Problems to Introduction to Quantum Gravity I

WS 18/19

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Sheet 9 — Handout: 18.12.2018 — To present: 07.01.2019

Problem 1: Covariant divergence

a) Conclude from $\nabla_\mu g_{\rho\sigma} = 0$ that $\nabla_\mu(-g) = 0$, where $g = \det g_{\mu\nu}$. Show furthermore that this implies $\partial_\mu \sqrt{-g} - \sqrt{-g}\Gamma^\nu_{\mu\nu} = 0$.

b) Show that $\nabla_\mu(\sqrt{-g}v^\mu) = \partial_\mu(\sqrt{-g}v^\mu)$ for arbitrary vectors v^μ.

c) Conclude that covariant derivatives can be partially integrated if the integrand has density weight one, i.e. contains one power of $\sqrt{-g}$, and no uncontracted indices.

Problem 2: Properties of the Lie derivative

a) Show that the Lie derivative is compatible with index contraction

b) Compute the Lie derivative of a density n object, such as $\sqrt{\det g}^n$. What happens if the object has additional tensor indices?

c) Show that the Lie derivative can be partially integrated if the integrand is a density 1 object with no (uncontracted) tensor indices (as it should be).

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Problem 3: Parallel transport on a Sphere

Consider a two-dimensional sphere S^2 parametrized by the coordinates $\theta \in (0, \pi)$ (vertical angle) and $\phi \in (-\pi, \pi)$ (horizontal angle).

a) Calculate the parallel transport of an arbitrary vector along the two curves

$$\gamma_\theta(s) = (\theta_0 + s, \phi_0), \quad s \in (a, b)$$

and

$$\gamma_\phi(s) = (\theta_0, \phi_0 + s), \quad s \in (a, b).$$

b) Calculate the parallel transport along a closed curve composed of components of γ_θ and γ_ϕ along the points: $(\pi/2, 0) \to (\pi/2, \phi^*) \to (\theta^*, \phi^*) \to (\theta^*, 0) \to (\pi/2, 0)$. Compare the parallel transported vector and the initial vector. What happens for an infinitesimal small loop? Compare with the definition of the Riemann tensor.

Problem 4: Christoffel symbols

Show that the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ transform as a connection under general coordinate transformations.

Problem 5: BONUS: Metric compatibility

Show that for a torsion free connection, i.e. $A^\rho_{\mu\nu} = A^\rho_{\nu\mu}$, the condition

$$D_\mu g_{\rho\sigma} = \partial_\mu g_{\rho\sigma} - A^\nu_{\mu\rho} g_{\nu\sigma} - A^\nu_{\mu\sigma} g_{\rho\nu} = 0$$

determines $A^\rho_{\mu\nu}$ to be equal to the Christoffel symbols $\Gamma^\rho_{\mu\nu}$.

Hint: Sum various $D_\mu g_{\rho\sigma} = 0$ with suitably interchanged indices.