Problems to Introduction to Quantum Gravity I WS 18/19

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Sheet 14 — Handout: 07.03.2019 — To present: 25.04.2019

Problem 1: ADM Poisson brackets in triad variables

Show that
$$\Big\{q_{ab}[E,K](x), P^{cd}[E,K](y)\Big\}_{\{K,E\}} = \delta^c_{(a}\delta^d_{b)}\delta^{(3)}(x,y) \tag{1}$$
i.e. that the Poisson brackets of the ADM variables q_{ab}, P^{ab} are reproduced by the Poisson brackets of the triad variables.

Hint: $0 = \{\delta^b_a, f\} = \{q_{ac}q^{cb}, f\} = q_{ac}\{q^{cb}, f\} + q^{cb}\{q_{ac}, f\}$ and $\{\det q, f\} = qq^{ab}\{q_{ab}, f\}$ for arbitrary phase space functions f.

BONUS question: Show that $\left\{P^{ab}[E,K](x), P^{cd}[E,K](y)\right\}_{\{K,E\}} = G_{ij}[...]$.

Problem 2: Hamiltonian constraint

Show that the Hamiltonian constraint (for $\kappa = 1$) can be written as
$$\mathcal{H}[N] = \int_\Sigma d^3x\, N\left(\beta^2\frac{{}^{(\beta)}E^{ai\,(\beta)}E^{bj}}{2\sqrt{q}}\epsilon^{ijk}F^k_{ab}({}^{(\beta)}A) - \frac{(1+\beta^2)}{\sqrt{q}}{}^{(\beta)}K^i_{[a}{}^{(\beta)}K^j_{b]}{}^{(\beta)}E^{ai\,(\beta)}E^{bj}\right) \tag{2}$$
up to terms proportional to the Gauß law. Here, ${}^{(\beta)}E^a_i = \frac{1}{\beta}E^a_i$, ${}^{(\beta)}K^i_a = \beta K^i_a$, and ${}^{(\beta)}A^i_a = \Gamma^i_a + \beta K^i_a$.

Hints: Remember that $\Gamma_{aij} = -\epsilon_{ijk}\Gamma^k_a$, for which the defining equation reads $\partial_a e^i_b - \Gamma^c_{ab}e^i_c + \Gamma_a{}^i{}_k e^k_b = 0$. We define the field strengths $R(\Gamma)_{abij} = 2\partial_{[a}\Gamma_{b]ij} + [\Gamma_a, \Gamma_b]_{ij}$ and $R_{abi} = 2\partial_{[a}\Gamma^i_{b]} + \epsilon^{ijk}\Gamma^j_a\Gamma^k_b$. What is the relation between R_{abi} and R_{abij}? Show that $R_{abij}e^i_c e^j_d = R_{abcd}$, where R_{abcd} is the Riemann tensor defined with the convention $[\nabla_a, \nabla_b]u_c = R_{abc}{}^d u_d$. Relate R^i_{ab} to F^i_{ab}.

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Problem 3: Gauß law in connection variables

Show that $-\int d^3x \Lambda_k \epsilon^{ijk} E^a_{[i} K_{a|j]} = \int d^3x \Lambda^k D_a E^a_k$.

Hint: This exercise suggests a special form of the covariant divergence of densitized vectors: $\nabla_a \left(\sqrt{q} v^a\right) = ?$

Problem 4: BONUS: Canonical connection variables

To show that A^i_a, E^b_j is a canonical pair, it was left to show that

$$\left\{\Gamma^i_a(x), K^j_b(y)\right\} + \left\{K^i_a(x), \Gamma^j_b(y)\right\} = 0. \tag{3}$$

This can be done either by brute force or by following the hints below. Boundary terms can be neglected throughout.

a) Show that the equation would be satisfied if Γ^i_a has a generating potential, i.e. $\Gamma^i_a(x) = \frac{\delta F}{\delta E^a_i(x)}$

b) Construct a candidate for F (The simplest possible will do.)

c) Show that the candidate for F is indeed a potential for Γ^i_a.

Special bonus question: What goes wrong if we try to construct similar variables in higher dimensions?

Special bonus question: What happens to the canonical variables at the boundary of the spatial slice if we keep track of all boundary terms?

Special Bonus question: Is there any other useful way to construct canonical connection variables?