**Problems to Introduction to Quantum Gravity I** WS 18/19

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*Sheet 8* — Handout: 11.12.2018 — To present: 17.12.2018

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## Problem 1: Covectors

Derive the transformation law for covector components from the definition $dx^i(\partial_j) = \delta^i_j$ for the dual basis. Conclude that the evaluation $w(v) = w_i v^i$ of a vector $v$ using a covector $w$ is independent on the choice of basis.

## Problem 2: Push forward and pull back

Compute the coordinate expression for the pull back of a vector field along a diffeomorphism $\Phi$ by using its inverse. Confirm that the evaluation of a vector $v$ using a covector $w$, $w(v)$, is invariant under pull backs, i.e. $(\Phi_* w)(\Phi_* v) = w(v)$. Show also that $\Phi_* \Phi^* = \Phi^* \Phi_* = \mathrm{Id}$.

Hint: $\delta^i_j = \frac{\partial x^i}{\partial x^j} = \frac{\partial (\Phi^{-1})^i(\Phi(x))}{\partial x^j} = \dots$

## Problem 3: Great circles

Show that the shortest path between two points on a sphere is a section of a great circle by solving the geodesic equation in suitable coordinates.

## Problem 4: BONUS: Geodesic equation

Derive the geodesic equation by varying the length functional

$$d(c) = \int_c ds := \int_c \sqrt{g_{ij} dx^i dx^j} := \int_a^b d\lambda \sqrt{g_{ij}(c(\lambda)) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$$

along a curve $c : [a, b] \to \mathcal{M}_n$. It is easiest to start with the case where the curve parameter $\lambda$ measures proper length, so that

$$\sqrt{g_{ij}(c(\lambda)) \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} = \mathrm{const}$$

as a function of $\lambda$ along the desired trajectory. What corrections to the geodesic equation are necessary if $\lambda$ does not measure proper length?