b_j \delta^{ij}$
- Unique up to rotation and orientation change: $e^a_i \to \pm R_i{}^j e^a_j$
- We choose positive orientation: $\det e^a_i = \sqrt{q}^{-1}$

$\Rightarrow$ Using e^a_i instead of q_{ab} introduces local gauge invariance
This is just the rotation of local frames.

i, j are indices in an internal space.

e^a_i is called a triad (dreibein).

Strategy:

1. Extend the phase space and prescribe new Poisson bracket
   Something like e^a_i and a conjugate, encoding both q_{ab} and P^{ab}
2. Introduce new constraint which kills new degrees of freedom
3. Relate gauge invariant part of new phase space to old phase space
4. Write old constraints in terms of new phase space variables

$\Rightarrow$ Leaves dynamics generated for q_{ab} and P^{ab} invariant

New variables:

- $E^a_i := \sqrt{q} e^a_i \quad \Rightarrow qq^{ab} = E^a_i E^b_j \delta^{ij}$
- K^i_a, with $K_{ab} := K^i_{(a} e^j_{b)} \delta_{ij}$
- $\left\{ K^i_a(x), E^b_j(y) \right\} = \kappa \delta^{(3)}(x, y) \delta^b_a \delta^i_j$

Relation to old phase space via $P^{ab} \leftrightarrow K_{ab}$

Need rotation generator:

- Should map $E^a_i \mapsto \Lambda_i{}^j E^a_j, \quad K^i_a \mapsto \Lambda^i{}_j K^j_a \quad$ with $\Lambda^i{}_j$ antisymmetric
  $\Rightarrow$ Needs to contain 1 power of E^a_i and K^i_a
- Should then enforce $K^i_{[a} e^j_{b]} \delta_{ij} = 0 \qquad$ *q_{ab} is already symmetric due to $e^i_{[a} e^j_{b]} \delta_{ij} = 0$*

$\Rightarrow$ Gauß constraint $G_{ij}[\Lambda^{ij}] := \frac{1}{\kappa} \int d^3x \, E^a_{[i} K_{a|j]} \Lambda^{ij} \approx 0$

Check:

$$0 \approx E^a_{[i} K_{a|j]} e^i_b e^j_c = E^a_i K_{aj} e^i_{[b} e^j_{c]} = \sqrt{q} K^i_{[b} e_{c]i} \tag{5.17}$$

Gauge transformations:

- $\{E^a_i, G_{kl}[\Lambda^{kl}]\} = \Lambda_i{}^j E^a_j$
- $\{K^i_a, G_{kl}[\Lambda^{kl}]\} = \Lambda^i{}_j K^j_a$

Algebra: $\{G_{ij}[\Lambda^{ij}], G_{kl}[\Omega^{kl}]\} = G_{ij}[(\Lambda\Omega - \Omega\Lambda)^{ij}]$

Objects with contracted internal indices are invariant:

$$\{E^a_i E^{bi}, G_{kl}[\Lambda^{kl}]\} = \Lambda_i{}^j E^a_j E^{bi} + E^a_i \Lambda^i{}_j E^{bj} = (\Lambda_{ji} + \Lambda_{ij}) E^{ai} E^{bj} = 0 \tag{5.18}$$

Similar for K^i_a.

9+9 DOF per point - 3 rotations - 3 constraints = 6+6 DOF per point as before (without imposing $\mathcal{H}$, $\mathcal{H}_a$)

Express ADM variables in terms of new variables:

- $\det E^a_i = \sqrt{q}^{(3-1)} = q$
- $q^{ab} = \frac{1}{\det E} E^a_i E^{bi}$
- $P^{ab} = \frac{1}{2\kappa}\sqrt{q}\left(K^{ab} - q^{ab}K\right)$, with $\sqrt{q} K_a{}^b = K^i_a E^b_i$

Non-vanishing Poisson bracket: (exercise)

$$\Big\{ q_{ab}[E,K](x), P^{cd}[E,K](y) \Big\}_{\{K,E\}} = \kappa \delta^c_{(a}\delta^d_{b)} \delta^{(3)}(x,y) \tag{5.19}$$

$\Rightarrow$ Dynamics generated by $\mathcal{H}$, $\mathcal{H}_a$ is invariant for q_{ab} and P^{ab}

Evolution of E^a_i, K^i_a depends on new arbitrary gauge parameters Λ_{ij}

Both E^a_i and K^i_a transform as internal vectors. In order to compute holonomies, i.e. exponentiate a Lie algebra to a group, we need one of the canonical variables to transform as a connection.

5.4 Connection variables

Need to construct connection out of canonical variables

Connection should transform as

$$\{\omega_{aij}, G_{kl}[\Lambda^{kl}]\} = -D_a \Lambda_{ij} := -\partial_a \Lambda_{ij} - [\omega_a, \Lambda]_{ij} = -\partial_a \Lambda_{ij} - \omega_{aik}\Lambda^k{}_j + \omega_{akj}\Lambda_i{}^k \tag{5.20}$$

Connection extends the covariant derivative to internal indices:

- $D_a v^i = \partial_a v^i + \omega_a{}^i{}_j v^j$
- $D_a v^i_b = \partial_a v^i + \omega_a{}^i{}_j v^j - \Gamma^c_{ab} v^i_c$

We can construct a connection by compatibility with e^i_a:

$$D_a e^i_b = \partial_a e^i_b + {\omega_a}^i{}_j e^j_b - \Gamma^c_{ab} e^i_c \stackrel{!}{=} 0 \tag{5.21}$$

Can be solved for ${\omega_a}^i{}_j = {\Gamma_a}^i{}_j := -e^b_j \left(\partial_a e^i_b - \Gamma^c_{ab} e^i_c\right) = -e^b_j \nabla_a e^i_b$

$\Gamma_{aij} = -\Gamma_{aji}$, because $-e^b_j \nabla_a e_{bi} = (\nabla_a e^b_j) e_{bi} - \nabla_a \delta^i_j = e^b_i \nabla_a e_{bj}$

Γ_{aij}: Spin connection

$$\{\Gamma_{aij}, G_{kl}[\Lambda^{kl}]\} = -{\Lambda_j}^k e^b_k \nabla_a e_{bi} - e^b_j \nabla_a \left({\Lambda_i}^k e_{bk}\right) = -\partial_a \Lambda_{ij} - \Gamma_{aik} {\Lambda^k}_j + \Gamma_{akj} {\Lambda_i}^k \tag{5.22}$$

$\Rightarrow$ correct transformation behaviour.

Different index structure of Γ_{aij} and E^a_i, but same number of DOF!

Isomorphism: Antisymmetric pair $[ij] \leftrightarrow k$ single internal index.

- Let $a_{ij} = -a_{ji}$
- Define $a^k = -\frac{1}{2}\epsilon^{ijk} a_{ij}$
- $a_{ij} = -\epsilon_{ijk} a^k$, because $-\epsilon_{ijk}(-)\frac{1}{2}\epsilon^{mnk} a_{mn} = \frac{1}{2}(\delta^m_i \delta^n_j - \delta^m_j \delta^n_i) a_{mn} = a_{ij}$

Define $\omega^i_a = -\frac{1}{2}\epsilon^{ijk}\omega_{ajk}$.

$\Rightarrow D_a v^i_b = \partial_a v^i_b + \epsilon^{ijk}\omega_{aj} v_{bk} - \Gamma^c_{ab} v^i_c$

$\Rightarrow 0 = \partial_a e^i_b + \epsilon^{ijk}\Gamma_{aj} e_{bk} - \Gamma^c_{ab} e^i_c$

$\Rightarrow \Gamma^i_a = \frac{1}{2}\epsilon^{ijk} e^b_k \left(\partial_b e_{aj} - \partial_a e_{bj} + e^c_j e^l_a \partial_b e_{cl}\right)$

Define $A^i_a := \Gamma^i_a(E) + K^i_a$

From now on: D_a acts with A^i_a

connection + vector = connection

The added vector transforms "homogeneously", i.e. with the commutator part of the connection transformation, while the affine part, producing the derivative in the transformation law, stays as it is.

Poisson brackets:

- $\left\{A^i_a(x), E^b_j(y)\right\} = \kappa \delta^{(3)}(x,y)\delta^b_a \delta^i_j$
- $\left\{E^a_i(x), E^b_j(y)\right\} = 0$
- $\left\{A^i_a(x), A^j_b(y)\right\} = \left\{\Gamma^i_a(x), K^j_b(y)\right\} + \left\{K^i_a(x), \Gamma^j_b(y)\right\} = 0$ (exercise)

$\rightarrow A^i_a$ and E^b_j are canonically conjugate.

Compute new Gauß law:

- $G_{ij}[\Lambda^{ij}] := \frac{1}{\kappa}\int d^3x E^a_{[i}K_{a|j]}\Lambda^{ij} \approx 0$

- $G_k[\Lambda^k] := G_{ij}[-\epsilon^{ijk}\Lambda_k] = -\frac{1}{\kappa}\int d^3x\Lambda_k\epsilon^{ijk}E^a_{[i}K_{a|j]} = \frac{1}{\kappa}\int d^3x\Lambda^k D_a E^a_k$ (exercise)

Action of Gauß law:

- $\{A^i_a(x), G_k[\Lambda^k]\} = \{A^i_a(x), -\int d^3y(D_b\Lambda^k)E^b_k\} = -D_a\Lambda^i(x)$
- $\{E^a_i(x), G_k[\Lambda^k]\} = \{E^a_i(x), \int d^3y\Lambda_k\epsilon^{kmn}A_{bm}E^b_n\} = \epsilon^{ikn}\Lambda_k E^a_n(x)$

For the quantum theory, it is interesting to perform a rescaling of the canonical variables. This changes spectra of the geometric operators.

Rescale the canonical variables: (interesting in quantum theory)

- ${}^{(\beta)}E^a_i := \frac{1}{\beta}E^a_i$
- ${}^{(\beta)}A^i_a := \Gamma^i_a + \beta K^i_a$

We still have: $\left\{{}^{(\beta)}A^i_a(x), {}^{(\beta)}E^b_j(y)\right\} = \kappa\delta^{(3)}(x,y)\delta^b_a\delta^i_j$

"Ashtekar-Barbero variables" (1986 ($\beta = i$), 1994 ($\beta \in \mathbb{R}$))

Drop ${}^{(\beta)}$ notation from now on.

Write all constraints in the new variables ($\kappa = 1$): (exercises) (all ${}^{(\beta)}$ implied!)

$$G_k[\Lambda^k] = \int d^3x\, \Lambda^k D_a E^a_k \tag{5.23}$$

$$\mathcal{H}_a[N^a] = \int d^3x\, E^{ai}\mathcal{L}_{\vec{N}}A_{ai} = \int d^3x N^a F^i_{ab}E^b_i + G_{ij}[...] + \text{boundary} \tag{5.24}$$

$$\mathcal{H}[N] = \int_\Sigma d^3x\, N\left(\beta^2\frac{E^{ai}E^{bj}}{2\sqrt{q}}\epsilon^{ijk}F^k_{ab} - \frac{(1+\beta^2)}{\sqrt{q}}K^i_{[a}K^j_{b]}E^{ai}E^{bj}\right) \tag{5.25}$$

Field strength $F^i_{ab} = 2\partial_{[a}A^i_{b]} + \epsilon^{ijk}A^j_a A^k_b$

$K^i_a = (A^i_a - \Gamma^i_a)/\beta$ *Again, the ${}^{(\beta)}$ subscript is implied on A^i_a and E^a_i.*

Remarks:

- G_k and $\mathcal{H}_a$ have a clear interpretation
- $\mathcal{H}$ looks complicated, hard to quantise
- $\beta = \pm i$ special: simplifies $\mathcal{H}$
- $\beta \notin \mathbb{R}$ has complicated reality conditions, hard to quantise
- Most quantisation progress so far for $\beta \in \mathbb{R}$

Canonical general relativity:

- The phase space of general relativity can be parametrised by a spatial metric and its momentum, related to the extrinsic curvature of the spatial slice
- The Hamiltonian and spatial diffeomorphism constraint encode the dynamics of the theory as well as the gauge structure
- The ADM phase space can be enlarged to contain an additional local SO(3) gauge redundancy and equipped with connection variables

6 Quantisation of constrained Hamiltonian systems

6.1 Quantisation without constraints

6.1.1 Abstract physical systems

Before discussing the algebraic properties of classical and quantum mechanics, let us consider what structure we need to do physics

Physical quantities $q_1, q_2, \ldots \in Q$.

Operations on physical quantities:

- Add and scale physical quantities (e.g. addition of mass, velocity, ...)
  $\rightarrow$ Vector space
- Multiply physical quantities, (e.g. torque = force x distance)
  $\rightarrow$ Algebra *Not necessarily commutative*

Definition 15. An algebra $\mathcal{A}$ over $\mathbb{C}$ is a $\mathbb{C}$-vector space equipped with a bilinear product $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$.

One may in principle forbid addition of structurally different quantities, e.g. with different dimensions, or the multiplication of all elements with all other elements.

"Less obvious" properties: *Those are used in our current physical theories*

- Adjoint element q^*: *real vs. complex objects*
  - $(q^*)^* = q$,
  - $(q_1 q_2)^* = q_2^* q_1^*$
  - $(cq)^* = \bar{c} q^*, \quad c \in \mathbb{C}$

  E.g. complex conjugation.

  up to here: *-algebra
- Norm: $\|\cdot\| : \mathcal{A} \rightarrow [0, \infty)$ *How large can a quantity be at most?*
  - $\|q_1 q_2\| \leq \|q_1\| \|q_2\|$
  - $\|q_1 + q_2\| \leq \|q_1\| + \|q_2\|$
  - $\|cq\| = |c| \|q\|$

  Existence of a norm requires us to work with bounded quantities, e.g. e^{iq}
- Completeness: all Cauchy series in Q converge in Q.

  Normed and complete algebra: Banach algebra
- *-compatibility of the norm: $\|q^* q\| = \|q\|^2, \quad \|q^*\| = \|q\|$

  up to here: C^*-algebra

Structure of $\underline{C^*\text{-algebra}}$!

All of these can in principle be dropped.

State of the system: map ω from Q to $\mathbb{C}$.

- Linearity: $\omega(c_1 q_1 + c_2 q_2) = c_1\omega(q_1) + c_2\omega(q_2), \quad c_1, c_2 \in \mathbb{C}$
  Motivated by classical evaluation of phase space functions at a phase space point, otherwise inconsistent with addition of phase space functions.

- Positivity: $\omega(q^* q) \geq 0$

Assigns values to physical quantities.

6.1.2 Algebraic structure of Hamiltonian mechanics

Canonical quantisation is a representation of the algebraic structure of classical mechanics on Hilbert spaces. It is therefore necessary to study this structure beforehand.

Functions on phase space form an algebra

- Vector space: addition $f_1 + f_2$ and multiplication $cf, \quad c \in \mathbb{C}$
- Bilinear product: multiplication $f_1 \cdot f_2 = f_2 \cdot f_1$, commutative!

Mathematical properties

- Norm: $\|f\| = \sup_{q,p} f(q,p)$ *limit to bounded smooth functions.*
- Adjoint element: complex conjugation, e.g. $(q + ip)^* = (q - ip)$
- Cauchy series converge: $\lim_{n\to\infty} f_n \to f$

  The set of bounded continuous functions is complete under the sup norm.

Additional structure:

- Poisson bracket $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $\to$ Poisson algebra
- Generator of dynamics: Hamiltonian H

States:

- $\omega_{q_0,p_0}(f) = f(q_0, p_0)$ *Evaluation on phase space point*
- Can be generalised to probability distributions

6.1.3 Algebraic structure of quantum mechanics

We need a suitable algebraic structure to represent / modify that of Hamiltonian mechanics.

Experiment:

- Quantities which are canonically conjugate at the classical level cannot be measured simultaneously with arbitrary precision

  $\rightarrow$ modify algebra to take canonical conjugacy into account

  $\rightarrow$ ordering important $qp \neq pq$

  because measurements in different orders $\rightarrow$ different results

- Superposition of classical states / solutions

  $\rightarrow$ linear structure *Photon takes all paths simultaneously, superposition of solutions*

Construction of quantum mechanics:

- Non-commutativity: physical quantities are operators
- Superposition: linear operators on vector space, for linear Schrödinger equation
- Adjoint element: scalar product in the vector space
- Scalar product + completeness + norm (derived from scalar product): Hilbert space

Definition 16. A Hilbert space $\mathcal{H}$ is a vector space with a scalar product that is also a complete metric space with a norm induced by the inner product.

Notation:

- Elements of $\mathcal{H}$: "kets" $|x\rangle$
- Scalar product: $\langle x, y\rangle = \langle x \mid y\rangle$
- Norm for Hilbert space elements: $\| \, |x\rangle \, \|^2 := \langle x, x\rangle \geq 0$, equality only if $|x\rangle = 0$.
- Adjointness: $\left\langle x, \hat{A}y \right\rangle = \left\langle \hat{A}^* x, y \right\rangle$
- Operator norm: $\sup_{|x\rangle\in\mathcal{H},\, \||x\rangle\|\neq 0} \frac{\|\hat{A}|x\rangle\|}{\||x\rangle\|} = c \quad \Leftrightarrow \quad \hat{A}$ has norm c

Boundedness:

- A bounded: $c \in [0, \infty)$
- A unbounded: otherwise, "$c = \infty$"

$\hat{A}$ self-adjoint (modulo domain issues): $\hat{A}^* = \hat{A}$

$\hat{A}$ self-adjoint $\Rightarrow$ $\hat{A}$ has real spectrum (eigenvalues in the finite dimensional case)

Inequalities for operator norm:

- $\|\hat{A}\hat{B}\| \leq \|\hat{A}\|\|\hat{B}\| \quad \forall \hat{A}, \hat{B} \in \mathcal{B}(\mathcal{H})$ *bounded linear operators*
- $\|\hat{A} + \hat{B}\| \leq \|\hat{A}\| + \|\hat{B}\|, \quad \forall \hat{A}, \hat{B} \in \mathcal{B}(\mathcal{H})$

6.1.4 Quantisation map

After having discussed what mathematical structure we want to use for quantum mechanics, we need to figure out how to construct a quantum system given a classical system.

Algebraic properties of Poisson bracket are mimicked by commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

- Bi-linearity: $[\lambda\hat{A} + \mu\hat{B}, \hat{C}] = \lambda[\hat{A}, \hat{C}] + \mu[\hat{B}, \hat{C}]$ and $[\hat{C}, \lambda\hat{A} + \mu\hat{B}] = \lambda[\hat{C}, \hat{A}] + \mu[\hat{C}, \hat{B}]$
- Antisymmetry: $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$
- Leibniz property: $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$ *ordering matters!*
- Jacobi identity: $[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0$

$\forall \hat{A}, \hat{B}, \hat{C} \in \mathrm{GL}(\mathcal{H}),\ \lambda, \mu \in \mathbb{C}$

Quantisation map (ideal version, see however subtleties):

Given a classical Hamiltonian system, we look for linear operators on a Hilbert space so that

$$[\hat{f}, \hat{g}] = i\hbar\,\widehat{\{f, g\}} \tag{6.1}$$

where $\hbar = \frac{h}{2\pi}$ is the Planck constant and f, g are phase space functions.

Dynamics:

Heisenberg picture:

$$\hat{f}(t) = e^{i\hat{H}(t-t_0)/\hbar}\hat{f}(t_0)e^{-i\hat{H}(t-t_0)/\hbar} = \hat{f}(t_0) + (t - t_0)\underbrace{\frac{-i}{\hbar}[\hat{f}, \hat{H}]}_{\widehat{\{f,H\}}} + \mathcal{O}\left((t - t_0)^2\right) \tag{6.2}$$

Possible outcome of individual measurements: spectrum of $\hat{f}$. *roughly: eigenvalues*

Expectation value over many measurements: $\left\langle \Psi \,\middle|\, \hat{f} \,\middle|\, \Psi \right\rangle := \left\langle \Psi, \hat{f}\Psi \right\rangle$ *for* $\langle \Psi \mid \Psi \rangle = 1$

The concrete outcome of the measurement is not determined, but we can additionally ask for the probability of a specific outcome by looking at the components of $|\Psi\rangle$ *in an eigenbasis of* $\hat{f}$.

Decompose $|\Psi\rangle = \sum_n c_n |f_n\rangle$ with $\hat{f}\,|f_n\rangle = f_n\,|f_n\rangle$.

Then, $|c_n|^2$ is the probability of the measure outcome f_n.

$\left\langle \Psi \,\middle|\, \hat{f} \,\middle|\, \Psi \right\rangle = \sum_n |c_n|^2 f_n$

Schrödinger picture:

$$|\Psi(t)\rangle = e^{-i\hat{H}(t-t_0)/\hbar}\,|\Psi(t_0)\rangle \tag{6.3}$$

Equivalence of dynamical pictures: $\left\langle \Psi(t_0) \,\middle|\, \hat{f}(t) \,\middle|\, \Psi(t_0) \right\rangle = \left\langle \Psi(t) \,\middle|\, \hat{f}(t_0) \,\middle|\, \Psi(t) \right\rangle$

The dynamics is either in the states or in the operators

6.1.5 GNS construction

Are there other structures than Hilbert spaces satisfying all needed properties for QM?

We now abstractly formulate what we want to achieve:

Consider the "free algebra" $\mathfrak{F}$ of all phase space functions with <u>no commutativity assumption</u>:

- Example element: $\mu f_1 f_2 + \gamma f_3 \neq \mu f_2 f_1 + \gamma f_3$, $\quad f_i$ phase space function, $\quad \mu, \gamma \in \mathbb{C}$

Identify $f_1 f_2 - f_2 f_1 = i\hbar\{f_1, f_2\}$

$\Rightarrow$ abstract "quantum" (unital) *-algebra $\mathfrak{B}$

This is an $\hbar$-deformation of the classical commutative algebra (deformation quantisation).

unital algebra: $\exists \mathbb{1} \in \mathcal{A}: \ \mathbb{1} f = f \mathbb{1} = f \ \forall\ f \in \mathcal{A}$

Not necessary here: C^*-property: Banach (= normed and complete) *-algebra with $\|f^* f\| = \|f\|^2$

Our task in quantum mechanics is to find representations of this algebra.

Theorem 3. (**Gelfand-Naimark-Segal**) Let ω be a positive, normed, and linear functional on a unital *-algebra $\mathcal{A}$. ω determines the GNS data $(\mathcal{H}_\omega, \rho_\omega, 0_\omega)$ consisting of a Hilbert space $\mathcal{H}_\omega$, a representation of $\mathcal{A}$ on $\mathcal{H}_\omega$, and a normed cyclic vector $|0_\omega\rangle \in \mathcal{H}_\omega$, so that

$$\omega(f) = \langle 0_\omega \,|\, \rho_\omega(f) \,|\, 0_\omega \rangle_{\mathcal{H}_\omega}, \quad \forall f \in \mathcal{A}. \tag{6.4}$$

The GNS data is unique up to unitary equivalence.

$|0\rangle$ cyclic $\Leftrightarrow \mathcal{B}(\mathcal{H})\,|0\rangle$ is dense in $\mathcal{H}$, $\quad \mathcal{B}(\mathcal{H})$ = *bounded linear operators on* $\mathcal{H}$.

This deep theorem tells us that we do not need to look beyond Hilbert spaces for finding a good arena for quantum physics. We recall however that the *-property is needed for this.

$\Rightarrow$ Hilbert space representations are all we need to consider!

Remarks:

- The $*$-property is needed to define $\langle a \mid b\rangle := \omega(a^*b)$ with $|b\rangle = \hat{b}\,|0_\omega\rangle$ and $\langle a| = \langle 0_\omega|\,\hat{a}^*$
- If the representation is by bounded operators, then the operator norm induces a C^*-norm on the quantum $*$-algebra $\mathfrak{B}$
- Related to Gelfand-Naimark theorem: Every C^*-algebra $\mathcal{A}$ is isometrically *-isomorphic to a C^*-algebra of bounded operators on a HS.

Example: $\omega(A^*BC) = \langle A \mid B \mid C\rangle$, where $|C\rangle = \hat{C}\,|0\rangle$, $|A\rangle = \hat{A}\,|0\rangle$

States are produced by operators acting on the cyclic vector $|0\rangle$, which may or may not have the interpretation of a vacuum = lowest energy eigenstate of the Hamiltonian.

6.1.6 Subtleties

Quantisation is not unique:

- Classical theory = limit of quantum theory *where the involved actions are $\gg \hbar$*

$\rightarrow$ We try to infer a general theory from a certain limit

Main issue: factor ordering

Example system:

- Classical phase space $\{x, p\} = 1$
- Hilbert space: $L^2(\mathbb{R}, dx)$
- Hilbert space element: $|\psi\rangle \Leftrightarrow$ square integrable function $\Psi(x)$, $\int_{-\infty}^{\infty} dx\, \overline{\Psi(x)}\Psi(x) < \infty$
- Operators: $\hat{x}\Psi(x) = x\Psi(x)$, $\quad \hat{p}\Psi(x) = -i\hbar\frac{d}{dx}\Psi(x)$

Question: What is $\widehat{xp}$? Different options:

- $\hat{x}\hat{p}$ or $\hat{p}\hat{x}$
- Symmetric ordering $\frac{1}{2}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right)$
- ...

$\rightarrow$ No good answer, needs additional input

Is it possible in principle to find operators for all phase space functions so that $[\hat{f}, \hat{g}] = i\hbar\widehat{\{f, g\}}$?

Theorem 4. (**Groenewold-van Hove**) It is not possible to find a consistent quantisation if we extend the allowed degree of polynomials in q, p beyond 2 in both variables.

This theorem tells us that we should in principle never expect to be able to represent all phase space functions as operators. Therefore, additional input is needed in constructing a canonical quantisation.

Main lesson:

- Need to choose point-separating subset of phase space functions to quantise consistently

Point-separating set of functions: We can tell points in phase space apart by knowing the value of all functions in the set.
e.g. q, p for one-dimensional system

For all phase space functions, can formalise with the Moyal product

$$f * g = f \cdot g + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i\hbar}{2}\right)^n \underbrace{\Pi^n(f,g)}_{\text{constructed from Poisson bracket}} \tag{6.5}$$

- Deformation quantisation: the star product is an $\hbar$-deformation of the usual multiplication.
  $f * g - g * f = i\hbar\{f, g\} + \mathcal{O}(\hbar^3)$
- Same as Hilbert space representation

Given a point-separating subset that we can quantise, is the quantisation unique?

To proof a theorem, simplify to bounded operators:

- Classical phase space functions: $e^{i\lambda x}, \; e^{i\mu p}, \quad \lambda, \mu \in \mathbb{R}$
- Operators: $\widehat{e^{i\lambda x}}, \widehat{e^{i\mu p}}$

Baker-Campbell-Hausdorff formula for $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$:

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}, \qquad e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A}, \hat{B}]} \tag{6.6}$$

Algebra: $\widehat{e^{i\lambda x}}\widehat{e^{i\mu p}} = e^{-i\hbar\lambda\mu}\widehat{e^{i\mu p}}\widehat{e^{i\lambda x}}$ Weyl-form of the canonical commutation relations (CCR)

Adjointness: $\widehat{e^{i\lambda x}}^* = \widehat{e^{-i\lambda x}}, \quad \widehat{e^{i\mu p}}^* = \widehat{e^{-i\mu p}}$

Note that the hat is over the complete exponential, not only over x and p. The derivation of the Weyl-from of the CCR is thus only formal here, but the result is what one would expect for well-behaved operators. We take this algebra here as a classical starting point and look for representations.

$\hat{U}(\lambda) := \widehat{e^{i\lambda x}}$ and $\hat{W}(\mu) := \widehat{e^{i\mu p}}$ are two one-parameter unitary groups:

- $\hat{U}(\lambda)^{-1} = \hat{U}(\lambda)^* = \hat{U}(-\lambda)$
- $\hat{W}(\mu)^{-1} = \hat{W}(\mu)^* = \hat{W}(-\mu)$

Theorem 5. (**Stone-von Neumann**) All jointly irreducible and weakly continuous one parameter unitary groups $\hat{U}(\lambda)$ and $\hat{W}(\mu)$ acting on a separable Hilbert space $\mathcal{H}$ and satisfying the Weyl-form of the canonical commutation relations are unitarily equivalent.

Separable Hilbert space: $\mathcal{H}$ admits a countable orthonormal basis.

$U(\lambda)$ and $W(\mu)$ jointly irreducible: The set $\{\hat{U}(\lambda) : \lambda \in \mathbb{R}\} \cup \{\hat{W}(\mu) : \mu \in \mathbb{R}\}$ is irreducible

irreducible: There are no invariant subspaces, more later in group theory

Weak continuity: $\lim_{\lambda\to 0} \left\langle \Psi \,\middle|\, \hat{U}(\lambda) \,\middle|\, \Psi' \right\rangle = \langle \Psi \mid \Psi' \rangle$

Weak continuity $\Rightarrow$ $\hat{x} := -i\frac{d}{d\lambda}\big|_{\lambda=0} \hat{U}(\lambda)$ exists (Stone's theorem)

This theorem tells us that under the given technical assumptions, all representations of the Weyl-form CCR are unitarily equivalent, i.e. the same up to a choice of basis.

The original theorems assumed strong continuity, however this can be relaxed to weak continuity

How can we avoid this theorem?

Drop weak continuity and separability!

Consider the Hilbert space of almost-periodic functions

- Spanned by $\{f_\lambda(x) = e^{i\lambda x} : \lambda \in \mathbb{R}\}$
- Scalar product: $\langle f_\lambda, f_{\lambda'} \rangle := \delta_{\lambda,\lambda'}$ (Kronecker delta!)

Note that in the Stone-von Neumann representation, $\langle f_\lambda, f_{\lambda'} \rangle_{\text{SvN}} = 2\pi\delta(\lambda, \lambda')$ (Dirac delta!)

This is only formal because plane waves are not square integrable, as the Dirac delta shows.

Action of operators:

- $\widehat{e^{i\lambda x}} \, |f_{\lambda'}\rangle = |f_{\lambda+\lambda'}\rangle$
- $\widehat{e^{i\mu p}} \, |f_\lambda\rangle = e^{i\hbar\mu\lambda} \, |f_\lambda\rangle$

No weak continuity:

$$\langle f_\lambda, f_\lambda \rangle = 1 \neq 0 = \lim_{\lambda'\to 0} \left\langle f_\lambda, \widehat{e^{i\lambda' x}} f_\lambda \right\rangle \tag{6.7}$$

$\Rightarrow \hat{x} := \frac{d}{d\lambda}\big|_{\lambda=0} \widehat{e^{i\lambda x}} = \lim_{\lambda\to 0} \frac{U(\lambda)-U(0)}{\lambda}$ $\quad$ <u>does not exist</u>

The differential quotient diverges when taking the expectation value w.r.t. $|f_\lambda\rangle$.

However, $\hat{p}$ exists, as $W(\mu)$ is weakly continuous.

What is the Hilbert space?

Complete w.r.t. norm induced by scalar product:

$\Rightarrow$ Bohr compactification of the real line, $\mathbb{R}_{\text{Bohr}}$.

This is a compact Abelian group that contains $\mathbb{R}$, but is much larger than it. In particular, a normalised Haar measure exists.

Important for loop quantum cosmology (later).

Main lessons:

- We can avoid uniqueness by dropping weak continuity
- Physics may be different in different representations.

  Here, need to regularise $\hat{x}$, e.g. via $\hat{x} \approx \frac{1}{\lambda}\sin(\lambda\hat{x}) = \frac{1}{2i\lambda}\left(\hat{U}(\lambda) - \hat{U}(-\lambda)\right)$

  $\rightarrow$ finite regularisation scale λ
- Separability can be regained by the dynamics / constraints via superselection

The above examples illustrate the finite-dimensional case. What about infinite dimensions?

What about field theories, i.e. infinite dimensional phase space?

$\rightarrow$ Uncountably many unitarily inequivalent representations in quantum field theory (by Haag's theorem)

The details are somewhat technical and won't be covered here.

Additional subtlety: Not possible to determine the representation using finite number of measurements (Fell's theorem)

Roughly, states written as expectation values w.r.t. density matrices in a representation obtained from a faithful state on a C^*-algebra are weakly dense in the set of all states on that algebra.

Quantisation of unconstrained systems:

- Algebraic structure of classical mechanics is mapped to quantum mechanics
- Factor ordering problems prohibit us from quantising all phase space functions consistently
- Representations of a certain sub-algebra of phase space functions are in general not unique, but may be under additional assumptions
- Dynamical input and physical intuition is likely needed to find the "correct" representation

6.2 Quantisation with constraints

6.2.1 Reduced quantisation

Solve all constraints classically

- Find enough Dirac observables and their Poisson brackets *or, equivalently:*
- Impose gauge conditions and compute Dirac bracket

$\Rightarrow$ standard quantisation

Pracitcal problem: Observable algebras / Dirac bracket are often too complicated to find representations

Conceptual problems for generally covariant systems:

- Fixes a classical notion of time (gauge condition)
- Quantum constraints define a notion of quantum general covariance
- Choice of gauge / observables not unique, may lead to inequivalent quantum theories

  The question of which quantum theory is the correct one then arises. This question is very subtle, since in all quantisations, one picks a preferred sub-algebra of observables. Mapping these observables to other ones is highly ambiguous at the level of operators.
- No good global gauge conditions in GR

  This means that no single gauge fixed classical quantisation is applicable in all situations

$\rightarrow$ Explore different methods of quantisation with constraints

6.2.2 Dirac quantisation

Idea: quantise first class constraints and impose on states

Set of constraints $\phi_\alpha(q^j, p_j) \approx 0$

Quantise unconstrained phase space:

- $|\Psi\rangle \in \mathcal{H}$, kinematical Hilbert space
- Operators, $\hat{q}^i$, $\hat{p}_i$

Constraint operators: $\hat{\phi}_\alpha := \phi_\alpha(\hat{q}^j, \hat{p}_j)$ in a specific ordering

Physical states: $\hat{\phi}_\alpha \left|\psi\right\rangle_{\text{phys}} = 0$

(Weak) Quantum Dirac observables: $[\hat{\mathcal{O}}, \hat{\phi}_\alpha] \approx 0, \quad \approx$ w.r.t. $\hat{\phi}_\alpha$

Is this consistent?

- Assume $\hat{\phi}_\alpha \left|\psi\right\rangle_{\text{phys}} = 0$
- $\Rightarrow \hat{\phi}_\alpha \hat{\phi}_\beta \left|\psi\right\rangle_{\text{phys}} = 0$

- $\Rightarrow \left(\hat\phi_\alpha\hat\phi_\beta - \hat\phi_\beta\hat\phi_\alpha\right)|\psi\rangle_{\text{phys}} = 0$
- $\Rightarrow \widehat{\{\phi_\alpha,\phi_\beta\}}\,|\psi\rangle_{\text{phys}} = 0$ (up to ordering problems)

Two cases:

- Only first class constraints: $\{\phi_\alpha,\phi_\beta\} = c_{\alpha\beta}{}^\gamma\phi_\gamma$, $\Rightarrow \widehat{c_{\alpha\beta}{}^\gamma\phi_\gamma}\,|\psi\rangle_{\text{phys}} = 0$
  Consistent (up to ordering)
- Second class constraints present: $\exists\,\alpha,\beta : \{\phi_\alpha,\phi_\beta\} \neq c_{\alpha\beta}{}^\gamma\phi_\gamma \quad \Rightarrow "\mathbb{1}"\,|\psi\rangle_{\text{phys}} = 0$
  Inconsistent

$\Rightarrow$ Only first class constraints can be treated using Dirac quantisation

Recall: Second class systems can be transformed to first class systems via gauge unfixing.

Subtleties for Dirac quantisation:

- Ordering problems may spoil the constraint algebra:
  $\widehat{c_{\alpha\beta}{}^\gamma\phi_\gamma} \neq \widehat{c_{\alpha\beta}{}^\gamma}\widehat{\phi_\gamma}$ in general, $\quad \widehat{c_{\alpha\beta}{}^\gamma\phi_\gamma}\,|\psi\rangle_{\text{phys}} \stackrel{?}{=} 0$
- On $\mathcal{H}_{\text{phys}}$, only quantum Dirac observables are well defined operators:
  $\hat f\,|\psi\rangle_{\text{phys}} \mapsto |\psi'\rangle_{\text{phys}} \quad \Rightarrow \quad (\hat f\hat\phi_\alpha - \hat\phi_\alpha\hat f)\,|\psi\rangle_{\text{phys}} = 0 \quad \Rightarrow \quad [\hat f,\hat\phi_\alpha] \propto \hat\phi_\beta$

  This may lead to problems when imposing the constraints iteratively. We will encounter this when quantising the constraints of GR.
- $[\hat\phi_\alpha,\hat\phi_\beta] = \widehat{c_{\alpha\beta}{}^\gamma}\widehat{\phi_\gamma}$ inconsistent with $\hat\phi_\alpha$ being self-adjoint and $\widehat{c_{\alpha\beta}{}^\gamma}$ anti-self-adjoint:
  $$[\hat\phi_\alpha,\hat\phi_\beta]^* = -[\hat\phi_\alpha,\hat\phi_\beta], \qquad \left(\widehat{c_{\alpha\beta}{}^\gamma}\widehat{\phi_\gamma}\right)^* = -\widehat{\phi_\gamma}\widehat{c_{\alpha\beta}{}^\gamma} \tag{6.8}$$
  Unless $\widehat{c_{\alpha\beta}{}^\gamma}$ commutes with $\hat\phi_\alpha$, but this is e.g. not true for general relativity.
- On $\mathcal{H}_{\text{phys}}$, the kinematical scalar product may not be well defined.
  $\rightarrow$ Need new scalar product, systematically e.g. via refined algebraic quantisation

Possible weakening of required structure:

- No scalar product on the kinematical representation space *i.e. no Hilbert space*
  $\rightarrow$ Scalar product only on $\mathcal{H}_{\text{phys}}$
- Quantum constraints not self-adjoint
  $\rightarrow$ Complex spectrum allowed as long as zero is included
- Solving constraints on the dual Hilbert space

  The dual Hilbert space $\mathcal{H}^*$ is much larger than $\mathcal{H}$ (for infinite-dimensional systems) and can contain a broader class of states which don't need to be normalisable w.r.t. to the naive transfer of the scalar product of $\mathcal{H}$

  An example is the solution to the spatial diffeomorphism constraint, where one (roughly) needs to average w.r.t. spatial diffeomorphisms. The dual Hilbert space can support the resulting state.

Variation: Master constraint: $M = \sum_\alpha \phi_\alpha^2$

- Different detection of Dirac observables: $[\hat{O}, [\hat{O}, M]] \approx 0$
- Technical advantages, e.g. existence of physical Hilbert space

6.2.3 Quantisation of second class systems

To quantise second class constraints, we need to weaken the condition on their imposition not to produce the above inconsistency

"Weak" imposition of constraints: $\left\langle \psi_\mathrm{w} \middle| \hat{\phi}_\alpha \middle| \psi'_\mathrm{w} \right\rangle = 0$

$|\psi_\mathrm{w}\rangle$ span the weak solution space $\mathcal{H}_\mathrm{w}$

In general: $\left\langle \psi_\mathrm{w} \middle| \hat{\phi}_\alpha \hat{\phi}_\beta \middle| \psi'_\mathrm{w} \right\rangle \neq 0 \Rightarrow$ No inconsistency, even for second class constraints

Weak imposition is therefore weaker than the above strong imposition a la Dirac

Examples:

- Gupta-Bleuler formalism for electrodynamics
- Virasoro constraints for quantum string

Quantisation of systems with constraints:

- The same problems as for quantisation without constraints also occur
- Additionally, there are several methods to impose the constraints
- Different schemes may lead to different quantum theories

7 Representation theory of SO(3)

The representation theory of SO(3) is key for loop quantum gravity. Much of the kinematical LQG structure rests on it.

Ashtekar-Barbero variables: local gauge invariance under SO(3)

Quantum theory: represent classical Poisson-algebraic structures
$\Rightarrow$ Need representation theory of SO(3)

Goal of section:

- Concept of Lie group / Lie algebra
- Representations of SO(3) / so(3)
- Invariant tensors and recoupling

Methodology:

- *Introduce concepts using $SO(3)$ as example*
- *Generalise to arbitrary Lie groups*

7.1 Lie groups

7.1.1 Group structure

v^i, w^i: vectors in 3-dim. Euclidean space

Rotation: $v^i \mapsto R^i{}_j v^j$.

Defining property: leaves scalar product (angles and lengths) invariant:

$$v^i w^j \delta_{ij} \mapsto R^i{}_k R^j{}_l \delta_{ij} v^k w^l \stackrel{!}{=} v^k w^l \delta_{kl} \tag{7.1}$$

$\Rightarrow R^i{}_k R^j{}_l \delta_{ij} = \delta_{kl} \quad \Leftrightarrow \quad R^T R = \mathbb{1}$

Products of rotations are rotations: $(R_1 R_2)^T (R_1 R_2) = R_2^T R_1^T R_1 R_2 = \mathbb{1}$

Inverse rotation $R^{-1} = R^T$.

Abstraction:

Definition 17. A group G is a non-empty set together with a multiplication operation satisfying the following properties:

1. Closure: $gh \in G, \quad \forall g, h \in G$
2. Associativity: $g(hk) = (gh)k, \quad \forall g, h, k \in G$
3. Identity element: $\exists e \in G : eg = ge = g, \quad \forall g \in G$
4. Inverse element: $\forall g \in G : \exists g^{-1} \in G : gg^{-1} = g^{-1}g = e$

Remarks:

- e is unique
- $gh \neq hg$ in general.
- Abelian group: $gh = hg$

Above: explicit realisation of rotations acting on vectors in $\mathbb{R}^3$

Group concept more general:

- Abstract concept: group
- Explicit realisation: representation on a vector space

Definition 18. A representation (ρ, V) of a group G on a vector space V is a map $\rho : G \to \mathrm{GL}(V)$ (from G to the linear invertible maps on V), so that

$$\rho(g)\rho(h) = \rho(gh) \quad \forall g, h \in G \tag{7.2}$$

Remarks:

- V is called "representation space"
- For $\dim V = n < \infty$, $GL(V) = GL(n)$ = invertible $n \times n$ matrices
- $\dim V$ is called the dimension of the representation
- Action of g on components of a vector $v = v^\alpha e_\alpha$, where e_α is a basis of V: $\rho(g)v = (\rho(g)^\alpha{}_\beta v^\beta) e_\alpha$
- We restrict to vector spaces over $\mathbb{R}$ or $\mathbb{C}$, but general fields are possible
- A representation is called faithful if ρ is injective *trivial rep.* $\rho(g) = \mathbb{1}$ *not faithful*
- (ρ, V) irreducible $\Leftrightarrow \nexists$ ρ-invariant subspace of V

Example for reducible representation:

$$\rho(g) = \begin{pmatrix} R(g) & 0_3 \\ 0_3 & R(g) \end{pmatrix} \tag{7.3}$$

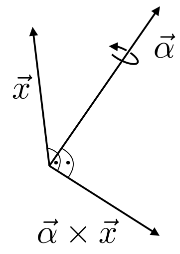
Question: are there more representations of the rotation group SO(3)?

Such questions are of crucial importance for physics!

Theoretical approach to elementary particle physics:

1. Find the underlying symmetry group
2. Study the representation theory of this group
3. Particles (= things that exist) transform under representations
4. Particle properties = representation properties

7.1.2 Manifold structure



Explicit realisation: Rotation around axis $\vec{\alpha}$ with angle $\alpha := |\vec{\alpha}|$:

$$x'^i = x^i \cos(\alpha) + \frac{\alpha_j x^j}{\alpha^2} \alpha^i (1 - \cos(\alpha)) + \epsilon^{ijk} \frac{\alpha_j x_k}{\alpha} \sin(\alpha) \tag{7.4}$$

Matrix realisation: $x'^i = R^i{}_j x^j$

$$R^i{}_j = \delta^i_j \cos(\alpha) + \frac{\alpha^i \alpha_j}{\alpha^2} (1 - \cos(\alpha)) - \epsilon^i{}_j{}^k \frac{\alpha_k}{\alpha} \sin(\alpha) \tag{7.5}$$

Conversely, reconstruct $\vec{\alpha}$ from $R^i{}_j$:

- $R^i{}_i = 1 + 2\cos\alpha$
- $R_{ij}\epsilon^{ijk} = -2(\alpha^k/\alpha)\sin\alpha$

Identification:

- Rotation around $\vec{n} = \vec{\alpha}/\alpha$ with angle $\alpha =$
- $=$ Rotation around $-\vec{n}$ with angle $2\pi - \alpha$

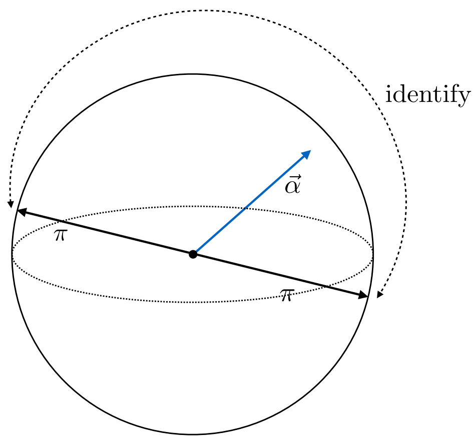


Figure 7.1: The topology of the rotation group SO(3) is depicted. SO(3) is topologically equivalent to a 3-ball with antipodal points on the boundary identified.

Topology:
SO(3) $\sim$ 3-Ball with antipodal identification $\sim \mathbb{RP}^3$

$\sim$ means diffeomorphic

SO(3) is connected, but not simply connected

- $\exists$ path from every point to every other point
- $\exists$ non-contractible loops = $\exists$ distinct paths between two points which are not continuously deformable into each other

$\Rightarrow$ SO(3) is a manifold

The group of permutations, or discrete symmetry groups, are not manifolds

Definition 19. A Lie group is a group G with the additional structure of a differentiable manifold, so that the group multiplication and inversion are smooth maps.

7.2 Lie Algebras

7.2.1 Infinitesimal Rotations

Infinitesimal transformations are a key technique in the study of Lie groups. The structure of infinitesimal transformations determines the group to a large extend.

Infinitesimal rotation: $R = \mathbb{1} + \Lambda$

$R^T R = \mathbb{1} \quad \Rightarrow \quad \Lambda^T = -\Lambda$

$$\Lambda = \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix} =: \alpha^i \Lambda_i \tag{7.6}$$

$$\Lambda_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Lambda_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.7}$$

$(\Lambda_i)_{jk} = -\epsilon_{ijk}$

We note the conceptual difference between the index i and jk.

Λ_i generates rotations around the i-axis.

Relation to finite transformation:

$R(\vec{\alpha}) = R(\vec{\alpha}/2)R(\vec{\alpha}/2) = \ldots = R(\vec{\alpha}/N)^N$

For $N \to \infty$: $R(\vec{\alpha}/N) \approx \mathbb{1} + \alpha^i \Lambda_i/N$

$\Rightarrow \quad R(\vec{\alpha}) = \lim_{N\to\infty} \left(\mathbb{1} + \alpha^i \Lambda_i/N\right)^N = \exp\left(\alpha^i \Lambda_i\right)$

Linear combinations of the Λ_i also generate rotations

What about products?

$(\Lambda_1 \Lambda_2)^T = \Lambda_2 \Lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq \alpha^i \Lambda_i$

Need antisymmetry:

$$\begin{aligned} & \left(\Lambda_i \Lambda_j - \Lambda_j \Lambda_i\right)_{mo} \\ = \; & \epsilon_{imn}\epsilon_{jno} - \epsilon_{jmn}\epsilon_{ino} \\ = \; & \delta_{io}\delta_{jm} - \delta_{ij}\delta_{om} - \delta_{oj}\delta_{mi} + \delta_{ij}\delta_{om} \\ = \; & -\epsilon^{ijk}\epsilon_{kmo} = \epsilon^{ijk}\left(\Lambda_k\right)_{mo} \end{aligned} \tag{7.8}$$

The commutator $[\Lambda_i, \Lambda_j] = \Lambda_i \Lambda_j - \Lambda_j \Lambda_i$ of two generators is again a generator

Abstraction:

7.2.2 Lie Algebras

Consider Λ_i as a specific realisation of the generators of an abstract algebra:

Definition 20. A Lie algebra is a vector space $\mathfrak{g}$ together with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x, y) \mapsto [x, y]$, called Lie bracket, that satisfies

1. Bi-linearity: $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y] \quad \forall x, y, z \in \mathfrak{g},\; a, b \in \mathbb{C}$
2. $[x, x] = 0 \quad \forall x \in \mathfrak{g}$.
3. Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in \mathfrak{g}$

1. + 2. $\Rightarrow [x, y] = -[y, x]$

Remarks:

- Lie bracket not associative: $[[x, y], z] \neq [x, [y, z]]$ in general
- Flexibility law: $[x, [y, x]] = [[x, y], x]$

We consider only finite-dimensional Lie algebras.

Choose basis $\{X_A\}$ of Lie algebra, $A = 1, \ldots, n = \dim \mathfrak{g}$

Lie bracket: $[X_A, X_B] = f^C{}_{AB}\ X_C$

$f^C{}_{AB}$ are called structure constants of $\mathfrak{g}$.

Properties:

- Antisymmetry: $f^C{}_{AB} = -f^C{}_{BA}$
- Jacobi identity: $f^D{}_{AE} f^E{}_{BC} + f^D{}_{BE} f^E{}_{CA} + f^D{}_{CE} f^E{}_{AB} = 0$

As before: Explicit realisation on vector space = representation.

Definition 21. A representation of a Lie algebra $\mathfrak{g}$ on a vector space V is a map $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, so that

$$\rho(x)\rho(y) - \rho(y)\rho(x) = \rho([x, y]) \ \ \forall x, y \in \mathfrak{g} \tag{7.9}$$

$\mathfrak{gl}(V)$ = endomorphisms of V = linear maps from V to V

These do not need to be invertible. This notation makes sense since the invertible general linear group $GL(V)$ is obtained by exponentiating $\mathfrak{gl}(V)$, which makes the maps invertible.

Idea: study representations of the Lie algebra (find the generators) and exponentiate to obtain the Lie group.

Notation: Lie group: SO(3), Lie algebra so(3), often $\mathfrak{so}(3)$.

7.2.3 Casimir operators

Physical interest: $\mathfrak{g}$-invariant objects

E.g. invariant under some symmetries encoded in $\mathfrak{g}$.

For so(3): $\Lambda_i \Lambda_j \delta^{ij} = -2\ \mathbb{1}$

$\mathbb{1}$ commutes with all elements of $\mathfrak{g}$.

For general Lie algebras: Need analogue of δ^{ij}

How can this be done?

The available structure so far are only the structure constants. We can define:

Killing-Cartan tensor: $g_{AB} := f^C{}_{DA} f^D{}_{CB}$

Restrict to g_{AB} non-degenerate (semi-simple Lie algebras)

Inverse: g^{AB}

Definition 22. The quadratic Casimir operator C of a Lie algebra is defined as $X_A X_B g^{AB}$.

Remarks:

- $[C, x] = 0 \; \forall x \in \mathfrak{g}$ (exercises)
- $f_{ABC} := g_{AD} f^D{}_{BC}$ is totally antisymmetric (exercises)
- Higher order (in X_A) Casimir operators exist in general
- Irreducible representation: $C \propto \mathbb{1}$
  Otherwise, a non-trivial invariant subspace exists where C acts differently, e.g. $C = diag(2, 1)$.

  Proportionality constant used to classify the representation.
  Example: mass and spin of a particle (Poincaré group).
  one quadratic and one quartic Casimir operator

7.3 Unitary irreducible representations of SO(3)

7.3.1 Simplifying facts

For quantum mechanics: representation on Hilbert space $\mathcal{H}$.

Additional structure: scalar product $\langle \cdot, \cdot \rangle$ *As opposed to vector space*

Physically plausible: action of symmetry leaves physics invariant

Unitary representation: $\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle \;\; \forall v, w \in \mathcal{H}$.

Simplifying facts from theory of Lie groups:

- Every continuous representation of a compact Lie group in a Hilbert space is equivalent to a unitary representation
  $\rightarrow$ *enough to consider unitary representations*
- Every continuous unitary representation of a compact Lie group is a direct orthogonal sum of irreducible sub-representations
  $\rightarrow$ *enough to consider irreducible representations, take orthogonal sums afterwards*
- Every continuous irreducible representation of a compact Lie group in a Hilbert space is finite-dimensional
  $\rightarrow$ *enough to consider finite-dimensional representations*

7.3.2 Classification of so(3) representations

Strategy:

1. Derive some necessary conditions on representations of algebra while assuming their existence

2. Verify existence by explicitly constructing the representation using the necessary conditions
3. Exponentiate to obtain the group representations

Unitary representation:
$\langle v, w\rangle = \langle \rho(g)v, \rho(g)w\rangle = \langle \rho(g)^*\rho(g)v, w\rangle \quad \Rightarrow \rho(g)^*\rho(g) = \mathbb{1}$

Infinitesimal rotation: (generic representation)

$\rho(g) = \mathbb{1} + \alpha^i\Lambda_i \Rightarrow \Lambda_i^* = -\Lambda_i$

Alternative definition: $\rho(g) = \mathbb{1} - i\alpha^i J_i \Rightarrow J_i^* = J_i$

Generators are hermitian (J_i) or anti-hermitian (Λ_i).

Choose generators J_i hermitian: $J_i^* = J_i$

$[J_i, J_j] = i\epsilon_{ijk}J_k$

Casimir operator: $J^2 := J_iJ_j\delta^{ij} = \lambda\mathbb{1}, \quad \lambda > 0.$

Analyse spectrum of another operator, J_3, that can be diagonal simultaneously.

Ladder operators: $J_\pm := J_1 \pm iJ_2 = J_\mp^*$

$[J_+, J_-] = 2J_3, \qquad [J_3, J_\pm] = \pm J_\pm, \qquad J^2 = J_\pm J_\mp \mp J_3 + J_3^2$

Let x_m be an eigenvector for J_3 with

$$J_3 |x_m\rangle = m |x_m\rangle\,, \quad \langle x_m \mid x_m\rangle = 1 \tag{7.10}$$

As a hermitian matrix, J_3 has to have $n = \dim\mathcal{H}$ eigenvalues

Properties of vectors $J_\pm |x_m\rangle$

- $J_3J_\pm |x_m\rangle = (J_\pm J_3 + [J_3, J_\pm]) |x_m\rangle = (m \pm 1)J_\pm |x_m\rangle$
- $\langle J_\pm x_m \mid J_\pm x_m\rangle = \lambda \mp m - m^2$

Either $J_\pm |x_m\rangle = 0$, or $(m \pm 1)$ is also an eigenvalue of J_3

Representation finite-dimensional: $\exists$ largest eigenvalue j.

$J_+ |x_j\rangle = 0, \qquad \lambda = j + j^2$

Use J_- to construct further eigenvectors: $J_- |x_j\rangle, (J_-)^2 |x_j\rangle, \ldots$

Finite number of eigenvalues:

$\exists N \geq 1 :\ (J_-)^{N-1} |x_j\rangle \neq 0,\ (J_-)^N |x_j\rangle = 0$

$J_3(J_-)^{N-1}\left|x_j\right\rangle = j'(J_-)^{N-1}\left|x_j\right\rangle$

$\Rightarrow \lambda = (j')^2 - j', \qquad j-(N-1) = j'$

$\Rightarrow N = 2j+1, \quad j = -j'$

$\Rightarrow j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ *j is called the "highest weight" of the representation*

Eigenvalues: $m = -j, -j+1, \dots, j-1, j$

Eigenvectors: $\left|x_j\right\rangle, J_-\left|x_j\right\rangle, \dots, (J_-)^{2j}\left|x_j\right\rangle$
Span subspace invariant under J_3, J_-.

J_+ invariance: $J_+(J_-)^p\left|x_j\right\rangle = J_+J_-(J_-)^{p-1}\left|x_j\right\rangle = (J^2+J_3-J_3^2)(J_-)^{p-1}\left|x_j\right\rangle \propto (J_-)^{p-1}\left|x_j\right\rangle$
$\Rightarrow$ subspace invariant under all generators

Representation irreducible $\Rightarrow \mathcal{H} = \mathrm{span}(\left|x_j\right\rangle, J_-\left|x_j\right\rangle, \dots, (J_-)^2 j\left|x_j\right\rangle)$

There cannot be any other eigenvectors and eigenvectors cannot be degenerate. Otherwise, we could build a new ladder from those, and thus an invariant subspace.

Up to here: assumed existence of representation.

Now: Compute matrix elements of $J_1, J_2, J_3 \Leftrightarrow J_+, J_-, J_3$

$J_3\left|x_m\right\rangle = m\left|x_m\right\rangle$

$J_\pm\left|x_m\right\rangle = \sqrt{j(j+1) \mp m - m^2}\,|x_{m\pm1}\rangle, \qquad \langle x_m, x'_m\rangle = \delta_{m,m'}$

The matrices $(J_i)_{mm'} = \langle x_m \mid J_i \mid x_{m'}\rangle$ provide representations of the Lie algebra so(3).

We don't check this explicitly here, but it can be done.

Examples:

- $j=0$:

  $J_1 = J_2 = J_3 = 0$, trivial representation, 1-dimensional, Casimir eigenvalue $j(j+1) = 0$

- $j=1$:

$$J_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_1 := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 := \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{7.11}$$

  Defining or adjoint representation, dimension 3, Casimir eigenvalue $j(j+1) = 2$

Adjoint rep.: G acts on its algebra as hXh^{-1}, $X \in \mathfrak{g}$, $h \in G$. The dimension of this representation is the same as the number of generators, here $2j+1=3$.

Equivalent to defining representation $\Lambda_1, \Lambda_2, \Lambda_3$.

Equivalent representation: $\exists$ vector space isomorphism $S: \; SJ_iS^{-1} = \Lambda_i$

- $\underline{j = 1/2}$:

$J_i = \sigma_i/2, \quad \sigma_i$: Pauli matrices

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{7.12}$$

Anti-commutator relation: $[\sigma_i, \sigma_j]_+ := \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1}$

- $\underline{j = 2}$:

5-dim. representation on traceless symmetric tensors T^{ij}, $T^{ij}\delta_{ij} = 0$

Infinitesimally:

$$\left(\mathbb{1} + \tau\alpha^i\Lambda_i\right) \otimes \left(\mathbb{1} + \tau\alpha^j\Lambda_j\right) \approx \mathbb{1} \otimes \mathbb{1} + \tau\alpha^i \left(\mathbb{1} \otimes \Lambda_i + \Lambda_i \otimes \mathbb{1}\right) =: \mathbb{1} \otimes \mathbb{1} + \tau\alpha^i t_i$$

$$t^i t_i = \Lambda^2 \otimes \mathbb{1} + \mathbb{1} \otimes \Lambda^2 + 2\Lambda_i \otimes \Lambda^i$$

Need action of $\Lambda_i \otimes \Lambda^i$, rest known:

We are now somewhat sloppy with the upper / lower indices:

$$\left(\Lambda_i \otimes \Lambda^i\right)^{jk}{}_{mn} = \epsilon^{ijm}\epsilon_{ikn} = \delta^j_k\delta^m_n - \delta^j_n\delta^m_k$$

$$\left(\Lambda_i \otimes \Lambda^i\right)^{jk}{}_{mn}T^{mn} = \delta^{jk}T^m{}_m - T^{kj} = -T^{jk}$$

$$\left(t^i t_i\right)^{jk}{}_{mn}T^{mn} = -2T^{jk} - 2T^{jk} - 2T^{jk} = -6T^{jk}$$

$j(j+1) = 6$, Casimir $C = -6\mathbb{1}$. *The normalisation of C is a choice of convention.*

- $\underline{j = n \in \mathbb{N}}$:

Traceless symmetric rank n tensors (exercises)

7.4 Group representations and SU(2)

Up to now: Lie algebra representations

Strategy for Lie group representation: exponentiate

$R(\vec{\alpha}) = \exp(\alpha^i t_i), \quad \alpha^i t_i = \frac{d}{d\tau}\big|_{\tau=0} \exp(\tau \alpha^i t_i)$

Already known:

- $j = 0$: 1-dim. rep., $t_i = 0$, $R(\alpha) = 1$
- $j = 1$: 3-dim. rep., $t_i = \Lambda_i$, $R^i{}_j = \delta^i_j \cos(\alpha) + \frac{\alpha^i \alpha_j}{\alpha^2}(1 - \cos(\alpha)) - \epsilon^i{}_j{}^k \frac{\alpha_k}{\alpha} \sin(\alpha)$
- $j \in \mathbb{N}$: derives from $j = 1$ (exercise)

New: $j = 1/2$

Hermitian generators: $J_j := i t_j$, $[J_j, J_k] = i\epsilon_{jkl} J_l$

$\alpha^i J_i = \frac{1}{2}\alpha^i \sigma_i$

We again compute powers of the generators to find a recurring pattern which allows us to evaluate the exponential explicitly:

$(\alpha^i \sigma_i)^2 = \alpha^i \alpha^j \sigma_i \sigma_j = \alpha^2 \mathbb{1}$

With $\alpha^i = \alpha n^i$, $n^i n_i = 1$:
$(-i\alpha^j J_j)^2 = -\left(\frac{\alpha}{2}\right)^2 \mathbb{1}, \quad (-i\alpha^j J_j)^3 = i\left(\frac{\alpha}{2}\right)^3 n^i \sigma_i$

$$\begin{aligned} U(\vec{\alpha}) &:= \exp(-i\alpha^j J_j) = \mathbb{1}\cos(\alpha/2) - i n^j \sigma_j \sin(\alpha/2) && (7.13) \\ &= \begin{pmatrix} \cos(\alpha/2) - i n_3 \sin(\alpha/2) & -i(n_1 - i n_2)\sin(\alpha/2) \\ -i(n_1 + i n_2)\sin(\alpha/2) & \cos(\alpha/2) + i n_3 \sin(\alpha/2) \end{pmatrix} && (7.14) \end{aligned}$$

Unitary, $U^\dagger U = \mathbb{1}$, and unimodular, $\det U = 1$, matrix

The adjoint on the representation space is simply transpose + conjugate, denoted by †

New feature: $U(2\pi\vec{n}) = -\mathbb{1}, \quad U(\alpha\vec{n}) = -U(-(2\pi - \alpha)\vec{n})$

- $U(\alpha\vec{n})$ representation of SO(3) only for α sufficiently small
- $0 \le \alpha \le 2\pi$
  SU(2) is a double cover of ("twice as large as") SO(3)

  There exists a group homomorphism from SU(2) to SO(3) which hits every point in SO(3) twice.

  For each element in SO(3), there are two SU(2) elements

The $j = 1/2$-rep. is called a two-valued representation of SO(3)

These reps. are needed for physics (fermions)!

Topological structure of U(α)

- $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}$
- Unitarity: $|a|^2 + |b|^2 = 1 = |c|^2 + |d|^2, \quad a = \lambda \bar{d}, \quad \bar{c} = -\lambda b$
- Unimodularity: $ad - bc = 1, \quad \Rightarrow \lambda = 1$
- Independent relation $|a|^2 + |b|^2 = 1 \quad \Rightarrow$ 3-sphere S^3

  3-Ball with radius 2π with boundary identified as one point corresponding to $-\mathbb{1}$ in the group

Comparison with SO(3):

${R^i}_j(\pi\vec{n}) = -{\delta^i}_j + 2n^i n_j$ depends on n^i up to a sign, $\rightarrow \mathbb{RP}3$.
Opposite points of the 3-ball boundary at Radius π are identified.

$U(2\pi\vec{n}) = -\mathbb{1}_2$ *does not have this dependence, thus a different topology.*

a, b map to $\vec{\alpha}$

- $|\operatorname{Re}(a)| \leq 1 \Rightarrow \exists! \alpha \in [0, 2\pi] : \; \cos(\alpha/2) = \operatorname{Re}(a)$
- $\operatorname{Im}(a) = -n_3 \sin(\alpha/2)$ determines n_3 (except for $\sin(\alpha/2) = 0$, where U is independent of $\vec{n}$.)
- b determines n_1, n_2

From now on: talk about representations of SU(2)

- SU(2) = universal covering group of SO(3)
- The Lie algebras su(2) and so(3) coincide
- SU(2) is simply connected (no non-contractible loops)
  The antipodal identification we had in $SO(3)$ is dropped for $SU(2)$
- $j = 1/2$ is the defining representation
- Representations with $j \in \mathbb{N}$ are not faithful, i.e. $\rho(g)$ is not injective

7.5 Recoupling theory

Generalisation of addition of angular momentum in quantum mechanics. What is the product (as opposed to direct sum) of two representations?

7.5.1 Dual representations

Element in representation space V: vector $v = v^\alpha e_\alpha$ for some basis e_α

Construct dual vector space V^* = space of linear functionals on V

$w = w_\alpha e^\alpha \in V^*$, with dual basis e^α, $e^\alpha(e_\beta) = \delta^\alpha_\beta$

$w(v) = w_\alpha v^\alpha$

Idea: Construct representation on V^* that leaves $w(v)$ invariant

$v^\alpha \mapsto \rho(g)^\alpha{}_\beta v^\beta$

$\Rightarrow \;\; w_\alpha \mapsto \rho(g^{-1})^\beta{}_\alpha w_\beta$

Then, $w_\alpha v^\alpha \mapsto w_\gamma \rho(g^{-1})^\gamma{}_\alpha \rho(g)^\alpha{}_\beta v^\beta = w_\gamma \rho(g^{-1}g)^\gamma{}_\beta v^\beta = w_\alpha v^\alpha$

Definition 23. Given a representation (ρ, V) of a group G, the dual representation (ρ^*, V^*) is defined as $\rho^*(g) = \rho(g^{-1})^T$.

Check representation property:
$\rho^*(g_1)\rho^*(g_2) = \rho(g_1^{-1})^T \rho(g_2^{-1})^T = \left(\rho(g_2^{-1})\rho(g_1^{-1})\right)^T = \left(\rho(g_2^{-1}g_1^{-1})\right)^T = \rho\left((g_1 g_2)^{-1}\right)^T = \rho^*(g_1 g_2)$

7.5.2 Intertwiners

Recall: representation irreducible = no invariant subspaces

Example for reducible representation: symmetric tensors T^{ij}

Trace part $T^{ij}\delta_{ij}$ is invariant under rotations, trivial $j = 0$ rep. .

Trace free part is an irreducible representation with $j = 2$.

$\Rightarrow$ the rep. on symmetric tensors decomposes into the direct sum of the $j = 0$ and $j = 2$ reps.

Dimensions: $6 = 5 + 1$

For general 2-tensors, we additionally have the 3-dimensional anti-symmetric part. Then, $3 \otimes 3 = 5 \oplus 3 \oplus 1$.

$\rightarrow$ Generalise this concept to arbitrary representations.

General (completely reducible) representation: $\rho(g) = \bigoplus_k \rho_{j_k}(g)$

Representation space decomposes: $V = \bigoplus_k V_{j_k}$

Needed: decomposition of vectors in sub-vectors $\vec{v}_k$:

$v^\alpha = \sum_k (c_k)^\alpha{}_\beta \; v_k^\beta$

Simplest case: $\rho(g) = \rho_{j_1}(g) \otimes \rho_{j_2}(g)$

c.f. angular momentum in quantum mechanics.

Solution: Clebsh-Gordan coefficients:

$$|j_1, m_1\rangle \, |j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{|j_1+j_2|} \sum_{m=-j}^{j} |j, m\rangle \underbrace{\langle j, m \mid j_1, m_1; j_2, m_2\rangle}_{C^{j,m}_{j_1,m_1;j_2,m_2}} \tag{7.15}$$

Remember that m here is the label we call α above, i.e. m belongs to a specific basis, while α is more general

$|j, m\rangle$ is a basis vector and thus corresponds to a lower index m, and vice versa for $\langle j, m|$

Remarks:

- Each j only once in decomposition of tensor product of two js
  This is true for $SU(2)$, but not for general groups!

- $C^{j,m}_{j_1,m_1;j_2,m_2}$ can be computed or looked up
- Invariance under the group action:

  From $\langle j, m \mid j_1, m_1; j_2, m_2\rangle = \left\langle j, m \,\middle|\, \rho(g^{-1})\rho(g) \,\middle|\, j_1, m_1; j_2, m_2\right\rangle$

  Action on basis elements: *considered as vectors with one non-zero component*

  - $\rho(g)v = (\rho(g)^\alpha{}_\beta v^\beta) e_\alpha = v^\beta \; (\rho(g)^\alpha{}_\beta e_\alpha)$

  - $\rho(g)w = (\rho(g^{-1})^\beta{}_\alpha w_\beta) e^\alpha = w_\beta \; \left(\rho(g^{-1})^\beta{}_\alpha e^\alpha\right)$

  $$\left(C^j_{j_1,j_2}\right)^\alpha{}_{\alpha_1\alpha_2} \mapsto \rho(g^{-1})^\alpha{}_\beta \left(C^j_{j_1,j_2}\right)^\beta{}_{\beta_1\beta_2} \rho(g)^{\beta_1}{}_{\alpha_1} \rho(g)^{\beta_2}{}_{\alpha_2} = \left(C^j_{j_1,j_2}\right)^\alpha{}_{\alpha_1\alpha_2}$$

  With $g \mapsto g^{-1}$, usual invariance condition on vector / co-vector indices

  $C^j_{j_1,j_2} \in \mathrm{Inv}(V_{j_1} \otimes V_{j_2} \otimes V_j^*)$, $\quad \mathrm{Inv}(\ldots)$ = space of g-invariant maps from $(\ldots)$ to $\mathbb{C}$

- $C^j_{j_1,j_2}$ is called an intertwiner (invariant map)
  It connects different representations in a g-invariant way

How to construct more intertwiners?
It would be good to have two-valent intertwiners and use those along with the Clebsh-Gordan coefficients to build up higher ones by contraction.

Two-valent intertwiners: (exercises)

- $\epsilon_j^{m,m'} := (-1)^{j-m}\delta^{m,-m'} \in \mathrm{Inv}(V_j^* \otimes V_j^*)$ *Note that* $m \in \{-j, -j+1, \ldots, j\}$
- $\epsilon_{j;m,m'} := (-1)^{j-m}\delta_{m,-m'} \in \mathrm{Inv}(V_j \otimes V_j)$
- Trivial intertwiner: $\delta^m_{m'} \in \mathrm{Inv}(V_j \otimes V_j^*)$
- These are the only two-valent intertwiners up to equivalence

3J-symbol: $\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = C^{j,m'}_{j_1,m_1;j_2,m_2}\epsilon_{j;m',m}$ *Many symmetries*

Fact for SU(2) (without proof):

$$\rho_{j_1}(g) \otimes \rho_{j_2}(g) = \bigoplus_{j=|j_1-j_2|}^{|j_1+j_2|} \rho_j(g) \tag{7.16}$$

In words: Every irreducible representation appears only once when decomposing the tensor product of two representations.

$\Rightarrow$ The Clebsh-Gordan coefficients constitute all three-valent intertwiners (up to equivalence and lower / upper index position)

Higher intertwiners can all be built from contracting three-valent intertwiners:

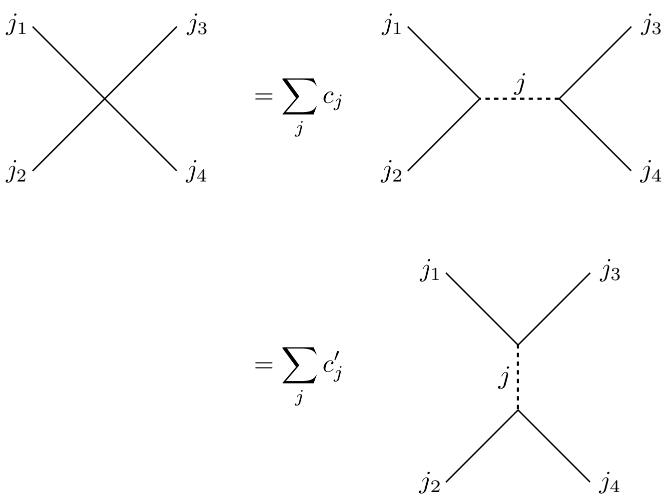
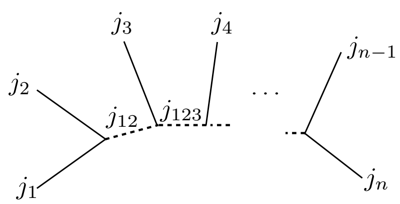


Figure 7.2: A 4-valent intertwiner can be written as a sum over an intermediate recoupling.

Similar for n-valent intertwiners:



We get all higher-valent intertwiners this way.

Many relations and asymptotics ($j \to \infty$) known among these intertwiners, including graphical calculus. Details not important for this lecture.

7.6 Harmonic analysis on SU(2)

The Hilbert space in loop quantum gravity will be build out of $SU(2)$ representations. For this, we need an orthogonality relation among different representations and a generalisation of the Fourier transform to functions on $SU(2)$.

7.6.1 Haar measure

So far no integration on groups. For scalar product, we would like to integrate functions of group elements (parallel transports of the connection).

Integration of a function $f : G \to \mathbb{C}$ over a Lie group G.

Need to define $\int_G d\mu(g) f(g)$

- Can use topology of the Lie group for σ-algebras to develop measure theory
- Need to select a certain measure μ

Invariance criterion: $\int_G d\mu(g) f(g) = \int_G d\mu(g) f(hg) = \int_G d\mu(h^{-1}g) f(g)$

Theorem 6. Let G be a compact Lie group and $f : G \to \mathbb{C}$. There exists a measure, the Haar measure μ_H, so that

- $\int_G d\mu_H(g) = 1$ (Normalisation)
- $\int_G d\mu_H(g) f(g) = \int_G d\mu(g) f(hg) = \int_G d\mu(g) f(gh)$ (Left- and right-invariance)

$\mu(g)$ can be explicitly computed, but its precise form is irrelevant for this lecture.

What is important for the quantum theory are only some orthogonality relations which we will state now.

Wigner matrices $(D^j)^m{}_n(g) := (\rho(g)_j)^m{}_n$
Just the matrix elements in the (j, m) basis

$d_j := 2j + 1 =$ dim. of representation space with label j

Important relation for later:

$$\int_{\mathrm{SU}(2)} d\mu_H(g) \overline{(D^j)^m{}_n(g)} (D^{j'})^{m'}{}_{n'}(g) = \frac{1}{d_j} \delta^{j,j'} \delta^{m,m'} \delta_{n,n'} \tag{7.17}$$

In words: 2 representations with $j \neq j'$ are orthogonal in the above sense

The scalar product in the Hilbert space will later be constructed using the Haar measure μ_H, so that this relation will translate into an orthogonality relation for different quantum states, *which will be labelled by irreducible representations j.*

$\to$ Orthogonality relations for the Hilbert space

$\to$ Similar orthonormal basis for intertwiner space

7.6.2 Peter-Weyl Theorem

The Wigner matrices provide us with an orthonormal set of functions on $SU(2)$. Do they form a complete basis?

Theorem 7. **(Peter and Weyl)** The matrices $\sqrt{d_j}(D^j)^m{}_n(g)$ form a complete orthonormal basis of of $L^2(\mathrm{SU}(2), d\mu_H)$. The theorem generalises to arbitrary compact Lie groups.

$\Rightarrow$ Analogue of Fourier transform

$$f \in L^2(\mathrm{SU}(2), d\mu_H) \quad \Rightarrow \quad f(g) = \sum_{j=0,\frac{1}{2},\dots}^{\infty} \sum_{m,n=-j}^{j} \underbrace{(f_j)_m{}^n}_{\text{Fourier components}} \underbrace{\sqrt{d_j}(D^j)^m{}_n(g)}_{\text{waves}} \tag{7.18}$$