Problems to Introduction to Quantum Gravity I WS 18/19

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Sheet 13 — Handout: 29.01.2019 — To present: 04.02.2019

Problem 1: Constraint algebra

Show that the Hamiltonian and spatial diffeomorphism constraints

$$\mathcal{H} = \frac{2\kappa}{\sqrt{q}}\left(P^{ab}P_{ab} - \frac{1}{2}P^2\right) - \frac{\sqrt{q}}{2\kappa}R \tag{1}$$

$$\mathcal{H}_a = -2q_{ac}\nabla_b P^{bc} \tag{2}$$

satisfy the hypersurface deformation algebra

$$\{\mathcal{H}[M], \mathcal{H}[N]\} = \mathcal{H}_a\left[q^{ab}\left(M\partial_b N - N\partial_b M\right)\right] \tag{3}$$

$$\{\mathcal{H}[M], \mathcal{H}_a[N^a]\} = -\mathcal{H}\left[\mathcal{L}_N M\right] \tag{4}$$

$$\{\mathcal{H}_a[M^a], \mathcal{H}_a[N^a]\} = -\mathcal{H}_a\left[\mathcal{L}_N M^a\right]. \tag{5}$$

Hint: For (3), use that

$$\delta R^{(3)} = -R^{ab}_{(3)}\delta q_{ab} - q^{ab}(\nabla_c\nabla^c\delta q_{ab}) + (\nabla^a\nabla^b\delta q_{ab}). \tag{6}$$

For the other Poisson brackets, use the properties of the Lie derivative generated by $\mathcal{H}_a$.

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Problem 2: BONUS: Variational identifies

For general spatial dimension D, show that

$$\delta R^{(D)} = -R^{ab}_{(D)}\delta q_{ab} - q^{ab}(\nabla_c\nabla^c\delta q_{ab}) + (\nabla^a\nabla^b\delta q_{ab}) \tag{7}$$

and

$$\delta(\nabla_c\nabla^c\Phi) = -(\nabla^a\nabla^b\Phi)\delta q_{ab} + (\nabla_c\nabla^c\delta\Phi) - (\nabla^b\Phi)(\nabla^a\delta q_{ab}) + \frac{1}{2}q^{ab}(\nabla^c\Phi)(\nabla_c\delta q_{ab}) \tag{8}$$

where Φ is a scalar field so that $\nabla_c\Phi = \partial_c\Phi$ is a co-vector.

Hint: Show that

$$\delta\Gamma^c_{ab} = \frac{1}{2}q^{cd}\left(\nabla_a q_{db} + \nabla_b q_{da} - \nabla_d q_{ab}\right) \tag{9}$$

and

$$\delta R^{(D)c}_{abd} = 2\nabla_{[a}\delta\Gamma^c_{b]d}. \tag{10}$$