b-divisors on toric varieties

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Workshop: Asymptotic invariants attached to linear series

Pedagogical University in Cracow

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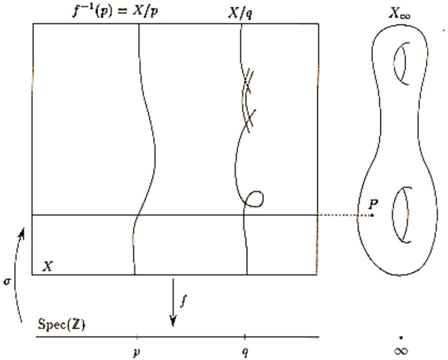
- One considers *arithmetic line bundles* $\overline{L} = (\mathcal{L}, (L_{\mathbb{C}}, ||\cdot||))$



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- **Question**: Does this result hold true in the mixed Shimura case?

Goal: Extend the line bundle of Jacobi forms $\mathcal{O}(D_{\mathcal{E}(N)})$ on the universal elliptic curve $\mathcal{E}(N)$ with its invariant metric to a toroidal compactification in a "meaningful way".

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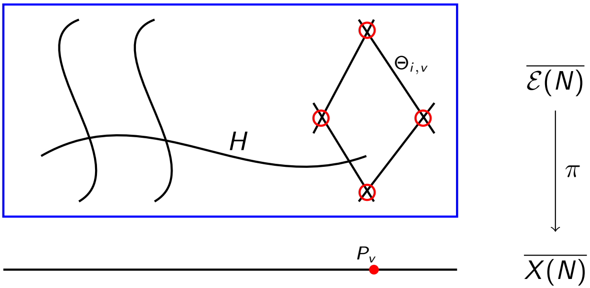

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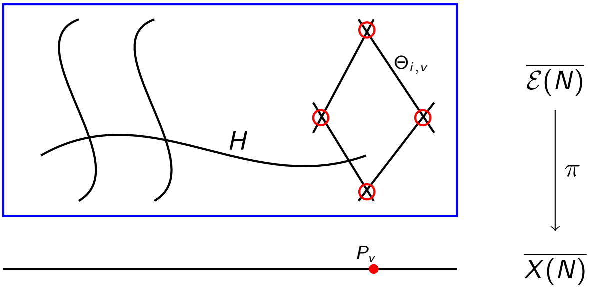

$$D_{\overline{\mathcal{E}(N)}} = \pi^* L + m \cdot H + \sum_{i,v} m_{i,v} \cdot \Theta_{i,v}.$$

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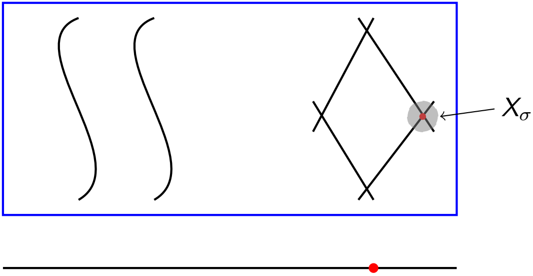
Advantages:

- Functoriality
- Chern-Weil theory
- Hilbert-Samuel Formula

} (Burgos, Kramer, Kühn, '14)

- Geometric Interpretation (B., '14)

Moreover,



$$\mathbb{D}^2 = D^2_{\overline{\mathcal{E}(N)}} - \#\text{cusps} \cdot N \cdot \text{R}$$
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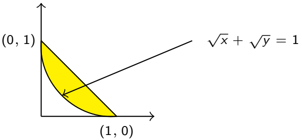
where

$\Delta =$



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Definition

We define

$$R(\Sigma) := \{\Sigma' \geq \Sigma \mid \Sigma' \text{ is a smooth subdivision}\}.$$

$R(\Sigma)$ is a directed set.
A toric b-divisor is an element $\mathbb{D}$ in the inverse limit

$$\varprojlim_{\Sigma' \in R(\Sigma)} \mathbb{T}\text{-}\mathbb{Q}\text{-Ca}\,(X_{\Sigma'}) \subset \mathfrak{X}_{\Sigma} := \varprojlim_{\Sigma' \in R(\Sigma)} X_{\Sigma'}.$$

Proposition

There is a one to one correspondence

$$\begin{aligned}
\{\text{toric b-divisors on} X_\Sigma\} \quad &\leftrightarrow \quad \{\mathbb{Q}\text{-valued, 1-homogeneous functions on } N\} \\
\mathbb{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)} \quad &\mapsto \quad \tilde{\phi}_{\mathbb{D}} \colon N^{\text{prim}} \to \mathbb{Q}, \\
& \qquad v \mapsto - \text{ coefficient of } D_{\langle v \rangle} \text{ in } \mathbb{D}.
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Example A: Consider $X_\Sigma = \mathbb{P}^2$

$$\tilde{\phi}_{\mathbb{D}}(a,b) = \begin{cases} \frac{ab}{a+b} & \text{if } a,b \geq 0 \\ \min(a,b) & \text{otherwise.} \end{cases}$$

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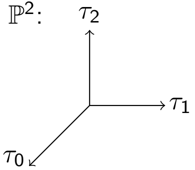

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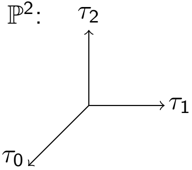

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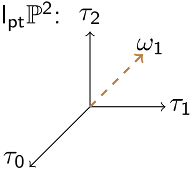

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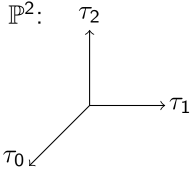

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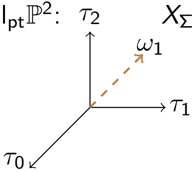

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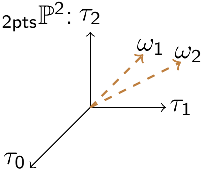

$D_{\Sigma''} = D_{\tau_0} - \frac{1}{2} D_{\omega_1} - \frac{2}{3} D_{\omega_2}$

Definition

A toric b-divisor $\mathbb{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ is *nef* if $D_{\Sigma'} \subset X_{\Sigma'}$ is nef for a cofinite subset $\Sigma' \in R(\Sigma)$.

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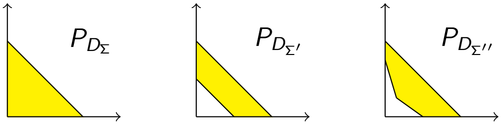
In the example above, we have that $\mathbb{D}$ is nef. Indeed, this can be verified by looking at the polytopes $P_{D_{\Sigma'}}$ for $\Sigma' \in R(\Sigma)$. We have for example



Integrability of toric b-divisors

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We say that a toric b-divisor $\mathbb{D} = (D_{\Sigma'})_{\Sigma' \in R(\Sigma)}$ is *integrable* if the limit (in terms of nets)

$$\mathbb{D}^n := \lim_{\Sigma' \in R(\Sigma)} D_{\Sigma'}^n < \infty.$$

In this case, we will call the limit $\mathbb{D}^n$ the *degree* of the toric b-divisor.

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Theorem (B., '15)

Let $\mathbb{D}$ as above be a nef toric b-divisor. Let $\tilde{\phi}_{\mathbb{D}} \colon N_{\mathbb{Q}} \to \mathbb{Q}$ be its associated 1-homogeneous $\mathbb{Q}$-concave function. Then there exists a unique concave 1-homogeneous function

$$\phi_{\mathbb{D}} \colon N_{\mathbb{R}} \to \mathbb{R}$$

extending $\tilde{\phi}_{\mathbb{D}}$. Moreover, $\mathbb{D}$ is integrable and its degree is given by

$$\mathbb{D}^n = n! \operatorname{vol}\left(\Delta_{\phi_{\mathbb{D}}}\right).$$

Example A:

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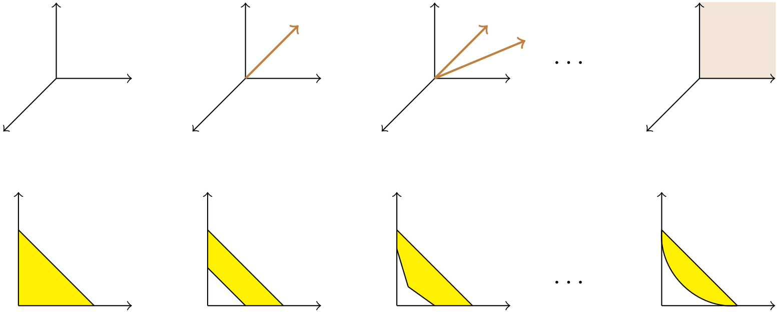

• Given a nef toric b-divisor $\mathbb{D}$, one can define a b-divisorial sheaf $\mathcal{O}_{X_\Sigma}(\mathbb{D})$. Its space of global sections

$$H^0(X_\Sigma, \mathbb{D}) = \{f \in \mathbb{C}(X_\Sigma) \mid \text{b-div}(f) + \mathbb{D} \geq 0\}$$

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Theorem (B., '15)

Let $\mathbb{D}$ be as above. We have a Hilbert–Samuel type formula

$$\mathbb{D}^n = \lim_{\ell \to \infty} \frac{h^0(X_\Sigma, \ell\mathbb{D})}{\ell^n / n!}.$$

- Let $\mathbb{D}$ be a nef toric b-divisor. Consider the graded algebra

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With notations as above, we have

$$\Delta_{\phi_{\mathbb{D}}} = \Delta_{R(\mathbb{D})}.$$

Work in progress

Goal: Prove a Hilbert-Samuel type formula for singular metrics on toroidal compactifications of universal abelian schemes and self fiber products of those.

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- On $\begin{pmatrix} \overline{\mathcal{X}}_2 \\ \downarrow \\ \overline{\mathcal{A}}_2 \end{pmatrix}$ we consider the theta line bundle with its invariant metric. The singularities of the metric are encoded in the concave function

$$\phi = \frac{xy^2 + y^2v + y^2\omega + 2xyv + xv^2 + yv^2 + zv^2}{xy + xz + xv + yv + zv + x\omega + y\omega + z\omega}.$$

Thank you!