

RINGS OF SIEGEL–JACOBI FORMS OF BOUNDED RELATIVE INDEX ARE NOT FINITELY GENERATED

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ABSTRACT. We show that the ring of Siegel–Jacobi forms of fixed degree and of fixed or bounded ratio between weight and index is not finitely generated. Our main tool is the theory of toroidal b -divisors and their relation to convex geometry. As a byproduct of our methods we recover a formula due to Tai for the asymptotic dimension of the space of Siegel–Jacobi forms of given ratio between weight and index.

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1. INTRODUCTION

Siegel modular varieties arise as moduli spaces of abelian varieties of a fixed dimension g equipped with a principal polarization and level structure. They carry distinguished line bundles whose global sections are Siegel modular forms of degree g . Siegel modular forms constitute an important class of automorphic forms and generalize the classical modular forms on finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$ to higher dimensions. The study of Siegel modular forms is of fundamental importance in number theory and algebraic geometry.

Siegel modular varieties carry universal abelian varieties, and these similarly carry distinguished line bundles. Their global sections are known as *Siegel–Jacobi forms*. Importantly, Siegel–Jacobi forms appear as Fourier coefficients of Siegel modular forms in higher degrees.

The systematic study of Siegel–Jacobi forms goes back to the 1980s. The case of degree one is dealt with in the book [16] by Eichler and Zagier and in many papers from Zagier’s school. For higher degree, foundations were laid by Dulinski [15] and Ziegler [43]. Later Kramer developed the arithmetic theory of Siegel–Jacobi forms [22]. An important aspect in his work is the consideration of toroidal compactifications of the universal abelian variety following the work of Mumford and its collaborators [20], [1], [34], and in the arithmetic setting by Faltings and Chai [17]. Around the same time Runge contributed by further studying the geometric aspects of Siegel–Jacobi forms [31]. Other work related to Siegel–Jacobi forms of higher degree has appeared in [38, 39, 40, 41, 42].

1.1. Statement of the main result. Let g be a positive integer. Let $\Gamma \subseteq \mathrm{Sp}(2g, \mathbb{Z})$ be a finite index subgroup. Siegel modular forms with respect to Γ are classified by weight, whereas Siegel–Jacobi forms with respect to Γ are classified by two invariants, namely *weight* and *index*. We denote by $J_{k,m}(\Gamma)$ the space of Siegel–Jacobi forms of weight k and index m with respect to Γ . Then $\bigoplus_{k,m} J_{k,m}(\Gamma)$ is naturally a bigraded \mathbb{C} -algebra which is known not to be finitely generated (see [16] for the case of degree one, and Proposition 4.7 below for the general case).

From now on we fix a neat subgroup $\Gamma \subseteq \mathrm{Sp}(2g, \mathbb{Z})$. Assume $g \geq 2$. Then Runge claims in [31, Theorem 5.5] that, for suitable integers r , the rings $\bigoplus_{m \leq 2rk} J_{k,m}(\Gamma)$ are finitely generated \mathbb{C} -algebras.

Our main aim in this paper is to disprove Runge’s claim. In fact we show that this algebra is *never* finitely generated. In Remark 6.11 we explain the oversight in Runge’s proof.

Theorem A. (Theorem 6.9 and Proposition 6.10) Let $g \geq 1$. For each $k > 0$ and $m > 0$ the graded algebra $\bigoplus_{\ell} J_{\ell k, \ell m}(\Gamma)$ is not finitely generated. Therefore for any $r \in \mathbb{Q}_{>0}$ the graded algebra $\bigoplus_{m \leq rk} J_{k,m}(\Gamma)$ is not finitely generated.

Our main tool to prove Theorem A is the theory of toroidal b-divisors (where b stands for birational). A b-divisor on a projective variety X can be thought of as an infinite

tower of divisors living over all smooth birational models of X , and compatible under pushforward (see Section 2.3 for a precise definition and further details). A b-divisor is said to be *Cartier* if it is determined on one such birational model, in other words, if all the elements in the tower can be obtained by pull back and push forward of a divisor on a single model.

If X carries a toroidal structure then the theory of b-divisors can be naturally simplified by considering only the toroidal blowups of X . The corresponding toroidal b-divisors can then be seen as conical functions on a conical polyhedral complex associated to X . This enables the study of toroidal b-divisors by techniques from convex geometry [3]. For example, if a toroidal b-divisor is Cartier then the corresponding conical function is piecewise linear.

To the neat subgroup Γ is associated a fibration of principally polarized abelian varieties

$$\pi: \mathcal{B}(\Gamma) \longrightarrow \mathcal{A}(\Gamma).$$

Here $\mathcal{A}(\Gamma)$ is the Siegel modular variety associated to Γ . For each pair of given integers k, m the complex variety $\mathcal{B}(\Gamma)$ carries a line bundle of Siegel–Jacobi forms $L_{k,m}$. It is endowed with a natural invariant metric, which we denote by h (see Section 4.2). As we will see in Propositions 5.23 and 5.27 one can choose a toroidal compactification $\overline{\mathcal{B}}(\Gamma)$ of $\mathcal{B}(\Gamma)$ in such a way that $L_{k,m}$ extends as an algebraic line bundle $\overline{L}_{k,m}$ on $\overline{\mathcal{B}}(\Gamma)$ and h extends as a *toroidal psh* metric on $\overline{L}_{k,m}$ (see Section 2.6 for the definition of toroidal psh metrics).

As follows from the work done in our paper [4], given a non-zero rational section s of $\overline{L}_{k,m}$ we have an associated toroidal b-divisor $\mathbb{D}(\overline{L}_{k,m}, s, h)$ on $\overline{\mathcal{B}}(\Gamma)$. This b-divisor does not depend on the actual choice of toroidal psh extension $\overline{L}_{k,m}$. The b-divisor $\mathbb{D}(\overline{L}_{k,m}, s, h)$ corresponds to a convex function on the conical polyhedral complex attached to $\overline{\mathcal{B}}(\Gamma)$.

The key point is that, via an explicit computation, we can show that this conical function is *not* piecewise linear. This implies that the toroidal b-divisor $\mathbb{D}(\overline{L}_{k,m}, s, h)$ is not Cartier (Corollary 6.3). We then use a characterization of Jacobi cusp forms in terms of the invariant metric (Proposition 4.11) to deduce that the algebra $\bigoplus_{\ell} J_{\ell k, \ell m}(\Gamma)$ is not finitely generated.

1.2. Asymptotic dimension formulae. Tai’s celebrated work [34] implies an asymptotic formula for the dimension of the space of Siegel–Jacobi forms of given ratio between weight and index. We recall that the main aim of [34] is to show that the moduli space \mathcal{A}_g of principally polarized abelian varieties of dimension g is of general type for $g \geq 9$. Tai’s proof proceeds via a study of the pushforward of the line bundle of Siegel–Jacobi forms to \mathcal{A}_g , and an application of Mumford’s version of the Hirzebruch proportionality principle [26] on the non-compact pure Shimura variety \mathcal{A}_g .

In the present work we arrive at a form of the proportionality principle on the universal abelian variety itself, via the machinery of b-divisors. This gives a new way to compute the asymptotic dimension of the space of Siegel–Jacobi forms, namely as the degree of a suitable toroidal b-divisor on a toroidal compactification of the universal abelian variety.

Let $g \in \mathbb{Z}_{\geq 1}$ and set $G = g(g+1)/2$ and $n = G + g$. The b-divisor $\mathbb{D}(\overline{L}_{k,m}, s, h)$ has a well-defined degree $\mathbb{D}(\overline{L}_{k,m}, s, h)^n$ in $\mathbb{R}_{\geq 0}$ (combine Remark 2.18 and Lemma 6.7).

Theorem B. (Corollary 7.4) Let $\mathbb{D}(\overline{L}_{k,m}, s, h)$ be the toroidal b-divisor on the toroidal compactification $\overline{\mathcal{B}}(\Gamma)$ associated to the line bundle $L_{k,m}$ of Siegel–Jacobi forms of weight k and index m , the rational section s and the invariant metric h . Then the Hilbert–Samuel

type formula

$$(1.1) \quad \mathbb{D}(\overline{L}_{k,m}, s, h)^n = \limsup_{\ell \rightarrow \infty} \frac{\dim J_{\ell k, \ell m}(\Gamma)}{\ell^n/n!}$$

holds, where $\mathbb{D}(\overline{L}_{k,m}, s, h)^n$ is the degree of the b-divisor $\mathbb{D}(\overline{L}_{k,m}, s, h)$.

Using Chern–Weil theory for line bundles with psh metrics as developed in [4] we can compute the degree on the left hand side in (1.1) explicitly using integrals of smooth differential forms on the open universal abelian variety $\mathcal{B}(\Gamma)$. Then by applying the equality in (1.1) we obtain an explicit asymptotic dimension formula. The resulting formula is compatible with the one implicit in Tai’s work [34], see Remark 7.7.

Corollary C. (Corollary 7.6) The asymptotic growth of the dimension of the space $J_{\ell k, \ell m}(\Gamma)$ is given by the following formulae:

$$\begin{aligned} \limsup_{\ell \rightarrow \infty} \frac{\dim J_{\ell k, \ell m}(\Gamma)}{\ell^n/n!} &= (-1)^G n! m^g k^G [\Gamma_0 : \Gamma] \prod_{k=1}^g \frac{\zeta(1-2k)}{(2k-1)!!} \\ &= (-1)^n n! m^g k^G 2^{G-g} [\Gamma_0 : \Gamma] \prod_{k=1}^g \frac{(k-1)! B_{2k}}{(2k)!} \\ &= V_g \cdot n! m^g k^G 2^{-G-1} \pi^{-G} [\Gamma_0 : \Gamma], \end{aligned}$$

where $\Gamma_0 = \mathrm{Sp}(2g, \mathbb{Z})$, where $B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k)$ are the Bernoulli numbers and

$$V_g = (-1)^n 2^{g^2+1} \pi^G \prod_{k=1}^g \frac{(k-1)! B_{2k}}{(2k)!}$$

is the symplectic volume computed by Siegel in [33, Section VIII].

Note that we are assuming that Γ is neat. In particular $-\mathrm{Id} \notin \Gamma$. In fact the above formulae are also true for arbitrary $\Gamma \subset \Gamma_0$ of finite index not containing $-\mathrm{Id}$. If Γ contains $-\mathrm{Id}$ the right hand side has to be multiplied by 2.

We mention that Theorem B and Corollary C generalize results proved in [10] for the case $g = 1$, $k = m = 4$ and $\Gamma = \Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$ a principal congruence subgroup. As we shall see in Remark 7.5, the lim sup in Theorem B and Corollary C is actually a lim for sufficiently divisible ℓ .

We expect that our method to compute asymptotic dimensions can be generalized to other spaces of automorphic forms on mixed Shimura varieties. Indeed, the b-divisorial approach to Chern-Weil theory from [4] continues to hold in these cases whenever the natural invariant metrics give rise to psh line bundles.

1.3. Outline of the paper. In Section 2 we set up the basic definitions, including those of b-divisors, toroidal b-divisors, and toroidal psh metrics (a special class of singular psh metrics). The main new result in this section is a characterisation of the Lelong numbers of a toroidal psh metric in terms of the conical convex function defining the toroidal psh metric, see Lemma 2.31.

In Section 3 we study properties of and relations between graded linear series, b-divisors and psh metrics. The relations are summarized in Diagram 3.5. We then show that these relations become stronger in the toroidal case, see Theorem 3.16 and Corollary 3.19. We describe an example to show that the toroidal assumption here is really necessary (Remark 3.18). The compatibility relations in the toroidal case are key ingredients in the proofs of Theorems A and B.

In Section 4 we recall the definitions of Siegel–Jacobi (cusp) forms. We discuss in detail the universal family $\pi: \mathcal{B}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$ of principally polarized abelian varieties, and describe Siegel–Jacobi forms as global sections of a line bundle $L_{k,m}$ over $\mathcal{B}(\Gamma)$. We also give an explicit description of the invariant metric h on this line bundle.

In Section 5 we describe the theory of toroidal compactifications of $\mathcal{A}(\Gamma)$ and $\mathcal{B}(\Gamma)$ and discuss extensions of $L_{k,m}$ and h over the compactifications. We prove that one can choose the toroidal compactifications and the extensions in such a way that the extended invariant metric is toroidal psh (Proposition 5.27). This leads to the observation that the b-divisor associated to the line bundle of Siegel–Jacobi forms with its invariant metric is toroidal (Corollary 5.29).

In Section 6 we give the proof of our main result. We start by giving an explicit formula for the relevant conical function (Lemma 6.1). We then observe that this function is not piecewise linear. It follows that the associated toroidal b-divisor is not Cartier (see Corollary 6.3), from which we deduce Theorem A.

In the final Section 7 we define the volume of a graded linear series and prove Theorem B. Among other things we use the Hilbert–Samuel formula for toroidal b-divisors from [3]. Finally, we use Chern–Weil theory for singular psh metrics as developed in [4] to compute the degree of our b-divisor in terms of the non-pluripolar volume of the positive current associated to our psh metric, leading to Corollary C.

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2. BASIC DEFINITIONS

In this section we recall the definitions of the basic tools we will use in the paper. Throughout the paper X will denote the complex manifold associated to a smooth projective variety over \mathbb{C} of dimension n .

2.1. Graded linear series. Let $F = K(X)$ be the field of rational functions on X . The following definition is taken from [19].

Definition 2.1. A *graded linear series* (on X) is a graded subalgebra $A \subset F[t]$. Let $L \subset F$ be a finite dimensional \mathbb{C} -vector space. The graded linear series A_L is the graded subalgebra of $F[t]$ generated by $L \cdot t$ in degree 1 (note that A_L is generated by finitely many elements of degree 1).

A graded linear series A is called of *integral type* if there is a finite dimensional linear subspace $L \subset F$ such that $A_L \subset A$ and A is finite over A_L .

A graded linear series A is called of *almost integral type* if there is a graded linear series A' of integral type such that $A \subset A'$.

If A is a graded linear series we write $A = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} A_\ell t^\ell$, where $A_\ell \subset F$.

Remark 2.2. If $X' \dashrightarrow X$ is a birational map then a graded linear series on X is a graded linear series on X' .

Let D be a Cartier divisor on X , with integer, rational or real coefficients. Then one writes

$$\mathcal{L}(D) = \{0 \neq f \in F \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\}$$

and

$$\mathcal{R}(D) = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} \mathcal{L}(\ell D) t^\ell \subset F[t].$$

The following result follows from [19, Theorems 3.7 and 3.8] (using that X is normal and projective).

Proposition 2.3. *The graded linear series $\mathcal{R}(D)$ is of almost integral type. If moreover D is very ample, then $\mathcal{R}(D)$ is of integral type.*

2.2. Toroidal structures.

Definition 2.4. A morphism $\pi: X' \rightarrow X$ of complex varieties is a *modification* if it is proper and birational.

Definition 2.5. A (*smooth*) *toroidal structure* on X is the choice of a smooth modification $\pi_1: X_1 \rightarrow X$ and a simple normal crossing divisor D on X_1 that contains the exceptional locus of π_1 . If (π_2, D') is another toroidal structure, we say that (π_2, D') is *above* (π_1, D) if π_2 factors as

$$\begin{array}{ccc} X_2 & & \\ \pi_{21} \downarrow & \searrow \pi_2 & \\ X_1 & \xrightarrow{\pi_1} & X \end{array}$$

and $\pi_{21}^{-1}(D) \subset D'$.

A toroidal structure will be denoted by the triple (X_1, π_1, D) or, if no confusion may arise, just by the divisor D .

The open subset $U = X_1 \setminus D$ can be identified with an open subset of X also denoted by U , and the inclusion $U \subset X_1$ is an example of a toroidal embedding without self intersection. We will use freely the theory of toroidal embeddings from [20]. For a concise description of what we will need the reader is referred to [3].

Remark 2.6. One can envisage more general toroidal structures where $U \subset X_1$ is an arbitrary toroidal embedding (without assuming X_1 smooth), but we will not need them in this paper.

Following [20], to any smooth toroidal embedding we can associate a conical polyhedral complex $\Pi(X_1, D)$. The rays of $\Pi(X_1, D)$ are in one to one correspondence with the irreducible components of D . If D_i is an irreducible component of D , we denote by ρ_{D_i} the corresponding ray. Given irreducible components D_1, \dots, D_n of D , then the set of cones of $\Pi(X_1, D)$ of dimension n having $\rho_{D_1}, \dots, \rho_{D_n}$ as edges is in bijection with the set of irreducible components of $D_1 \cap \dots \cap D_n$. Each irreducible component of an intersection of components of D is called a *stratum* of the toroidal structure. Thus the set of strata is in bijection with the set of cones. This bijection reverses dimensions and inclusions of closures. The conical polyhedral complex $\Pi(X_1, D)$ has a canonical integral structure and we denote by v_{D_i} the primitive generator of ρ_{D_i} .

If (X_2, π_2, D') is a toroidal structure above (X_1, π_1, D) , then there is a continuous map

$$r_{D, D'}: \Pi(X_2, D') \rightarrow \Pi(X_1, D),$$

that sends cones to cones, is linear on each cone and is compatible with the integral structure on each cone. Moreover it is functorial in the sense that, if (X_3, π_3, D'') is a toroidal structure above (X_2, π_2, D') , then

$$(2.1) \quad r_{D, D'} \circ r_{D', D''} = r_{D, D''}.$$

The map $r_{D, D'}$ is constructed in greater generality in [35, Theorem 1.1], but the case we need admits a particularly simple description as follows.

Let $\pi_{12}: X_2 \rightarrow X_1$ be the map of smooth modifications, Y a stratum of (X_2, D') and Z the minimal stratum of (X_1, D) containing $\pi_{12}(Y)$. Let D'_1, \dots, D'_r be the set of components of D' such that Y is an irreducible component of $D'_1 \cap \dots \cap D'_r$ and similarly let D_1, \dots, D_s be the set of components of D containing Z . Let σ be the cone corresponding

to Y and τ the cone corresponding to Z . Then the restriction $r_{D,D'}|_{\sigma}: \sigma \rightarrow \tau$ is the unique linear map satisfying

$$r_{D,D'}(v_{D'_i}) = \sum_{j=1}^s \text{ord}_{D'_i}(\pi_{12}^* D_j) v_{D_j}.$$

It is clear that these maps for the different cones of $\Pi(X_2, D')$ glue together to give a map with the listed properties.

To prove the functoriality (2.1), let $\pi_{23}: X_3 \rightarrow X_2$ and $\pi_{12}: X_2 \rightarrow X_1$ be maps of modifications. Let P be a component of D'' , let D'_1, \dots, D'_r be the set of components of D' containing the image of P in X_2 and let D_1, \dots, D_s be the set of components of D containing the image of P in X_1 . For each j , $\pi_{12}^{-1}(D_j) \subset D'$, we have

$$\text{ord}_P((\pi_{12} \circ \pi_{23})^* D_j) = \sum_{i=1}^r \text{ord}_P((\pi_{23})^* D'_i) \text{ord}_{D'_i}(\pi_{12}^* D_j).$$

Therefore

$$\begin{aligned} r_{D,D''}(v_P) &= \sum_{j=1}^s \text{ord}_P((\pi_{12} \circ \pi_{23})^* D_j) v_{D_j} \\ &= \sum_{j=1}^s \sum_{i=1}^r \text{ord}_P(\pi_{23}^* D'_i) \text{ord}_{D'_i}(\pi_{12}^* D_j) v_{D_j} \\ &= \sum_{i=1}^r \text{ord}_P(\pi_{23}^* D'_i) r_{D,D'}(v_{D'_i}) \\ &= r_{D,D'} \left(\sum_{i=1}^r \text{ord}_P(\pi_{23}^* D'_i) v_{D'_i} \right) \\ &= r_{D,D'} \circ r_{D',D''}(P). \end{aligned}$$

Let (X_1, π_1, D) be a toroidal structure on X . Among the modifications of X there is a distinguished class: the *allowable modifications* of the toroidal embedding $X_1 \setminus D \hookrightarrow X_1$. See for instance [20, Ch. II, § 2, Definition 3] for the precise definition.

The allowable modifications are in one-to-one correspondence with the subdivisions of the conical polyhedral complex Π associated to D . Any such modification will be called toroidal (with respect to the toroidal structure D). Note that, if (X_2, π_2, D') is toroidal with respect to (X_1, π_1, D) , then D' is above D and the map $r_{D,D'}$ defined above induces a homeomorphism between the underlying topological spaces $|\Pi(X_2, D')| \rightarrow |\Pi(X_1, D)|$, compatible with the fact that $\Pi(X_2, D')$ is a subdivision of $\Pi(X_1, D)$.

We end this section with two definitions.

Definition 2.7. Let (X_1, π_1, D) be a smooth toroidal structure on X . A *toroidal exceptional prime divisor* of (X_1, π_1, D) is an irreducible component of any exceptional divisor of a toroidal modification of (X_1, π_1, D) .

Definition 2.8. The *set of rational points* $\Pi(X_1, D)(\mathbb{Q})$ is the set of points of $\Pi(X_1, D)$ that have rational coordinates when expressed in terms of the primitive vectors v_{D_j} .

2.3. b-divisors. In this section we discuss Weil and Cartier \mathbb{R} -b-divisors on X as well as their toroidal counterparts. This is essentially Shokurov's notion of birational divisors (or b-divisors) [32]. This section is purely algebraic, so, if preferred, the reader can work with finite-type algebraic varieties over any field of characteristic zero.

For other terminologies and properties concerning \mathbb{b} -divisors we refer to [6] and [7] (see also [3] and [2] for the toroidal and the toric cases, respectively).

We write $\text{Div}_{\mathbb{R}}(X)$ for the set of Weil divisors on X with real coefficients, viewed as a real vector space. We endow it with the direct limit topology with respect to its finite dimensional subspaces. Explicitly, a sequence of divisors $(D_i)_{i \geq 0}$ converges to a divisor D in $\text{Div}_{\mathbb{R}}(X)$ if there is a divisor A , such that $\text{supp}(D_i) \subset \text{supp}(A)$ for all $i \geq 0$ and $(D_i)_{i \geq 0}$ converges to D in the finite dimensional vector space of real divisors with support contained in $\text{supp}(A)$.

Definition 2.9. The set of *models of X* is

$$R(X) := \{ \pi: X_{\pi} \rightarrow X \mid \pi \text{ is a smooth modification} \}.$$

If (X_1, π_1, D) is a smooth toroidal structure on X then the set of *toroidal models of X* (with respect to D) is

$$R^{\text{tor}}(X, D) := \{ \pi: X_{\pi} \rightarrow X \mid \pi \text{ is a smooth modification, toroidal w.r.t. } D \}.$$

In particular if $\pi \in R^{\text{tor}}(X, D)$ then it factors through π_1 .

We view both $R(X)$ and $R^{\text{tor}}(X, D)$ as full subcategories of the category of complex varieties over X , in particular morphisms are over X . Maps of models are unique if they exist, and are necessarily proper and birational. Hironaka's resolution of singularities implies that $R(X)$ is a directed set, where we set $\pi' \geq \pi$ if there exists a morphism $\mu: X_{\pi'} \rightarrow X_{\pi}$. Similarly $R^{\text{tor}}(X, D)$ is directed by the existence of a smooth common refinement of any two subdivisions.

Consider a pair $\pi' \geq \pi$ in $R(X)$, and let $\mu: X_{\pi'} \rightarrow X_{\pi}$ be the corresponding modification. We have a pullback map

$$\mu^*: \text{Div}_{\mathbb{R}}(X_{\pi}) \longrightarrow \text{Div}_{\mathbb{R}}(X_{\pi'})$$

and a pushforward map

$$\mu_*: \text{Div}_{\mathbb{R}}(X_{\pi'}) \longrightarrow \text{Div}_{\mathbb{R}}(X_{\pi})$$

between the associated divisor groups. Both maps are continuous.

Definition 2.10. The group of *Cartier \mathbb{R} - b -divisors on X* is the direct limit

$$\text{C-b-Div}_{\mathbb{R}}(X) := \varinjlim_{\pi \in R(X)} \text{Div}_{\mathbb{R}}(X_{\pi}),$$

in the category of topological vector spaces, with maps given by the pullback maps. The resulting topology is called the *strong* topology.

The group of *Weil \mathbb{R} - b -divisors on X* is the inverse limit

$$\text{W-b-Div}_{\mathbb{R}}(X) := \varprojlim_{\pi \in R(X)} \text{Div}_{\mathbb{R}}(X_{\pi}),$$

in the category of topological vector spaces, with maps given by the pushforward maps. The resulting topology is called the *weak* topology.

Usually we will denote \mathbb{R} - b -divisors with a blackboard bold font (\mathbb{D}) in order to distinguish them from classical \mathbb{R} -divisors that will be denoted with a slanted font (D).

Remark 2.11. As a set, $\text{C-b-Div}_{\mathbb{R}}(X)$ can be seen as the disjoint union of the sets $\text{Div}_{\mathbb{R}}(X_{\pi})$ modulo the equivalence relation which sets two divisors equal if they coincide after pullback to a common modification. The set $\text{W-b-Div}_{\mathbb{R}}(X)$ can be seen as the subset of $\prod_{\pi \in R(X)} \text{Div}_{\mathbb{R}}(X_{\pi})$ given by the elements $\mathbb{D} = (D_{\pi})_{\pi \in R(X)}$ satisfying the compatibility condition that for each $\pi' \geq \pi$ we have $\mu_* D_{\pi'} = D_{\pi}$, where μ is the corresponding modification.

For any variety Y and normal crossings divisor E on Y , we denote by $\mathrm{Div}_{\mathbb{R}}(Y, E)$ the real vector space of \mathbb{R} -divisors on Y whose support is contained in E . We endow it with the Euclidean topology.

Let (X_1, π_1, D) be a toroidal structure; we make analogous definitions of b-divisors.

Definition 2.12. The group of *toroidal Cartier \mathbb{R} -b-divisors on X* (with respect to D) is the direct limit

$$\mathrm{C}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X, D)^{\mathrm{tor}} := \varinjlim_{\pi \in R^{\mathrm{tor}}(X, D)} \mathrm{Div}_{\mathbb{R}}(X_{\pi}, \mu_{\pi}^{-1}(D)),$$

where $\mu_{\pi}: X_{\pi} \rightarrow X_1$ is the unique map of modifications. The limit is again in the category of topological vector spaces, with maps given by the pullback maps.

The group of *toroidal Weil \mathbb{R} -b-divisors on X* (with respect to D) is the inverse limit

$$\mathrm{W}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X, D)^{\mathrm{tor}} := \varprojlim_{\pi \in R^{\mathrm{tor}}(X, D)} \mathrm{Div}_{\mathbb{R}}(X_{\pi}, \mu_{\pi}^{-1}(D)),$$

also in the category of topological vector spaces, with maps given by the pushforward maps.

We recall from [3] that the space $\mathrm{W}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X, D)^{\mathrm{tor}}$ can be identified with the space of all conical real valued functions on $\Pi(X_1, D)(\mathbb{Q})$, while $\mathrm{C}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X, D)^{\mathrm{tor}}$ can be identified with the space of all continuous conical piecewise linear functions on $\Pi(X_1, D)$ with rational domains of linearity.

Clearly $R(X, D)^{\mathrm{tor}} \subset R(X)$. Moreover, for $\pi \in R(X, D)^{\mathrm{tor}}$ there are canonical maps

$$\begin{aligned} \mathrm{Div}_{\mathbb{R}}(X_{\pi}, \mu_{\pi}^{-1}(D)) &\longrightarrow \mathrm{Div}_{\mathbb{R}}(X_{\pi}), \\ \mathrm{Div}_{\mathbb{R}}(X_{\pi}) &\longrightarrow \mathrm{Div}_{\mathbb{R}}(X_{\pi}, \mu_{\pi}^{-1}(D)), \end{aligned}$$

where the last one sends any prime divisor not contained in $\mu_{\pi}^{-1}(D)$ to zero. Hence there is a canonical inclusion

$$\mathrm{C}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X, D)^{\mathrm{tor}} \longrightarrow \mathrm{C}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X)$$

and a canonical projection

$$\mathrm{W}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X) \longrightarrow \mathrm{W}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X, D)^{\mathrm{tor}}.$$

We now construct a section

$$(2.2) \quad \iota: \mathrm{W}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X, D)^{\mathrm{tor}} \longrightarrow \mathrm{W}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X),$$

of the canonical projection. Let $\mathbb{D} \in \mathrm{W}\text{-b-}\mathrm{Div}_{\mathbb{R}}(X, D)^{\mathrm{tor}}$ be a toroidal Weil b-divisor and choose a smooth modification $\pi \in R(X)$ above π_1 and a prime divisor P of X_{π} . We have to define $\mathrm{ord}_P(\iota(\mathbb{D}))$ for any such P .

Let $\varphi_{\mathbb{D}}$ be the conical function on $\Pi(X_{\pi_1}, D)(\mathbb{Q})$ corresponding to the Weil b-divisor \mathbb{D} . The function $\varphi_{\mathbb{D}}$ is characterized by the following property: If E is a toroidal exceptional prime divisor in some toroidal modification of (X_{π_1}, D) , and $v_E \in \Pi(X_{\pi_1}, D)$ is the corresponding primitive vector, then

$$\mathrm{ord}_E \mathbb{D} = -\varphi_{\mathbb{D}}(v_E).$$

Let (X_2, π_2, D') be a toroidal structure above (X_1, π_1, D) that has a map $f: X_2 \rightarrow X_{\pi}$ over X and such that $f^{-1}(P) \subset D'$. Let \widehat{P}_2 be the strict transform of P in X_2 . It is an irreducible component of D' . Let $v_{\widehat{P}_2} \in \Pi(X_2, D')$ be the primitive vector corresponding to \widehat{P}_2 . Then we define

$$\mathrm{ord}_{\iota(\mathbb{D})} P = -\varphi_{\mathbb{D}}(r_{D, D'}(v_{\widehat{P}_2})).$$

To see that this is independent of the choices, let (X_3, π_3, D'') be another toroidal structure above D and let \widehat{P}_3 be the strict transform of P in X_3 and $v_{\widehat{P}_3}$ the corresponding primitive vector. Since $r_{D', D''}(v_{\widehat{P}_3}) = v_{\widehat{P}_2}$, from (2.1) we obtain

$$r_{D, D''}(v_{\widehat{P}_3}) = r_{D, D'}(v_{\widehat{P}_2}).$$

Thus $\text{ord}_{\iota(\mathbb{D})}(P)$ does not depend on the choice of D' .

We will use the following terminology.

Definition 2.13. A b-divisor $\mathbb{D} \in \text{W-b-Div}_{\mathbb{R}}(X)$ is called *toroidal with respect to* (X_1, π_1, D) if it belongs to $\iota(\text{W-b-Div}_{\mathbb{R}}(X, D)^{\text{tor}})$. It is called *toroidal* if it is toroidal with respect to some toroidal structure. This gives rise to the set of toroidal b-divisors

$$\text{W-b-Div}_{\mathbb{R}}(X)^{\text{tor}} = \varinjlim_{(X_1, \pi_1, D)} \iota(\text{W-b-Div}_{\mathbb{R}}(X, D)^{\text{tor}}) \subset \text{W-b-Div}_{\mathbb{R}}(X).$$

Here the direct limit is taken with respect to the order $(X_2, \pi_2, D') \geq (X_1, \pi_1, D)$ iff D' is above D and the maps are just the pullback maps of divisors along the unique map of modifications.

Remark 2.14. By resolution of singularities, every Cartier b-divisor is toroidal with respect to some toroidal structure. So, toroidal b-divisors are in between Cartier and Weil b-divisors.

Definition 2.15. A Cartier b-divisor is called *nef* if it is nef in any modification of X where it is determined, and a Weil divisor is called *nef* if it belongs to the closure (with respect to the weak topology) of the space of nef Cartier b-divisors.

The following is a recent result by Dang and Favre:

Theorem 2.16 ([11, Theorem A]). *Any nef b-divisor is the limit of a decreasing sequence of nef Cartier b-divisors.*

Remark 2.17. If moreover the nef b-divisor is toroidal then by [3, Lemma 5.9] the sequence can be taken to consist of toroidal nef Cartier divisors (with respect to the same toroidal structure).

Remark 2.18. Using any approximating decreasing sequence of nef Cartier b-divisors, each nef b-divisor \mathbb{D} on X has a well-defined degree \mathbb{D}^n in $\mathbb{R}_{\geq 0}$; see [11, Theorem 3.2].

2.4. Psh functions, psh metrics, Lelong numbers. We briefly recall the notion of plurisubharmonic (psh) functions and metrics. A more thorough discussion with examples can be found in [4]. The following definition is taken from [29, Chapter 3].

Let X be a complex manifold.

Definition 2.19. Let U be an open coordinate subset of X , identified with an open subset of \mathbb{C}^n . A function $\varphi: U \rightarrow \mathbb{R} \cup \{-\infty\}$ is called *plurisubharmonic (psh)* if the following two conditions are satisfied:

- (1) the function φ is upper-semicontinuous and not identically equal to $-\infty$ on any connected component of U ;
- (2) for every $z \in U$ and every $a \in \mathbb{C}^n$ the function

$$\mathbb{C} \rightarrow \mathbb{C}, \quad \zeta \mapsto \varphi(z + a\zeta) \in \mathbb{R} \cup \{-\infty\}$$

is either identically $-\infty$, or subharmonic in each connected component of the open set $\{\zeta \in \mathbb{C} \mid z + a\zeta \in U\}$.

Now let $U \subset X$ be an arbitrary open subset. A function $\varphi: U \rightarrow \mathbb{R} \cup \{-\infty\}$ is called *psh* if U can be covered by open coordinate subsets U_i so that each restriction $\varphi|_{U_i}$ is psh.

We next discuss the notion of psh metrics. Let L be a line bundle on X .

Definition 2.20. Let $\{(U_i, s_i)\}$ be a trivialization of L , with transition functions $\{g_{ij}\}$. A *hermitian metric* h on L is a collection of measurable functions

$$\varphi_i: U_i \rightarrow \mathbb{R} \cup \{\pm\infty\},$$

such that the identities

$$(2.3) \quad e^{-\varphi_i} = |g_{ij}| e^{-\varphi_j}$$

hold on all $U_i \cap U_j$. The *norm* $h(s_i)$ is given by the formula

$$\varphi_i(z) = -\log h(s_i(z)), \quad z \in U_i.$$

From the identities (2.3) we find

$$\log h(s_i) - \log h(s_j) = \log |s_i/s_j|$$

on $U_i \cap U_j$. More generally, when s is any generating section of L locally near a point $z \in X$ we define its norm $h(s)$ via

$$\log h(s(z)) = \log |s(z)/s_i(z)| + \log h(s_i(z)) = \log |s(z)/s_i(z)| - \varphi_i(z)$$

whenever $z \in U_i$. This is easily seen to be independent of the choice of i .

We call the metric h *singular* (resp. *psh*, *continuous*, *smooth*) if the functions φ_i are all locally integrable (resp. psh, continuous, smooth).

An important measure of the singularities of a psh function is given by its Lelong numbers.

Definition 2.21. Let $U \subset X$ be an open coordinate subset, let φ be a psh function on U , and let $x \in U$ be a point. Then the *Lelong number* of φ at x is given as

$$\nu(\varphi, x) = \sup \{ \gamma \geq 0 \mid \varphi(z) \leq \gamma \log |z - x| + O(1) \text{ near } x \}.$$

The notion of Lelong number readily generalizes to the context of psh metrics. Let L be a line bundle on X equipped with a psh metric h and let $x \in X$ be a point. Choose an open coordinate subset $U \subset X$ and a generating section s of L on U . Then we put

$$\nu(h, x) = \nu(-\log h(s), x).$$

It is readily verified that this is independent of the choice of open set U and generating section s .

Let now $\pi: X' \rightarrow X$ be a smooth modification of X . Let (L, h) be a line bundle with a psh metric on X . It is easy to see that then (π^*L, π^*h) is a line bundle with a psh metric on X' . For $x \in X'$ we then put

$$\nu(h, x) = \nu(\pi^*h, x).$$

We note that if P is a prime divisor of X' and x, y are very general points on P , then the equality

$$\nu(h, x) = \nu(h, y)$$

holds. This leads to the following definition.

Definition 2.22. Let (L, h) be a line bundle equipped with a psh metric on X . Let $\pi \in R(X)$ be a smooth modification, and let P be a prime divisor of X_π . Then we write

$$\nu(h, P) = \inf_{x \in P} \nu(h, x).$$

The number $\nu(h, P)$ from Definition 2.22 allows the following alternative description.

Lemma 2.23. *Let (L, h) be a line bundle provided with a psh metric on X , let $\pi \in R(X)$ and let P be a prime divisor of X_π . For every point $x \in P$, for every generating section s of L around x and for every local equation g for P at x , the equality*

$$\nu(h, P) = \sup\{\gamma \geq 0 \mid -\log(h(s)) - \gamma \log |g| \text{ bounded above near } x\}$$

holds.

Proof. This follows from [5, Proposition 10.5]. \square

2.5. Some notions of convex analysis. In this section we recall some notions from convex analysis and prove some lemmas that will be useful later. Our basic reference for convex analysis is [30].

We fix a finite dimensional real vector space $N_{\mathbb{R}}$ and let $M_{\mathbb{R}}$ be its dual. We recall the definition of closed convex function (see [30, Theorem 7.1]).

Definition 2.24. Let $C \subseteq N_{\mathbb{R}}$ be a convex set. A convex function $f: C \rightarrow \mathbb{R} \cup \{\infty\}$ is called a *closed convex function* if the equivalent conditions

- (1) the epigraph $\{(x, y) \in C \times \mathbb{R} \mid y \geq f(x)\} \subset N_{\mathbb{R}} \times \mathbb{R}$ is a closed subset;
- (2) the function $\tilde{f}: N_{\mathbb{R}} \rightarrow \mathbb{R} \cup \{\infty\}$, given by $\tilde{f}(x) = f(x)$ for $x \in C$ and $\tilde{f}(x) = \infty$ if $x \notin C$ is lower semicontinuous;

are satisfied.

Note that if C is closed then condition 2 is equivalent to f being lower semicontinuous.

Definition 2.25. Let $C \subset N_{\mathbb{R}}$ be a convex set. The *recession cone* of C is the set of direction vectors of all rays contained in C :

$$\text{rec}(C) = \{y \in N_{\mathbb{R}} \mid C + y \subset C\}.$$

The set $\text{rec}(C)$ is a convex cone, which is closed if C is closed, and polyhedral if C is polyhedral.

Definition 2.26. Let $C \subset N_{\mathbb{R}}$ be a convex set and $\sigma = \text{rec}(C)$ its recession cone. Let $g: C \rightarrow \mathbb{R}$ be a closed convex function on C . The *recession function* of g is the function $\text{rec}(g): \sigma \rightarrow \mathbb{R} \cup \{\infty\}$ given, for any $x \in C$ and any $y \in \sigma$ by

$$(2.4) \quad \text{rec}(g)(y) = \lim_{\lambda \rightarrow +\infty} \frac{g(x + \lambda y) - g(x)}{\lambda}.$$

Lemma 2.27. *Let $C \subset N_{\mathbb{R}}$ be a convex set, $\sigma = \text{rec}(C)$ its recession cone and $g: C \rightarrow \mathbb{R}$ a closed convex function on C .*

- (1) *The recession function $\text{rec}(g)$ is well defined. That is, the limit (2.4) exists and does not depend on x . Moreover $\text{rec}(g)$ is a closed convex function.*
- (2) *If the function g is bounded above, then $\text{rec}(g)$ takes non-positive values. In particular it takes finite values.*
- (3) *If the function g is Lipschitz continuous with constant α then $\text{rec}(g)$ is also Lipschitz continuous with the same constant.*

Proof. The statement in (1) is the content of [30, Theorem 8.5]. To prove (2), let $x \in C$ and $y \in \text{rec}(C)$ and consider two real numbers $\lambda, \mu > 0$. We have

$$x + \lambda y = \frac{\mu}{\lambda + \mu}x + \frac{\lambda}{\lambda + \mu}(x + (\lambda + \mu)y).$$

The convexity of g yields

$$(2.5) \quad g(x + \lambda y) \leq \frac{\mu}{\lambda + \mu}g(x) + \frac{\lambda}{\lambda + \mu}g(x + (\lambda + \mu)y).$$

Since g is assumed to be bounded above, say by a constant B , equation (2.5) implies

$$g(x + \lambda y) \leq \inf_{\mu} \frac{\mu}{\lambda + \mu} g(x) + \frac{\lambda}{\lambda + \mu} B = g(x),$$

which implies that $\text{rec}(g)$ takes non-positive values, and in particular finite values.

Statement (3) follows from the computation

$$|\text{rec}(g)(u) - \text{rec}(g)(u')| = \lim_{\lambda \rightarrow +\infty} \frac{1}{\lambda} |g(v + \lambda u) - g(v + \lambda u')| \leq \alpha \|u - u'\|.$$

□

It is clear that $\text{rec}(g)$ is conical, in the sense that, for all real $\lambda \geq 0$ and all $y \in \sigma$

$$(2.6) \quad \text{rec}(g)(\lambda y) = \lambda \text{rec}(g)(y).$$

Recall that, if $\sigma \subset N_{\mathbb{R}}$ is a cone, the dual cone $\sigma^{\vee} \subset M_{\mathbb{R}}$ is given by

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} \mid m(x) \geq 0, \forall x \in \sigma\}.$$

Lemma 2.28. *Let C be a convex set and $\sigma = \text{rec}(C)$ its recession cone. Let $g: C \rightarrow \mathbb{R}$ be a bounded-above closed convex function on C and $\text{rec}(g)$ its recession function.*

(1) *For every $x \in C$ the function on σ given by*

$$y \mapsto g(x + y) - \text{rec}(g)(y)$$

is bounded above.

(2) *Assume that C is closed and g is Lipschitz continuous. Then, for every $m \in \text{int}(\sigma^{\vee})$ the function*

$$y \mapsto g(x + y) - \text{rec}(g)(y) + m(y)$$

is bounded below.

(3) *Assume that σ is full dimensional. Then for every $0 \neq m \in \sigma^{\vee}$ the function*

$$y \mapsto g(x + y) - \text{rec}(g)(y) + m(y)$$

is not bounded above.

Proof. Let $\lambda > 0$. The convexity of g yields

$$g(x + y) \leq \frac{1}{\lambda} g(x + \lambda y) + \left(1 - \frac{1}{\lambda}\right) g(x)$$

which implies

$$g(x + y) - g(x) \leq \frac{g(x + \lambda y) - g(x)}{\lambda}.$$

Hence $g(x + y) - \text{rec}(g)(y) \leq g(x)$ proving (1).

We prove (2) by contradiction. If the function in (2) is not bounded below, we can find a sequence of points $y_i \in \sigma$ for $i \geq 1$ satisfying $\|y_i\| \geq i$ and

$$(2.7) \quad g(x + y_i) - \text{rec}(g)(y_i) + m(y_i) \leq -i.$$

After choosing a subsequence we can further assume that the sequence $(y_i/\|y_i\|)$ converges to a point $y_0 \in \sigma$. We write $\lambda_i = \|y_i\|$ and $y_i^0 = y_i/\lambda_i$. Equation (2.7) implies that

$$(2.8) \quad \limsup_{i \rightarrow \infty} \frac{1}{\lambda_i} g(x + \lambda_i y_i^0) - \text{rec}(g)(y_i^0) + m(y_i^0) \leq 0.$$

Using that g is Lipschitz continuous we get

$$(2.9) \quad \lim_{i \rightarrow \infty} \frac{g(x + \lambda_i y_i^0) - g(x + \lambda_i y_0)}{\lambda_i} = 0.$$

By Lemma 2.27.2 the function $\text{rec}(g)$ is (Lipschitz) continuous. Since m is also continuous we obtain

$$(2.10) \quad \lim_{i \rightarrow \infty} -\text{rec}(g)(y_i^0) + m(y_i^0) + \text{rec}(g)(y_0) - m(y_0) = 0.$$

By Lemma 2.27.1 we know that

$$(2.11) \quad \lim_{i \rightarrow \infty} \frac{1}{\lambda_i} g(x + \lambda_i y_0) - \text{rec}(g)(y_0) = 0.$$

Since $m \in \text{int}(\sigma^\vee)$ and $0 \neq y_0 \in \sigma$ we deduce

$$(2.12) \quad m(y_0) > 0.$$

Summing up equations (2.9) to (2.12) we obtain

$$\lim_{i \rightarrow \infty} \frac{1}{\lambda_i} g(x + \lambda_i y_i^0) - \text{rec}(g)(y_i^0) + m(y_i^0) = m(y_0) > 0,$$

contradicting the inequality (2.8).

We next prove (3). The fact that σ is full dimensional implies that $\{0\}$ is a face of σ^\vee . Since $0 \neq m \in \sigma^\vee$, there is a $y_0 \in \sigma$ such that $m(y_0) = \varepsilon > 0$. Fix $x \in C$ and consider the one variable bounded above convex function

$$g_0(t) = g(x + ty_0)$$

Since g_0 is a bounded above convex function it is Lipschitz continuous for $t \geq 1$. The recession function of g_0 is

$$\text{rec}(g_0)(t) = \text{rec}(g)(ty_0).$$

We can apply (2) to this function to deduce that $g_0(t) - \text{rec}(g_0)(t) + (\varepsilon/2)t$ is bounded below. Therefore

$$g(x + ty_0) - \text{rec}(g)(ty_0) + m(ty_0) = g_0(t) - \text{rec}(g_0)(t) + \varepsilon t$$

can not be bounded above. □

2.6. Toroidal psh metrics and their Lelong numbers. We fix now a toroidal structure (X_1, π_1, D) on X and write $U = X_1 \setminus D$. Recall that we can view U as an open subset of X .

Definition 2.29. (Compare with [4, Definition 3.10]) A psh metric h on a line bundle L on X is called *toroidal with respect to D* if the following is satisfied.

- h is locally bounded on U .
- for every point $p \in D$ there is an open coordinate polydisc W with coordinates (z_1, \dots, z_d) with $|z_i| < e^{-c}$ for some constant c such that

$$W \cap D = \{z_1 \cdots z_r = 0\},$$

a generating section s of L on W , a bounded function γ on W and a bounded above convex Lipschitz continuous function g on the cone $\mathbb{R}_{\geq c}^r$ such that

$$-\log h(s)(z_1, \dots, z_d) = \gamma(z_1, \dots, z_d) + g(-\log |z_1|, \dots, -\log |z_r|).$$

We call D a *singularity divisor* of h .

Remark 2.30. If h is toroidal with respect to D and D' is a toroidal structure above D then h is also toroidal with respect to D' .

Let L be a line bundle with toroidal psh metric h (with respect to D). We now explain how to compute the Lelong numbers of h along toroidal exceptional divisors.

Let $\pi \in R(X, D)^{\text{tor}}$ be a toroidal model of (X, D) , $\mu: X_\pi \rightarrow X_1$ the corresponding map of modifications and P a prime toroidal exceptional divisor in X_π . Then P corresponds to a rational ray ρ_P in a cone σ of $\Pi(X_1, D)$. Let $y \in P$ be a generic point and let $x \in X_1$ be the image of y . Let W be a coordinate neighborhood of x as in Definition 2.29, s a generating section of L around x and g the convex function of the same definition. The cone σ is smooth and can be identified with $\mathbb{R}_{\geq 0}^r$ using its integral structure. This identification is canonical up to the ordering of the variables that can be fixed by the choice of coordinates. Then the function $\text{rec}(g)$ is canonically a function on σ .

Lemma 2.31. *Let v_P be the primitive generator of ρ_P . Then*

$$\nu(h, P) = -\text{rec}(g)(v_P)$$

Proof. Choose a small coordinate neighborhood V of y with coordinates (x_1, \dots, x_d) such that x_1 is a local equation for P . Since y is generic and V small we can assume that

$$(2.13) \quad \pi^{-1}(D) \cap V = P \cap V.$$

Let $v_\rho = (a_1, \dots, a_r)$. Then the map π can be written as

$$\pi(x_1, \dots, x_d) = (x_1^{a_1} u_1, \dots, x_1^{a_r} u_r, u_{r+1}, \dots, u_d),$$

where the u_i 's are functions on V . The condition (2.13) implies that u_i does not vanish on V for $i = 1, \dots, r$. Then

$$-\log h(s) = \gamma + g(-\log |u_1| - a_1 \log |x_1|, \dots, -\log |u_r| - a_r \log |x_1|).$$

Then Lemma 2.28.1 implies that

$$-\log h(s) - \text{rec}(g)(v_\rho)(-\log |x_1|)$$

is bounded above, but by Lemma 2.28.3 for every $\epsilon > 0$

$$-\log h(s) - \text{rec}(g)(v_\rho)(-\log |x_1|) + \epsilon(-\log |x_1|)$$

is not bounded above. By Lemma 2.23 we obtain that

$$\nu(h, P) = -\text{rec}(g)(v_\rho)$$

as required. □

3. PSH METRICS, B-DIVISORS AND GRADED LINEAR SERIES

In this section we relate graded linear series, b-divisors and psh metrics to one another. Recall that X denotes the complex manifold associated to a smooth projective variety over \mathbb{C} of dimension d .

3.1. From psh-metrics to graded linear series. Let $F = K(X)$ denote the field of rational functions on X . Let $U \subset X$ be a dense Zariski open subset and (L, h) a line bundle on X with a psh metric (see Definition 2.20). We assume furthermore that $(L, h)|_U$ is locally bounded (for example continuous or smooth). Following Nyström [37, Section 2.8] we define

$$H^0(X, L, h) = \{s \in H^0(X, L) \mid h(s) \text{ bounded}\}.$$

For s a non-zero rational section of L we define

$$\mathcal{L}(L, s, h) = \{f \in F \mid fs \in H^0(X, L, h)\},$$

$$\mathcal{R}(L, s, h) = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} \mathcal{L}(L^{\otimes \ell}, s^{\otimes \ell}, h^{\otimes \ell}) t^\ell \subset F[t].$$

Note that the latter is a graded linear series in the sense of Definition 2.1.

3.2. From graded linear series to b-divisors. Let $A = \bigoplus_{\ell \geq 0} A_\ell t^\ell \subset F[t]$ be a graded linear series of almost integral type, $\pi \in R(X)$ and X_π the corresponding model of X .

Lemma 3.1. *The set*

$$S = \{D \in \text{Div}_{\mathbb{R}}(X_\pi) \mid \forall \ell \geq 0, \forall f \in A_\ell \setminus \{0\}, \ell D + \text{div}(f) \geq 0\}$$

is not empty.

Proof. Since A is of almost integral type there is a graded linear series A' of integral type containing A . Every graded linear series of integral type is finitely generated. Therefore there are nonzero elements $f_i \in F$, $i = 1, \dots, r$ and positive integers ℓ_i , $i = 1, \dots, r$ such that A is contained in the subalgebra of $F[t]$ generated by the set $f_i t^{\ell_i}$, $i = 1, \dots, r$. Therefore

$$S \supset \{D \in \text{Div}_{\mathbb{R}}(X_\pi) \mid \ell_i D + \text{div}(f_i) \geq 0, i = 1, \dots, r\}. \quad \square$$

Definition 3.2. Let A be a graded linear series on X of almost integral type. We define

$$\text{b-div}(A) = (\text{b-div}(A)_\pi)_{\pi \in R(X)} \in \text{W-b-Div}_{\mathbb{R}}(X)$$

by

$$\text{b-div}(A)_\pi = \inf\{E \in \text{Div}_{\mathbb{R}}(X_\pi) \mid \ell E + \text{div}(f) \geq 0, \forall \ell \geq 0 \forall f \in A_\ell\},$$

where the infimum is defined componentwise. In other words, for every prime divisor P on X_π we put

$$\text{ord}_P(\text{b-div}(A)_\pi) = \sup\left\{\frac{-1}{\ell} \text{ord}_P(f) \mid \ell \geq 0, f \in A_\ell\right\}.$$

Note that $\text{b-div}(A)$ is well defined thanks to Lemma 3.1.

3.3. From psh metrics to b-divisors.

Definition 3.3. Let (L, h) be a line bundle on X together with a psh metric such that there is a dense open Zariski subset $U \subset X$ on which h is locally bounded, and let s be a non-zero rational section of L . The Weil \mathbb{R} -b-divisor $\mathbb{D}(L, s, h)$ is defined, for every $\pi \in R(X)$ and prime divisor P on X_π , by

$$\text{ord}_P \mathbb{D}(L, s, h)_\pi = \text{ord}_P \text{div}(s) - \nu(h, P).$$

Here $\nu(h, P)$ denotes the Lelong number from Definition 2.21.

Proposition 3.4. *The b-divisor $\mathbb{D}(L, s, h)$ is nef.*

Proof. This is proved in [4, Theorem 5.18]. □

Proposition 3.5. *Let (X_1, π_1, D) be a toroidal structure on X . Assume that the psh metric h is toroidal with respect to D in the sense of Definition 2.29.*

- (1) *The b-divisor $\mathbb{D} = \mathbb{D}(L, s, h) - \text{div}(s)$ is toroidal with respect to D . Note that this divisor only depends on the singularities of h and not on the particular section s .*
- (2) *Let σ be a cone of $\Pi(X_1, D)$ and Y the corresponding stratum. Let $\varphi_{\mathbb{D}}$ be the conical function on $\Pi(X_1, D)$ corresponding to \mathbb{D} . Choose a generic point x of Y and a small enough coordinate neighborhood W of x with $W \cap D$ given by the equation $z_1 \cdots z_r = 0$. After changing s if needed, we can assume that s is a generating section in W . Therefore, by the definition of toroidal psh metric, there is a bounded above convex Lipschitz continuous function g on a cone $\mathbb{R}_{\geq c}^r$ such that*

$$-\log h(s)(z_1, \dots, z_d) - g(-\log |z_1|, \dots, -\log |z_r|)$$

is bounded on $W \cap D$. Then

$$\varphi_{\mathbb{D}}|_{\sigma} = -\operatorname{rec}(g).$$

Proof. Let \mathbb{D}' be the projection of \mathbb{D} onto $W\text{-b-Div}_{\mathbb{R}}(X, D)^{\text{tor}}$. In order to prove (1) we have to show that

$$(3.1) \quad \iota(\mathbb{D}') = \mathbb{D},$$

where ι is the section (2.2) from toroidal b-divisors to general b-divisors. And for (2) we have to show that

$$(3.2) \quad \varphi_{\mathbb{D}'}|_{\sigma} = -\operatorname{rec}(g).$$

In view of the definition of $\mathbb{D}(L, s, h)$ using Lelong numbers, equation (3.2) is just a reformulation of Lemma 2.31. So it only remains to prove equation (3.1).

Let $\pi: X_{\pi} \rightarrow X$ be a smooth modification and P a prime divisor of X_{π} . We have to check that

$$(3.3) \quad \operatorname{ord}_P \mathbb{D} = \operatorname{ord}_P \iota(\mathbb{D}').$$

Let (X_2, π_2, D') be a toroidal structure above D with a map $f: X_2 \rightarrow X_{\pi}$ and such that $f^{-1}(P) \subset D'$. Let \widehat{P} be the strict transform of P in X_2 . In order to prove (3.3) it is enough to prove $\operatorname{ord}_{\widehat{P}} \mathbb{D} = \operatorname{ord}_{\widehat{P}} \iota(\mathbb{D}')$. Let v_P be the primitive vector in $\Pi(X_2, D')$ corresponding to \widehat{P} , σ the minimal cone of $\Pi(X_1, D)$ containing $r(D, D')(v_P)$, Y the stratum of X_1 corresponding to σ , x a generic point of Y , and W a small enough coordinate neighborhood of x with coordinates z_1, \dots, z_d such that $z_1 \cdots z_r = 0$ is an equation of $D \cap W$. Write D_i for the divisor $z_i = 0$, $i = 1, \dots, r$, and v_i for the corresponding primitive vector of $\Pi(X_1, D)$. Then the cone σ is generated by v_1, \dots, v_r .

Let $\mu: X_2 \rightarrow X_1$ be the map of modifications. Write $a_i = \operatorname{ord}_{\widehat{P}} \mu^* z_i$. Then

$$(3.4) \quad r_{D, D'}(v_P) = \sum_{i=1}^r a_i v_i,$$

and therefore

$$\operatorname{ord}_P \iota(\mathbb{D}') = \operatorname{ord}_{\widehat{P}} \iota(\mathbb{D}') = \sum_{i=1}^r a_i \varphi_{\mathbb{D}'}(v_i).$$

Let y be a generic point of \widehat{P} above x and choose a small enough coordinate neighborhood V with coordinates (x_1, \dots, x_d) such that x_1 is a local equation for \widehat{P} . As in the proof of Lemma 2.31, the map μ can be written as

$$\mu(x_1, \dots, x_d) = (x_1^{a_1} u_1, \dots, x_1^{a_r} u_r, u_{r+1}, \dots, u_d),$$

where the u_i 's are functions on V , and u_i does not vanish on V for $i = 1, \dots, r$. Choose a rational section s of L that generates L around x , so that

$$-\log h(s) = \gamma + g(-\log |z_1|, \dots, -\log |z_r|)$$

and $\varphi_{\mathbb{D}'}|_{\sigma} = -\operatorname{rec}(g)$. Then around y we have

$$-\log h(s) = \gamma + g(-\log |u_1| - a_1 \log |x_1|, \dots, -\log |u_r| - a_r \log |x_1|).$$

Hence

$$\operatorname{ord}_{\widehat{P}}(\mathbb{D}') = -\nu(h, P) = \operatorname{rec}(g) \left(\sum_{i=1}^r a_i v_i \right) = -\varphi_{\mathbb{D}'}(r_{D, D'}(v_{\widehat{P}})) = \operatorname{ord}_P \iota(\mathbb{D}'),$$

proving the result. \square

Proposition 3.5 has the following consequence

Corollary 3.6. *With the hypothesis of Proposition 3.5, if the b -divisor $\mathbb{D}(L, s, h)$ is Cartier, then for every cone $\sigma \in \Pi(X_1, D)$ and every decomposition*

$$-\log(h(s)(z_1, \dots, z_d)) = \gamma + g(-\log|z_1|, \dots, -\log|z_r|),$$

with γ locally bounded and g bounded above, convex and Lipschitz continuous, the function $\text{rec}(g|_\sigma)$ is piecewise linear.

Proof. If \mathbb{D} is Cartier, we know that $\varphi_{\mathbb{D}}|_\sigma$ is piecewise linear. Hence the Corollary is an immediate consequence of Proposition 3.5.2. \square

3.4. From b -divisors to graded linear series. Let \mathbb{D} be a Weil \mathbb{R} - b -divisor (not necessarily toroidal). We define

$$\begin{aligned} \mathcal{L}(\mathbb{D}) &= \{0 \neq f \in F \mid \mathbb{D}_\pi + \text{div}(f) \geq 0, \forall \pi \in R(X)\} \cup \{0\}, \\ \mathcal{R}(\mathbb{D}) &= \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} \mathcal{L}(\ell \mathbb{D}) t^\ell \subset F[t]. \end{aligned}$$

The latter is a graded linear series.

Lemma 3.7. *The graded linear series $\mathcal{R}(\mathbb{D})$ is of almost integral type.*

Proof. Let $\pi \in R(X)$. From the definition it follows that $\mathcal{R}(\mathbb{D}) \subset \mathcal{R}(\mathbb{D}_\pi)$. By Proposition 2.3 we know $\mathcal{R}(\mathbb{D}_\pi)$ is of almost integral type, so it is contained in a graded linear series A of integral type. \square

3.5. Summarizing the relations. Combining the previous subsections we obtain a diagram

$$(3.5) \quad \begin{array}{ccc} & & \text{graded linear series} \\ & \nearrow \mathcal{R} & \uparrow \\ \text{psh-metrics} & & \mathcal{R} \\ & \searrow \mathbb{D} & \downarrow \text{b-div} \\ & & \text{b-divisors} \end{array}$$

This diagram is in general not commutative. Much of the remainder of this section will be taken up with verifying certain weaker relations in this diagram.

Lemma 3.8. *Let \mathbb{D} and \mathbb{D}' be b -divisors and A and A' graded linear series.*

- (1) $\mathbb{D} \leq \mathbb{D}' \implies \mathcal{R}(\mathbb{D}) \subset \mathcal{R}(\mathbb{D}')$.
- (2) $A \subset A' \implies \text{b-div}(A) \leq \text{b-div}(A')$.
- (3) $\text{b-div}(\mathcal{R}(\mathbb{D})) \leq \mathbb{D}$.
- (4) $A \subset \mathcal{R}(\text{b-div}(A))$.

The same is true if we replace b -divisors by toroidal b -divisors.

Proof. These follow directly from the definitions. \square

Lemma 3.9. *Let (L, h, s) be a line bundle on X with a psh metric h and a non-zero rational section s of L . Then*

$$\mathcal{R}(L, s, h) \subset \mathcal{R}(\mathbb{D}(L, s, h)).$$

Proof. The elements of $\mathcal{R}(L, s, h)_\ell$ are rational functions f with $h(fs^{\otimes \ell})$ bounded, while the elements of $\mathcal{R}(\mathbb{D}(L, s, h))_\ell$ are rational functions $f \in \mathcal{L}(\ell \text{div}(s))$ with enough zeroes to cancel the Lelong numbers of h . The result then follows from the fact that bounded functions have zero Lelong numbers; in what follows we give some more details.

Let $f \in \mathcal{R}(L, s, h)_\ell$. From Definition 3.3 we can write $\mathbb{D} = \mathbb{D}(L, s, h) = (\mathbb{D}_\pi)_{\pi \in R(X)}$ with

$$\mathbb{D}_\pi = \pi^* \operatorname{div}(s) - \sum_P \nu(h, P)P,$$

where the sum is over all prime divisors P on X_π and $\nu(h, P)$ denotes the Lelong number of the metric h at a very general point of P . Let x be a very general point of P , s_0 a local generating section of L at x , and g a local equation of P at x . By Lemma 2.23 we have

$$(3.6) \quad \nu(h, P) = \sup\{\gamma \geq 0 \mid -\log(h(s_0)|g|^\gamma) \text{ bounded above near } x\}.$$

Since $h(fs^{\otimes \ell})$ is bounded we know that $-\log(h(s_0)^\ell |g|^{\operatorname{ord}_P(fs^{\otimes \ell})})$ is bounded below. Therefore, for every $\epsilon > 0$, $-\log(h(s_0)^\ell |g|^{\epsilon + \operatorname{ord}_P(fs^{\otimes \ell})})$ is *not* bounded above. Hence $\operatorname{ord}_P(fs^{\otimes \ell}) \geq \ell\nu(h, P)$ and therefore

$$\operatorname{div}(f) + \ell\mathbb{D}_\pi = \operatorname{div}(f) + \ell\pi^* \operatorname{div}(s) - \ell \sum_P \nu(h, P)P \geq 0,$$

so $f \in \mathcal{R}(\mathbb{D}(L, s, h))_\ell$. □

In the toroidal case we also have an inclusion in the reverse direction.

Lemma 3.10. *Let (X_π, D) be a toroidal structure on X and let h be a psh metric on a line bundle L on X that is toroidal with respect to D . Then for every real number $\epsilon > 0$ there is an inclusion*

$$\mathcal{R}(\mathbb{D}(L, s, h) - \epsilon D) \subset \mathcal{R}(L, s, h).$$

Proof. Let $f \in \mathcal{R}(\mathbb{D}(L, s, h) - \epsilon D)_\ell$ and let $p \in X_\pi$. We need to show that $\|fs^{\otimes \ell}\|$ is bounded near p , or equivalently that $-\log\|fs^{\otimes \ell}\|$ is bounded below near p . Let s' be another rational section of L with $s = vs'$ for some rational function v and let $f' = fv^\ell$. Then

$$f \in \mathcal{R}(\mathbb{D}(L, s, h) - \epsilon D)_\ell \iff f' \in \mathcal{R}(\mathbb{D}(L, s', h) - \epsilon D)_\ell,$$

and $\|fs^{\otimes \ell}\|$ is bounded if and only if $\|f's'^{\otimes \ell}\|$ is bounded. Therefore we can assume that s is a generating section around p .

Choose a small enough coordinate system W around p as in Definition 2.29 and let g and γ be the functions appearing in that definition. By Lemma 2.31, the condition $f \in \mathcal{R}(\mathbb{D}(L, s, h) - \epsilon D)_\ell$ implies that, on W , we can write

$$f = z_1^{m_1} \cdots z_r^{m_r} f_0$$

where f_0 is holomorphic, not divisible by z_1, \dots, z_r and the exponents m_i are integers with the property that for every primitive vector $u = (u_1, \dots, u_r)$ corresponding to a prime divisor P_u in an allowable modification, we have

$$(3.7) \quad \begin{aligned} \sum_{i=1}^r m_i u_i &= \operatorname{ord}_{P_u} f \\ &\geq -\ell \operatorname{ord}_{P_u}(\mathbb{D}(L, s, h) - \epsilon D) \\ &= -\ell \operatorname{rec}(g)(u) + \ell\epsilon \sum_{i=1}^r u_i. \end{aligned}$$

Since $\operatorname{rec}(g)$ is conical and continuous on $\mathbb{R}_{\geq 0}^r$ we deduce that, for every vector $u \in \mathbb{R}_{\geq 0}^r$ the inequality (3.7) holds. We compute

$$-\log\|fs^{\otimes \ell}\| = -\log|f_0| + \sum_{i=1}^r m_i(-\log|z_i|) + \ell\gamma + \ell g(-\log|z_1|, \dots, -\log|z_r|).$$

Using that f_0 is holomorphic, so $-\log |f_0|$ is bounded below, and that γ is bounded near p , the condition (3.7) implies that there is a constant B with

$$-\log \|f s^{\otimes \ell}\| \geq B + \ell g - \ell \operatorname{rec}(g) + \ell \varepsilon \sum_{i=1}^r (-\log |z_i|).$$

By Lemma 2.28.2 the quantity on the right is bounded below, hence we obtain the result. \square

Corollary 3.11. *Take the assumptions of Lemma 3.10. If for every irreducible component D_i of D the condition $\operatorname{ord}_{D_i} \mathbb{D}(L, s, h) > 0$ holds then, for every $\varepsilon > 0$,*

$$\mathcal{R}((1 - \varepsilon)\mathbb{D}(L, s, h)) \subset \mathcal{R}(L, s, h).$$

3.6. The case of a divisor generated by global sections.

Proposition 3.12. *Let \mathbb{D} be a \mathbb{Q} -Cartier b -divisor such that there is an $e \in \mathbb{Z}_{>0}$ with $e\mathbb{D}$ a globally generated integral Cartier divisor on some proper modification X_π of X . Then*

$$\mathfrak{b}\text{-div}(\mathcal{R}(\mathbb{D})) = \mathbb{D}.$$

Proof. By Lemma 3.8 we know that $\mathfrak{b}\text{-div}(\mathcal{R}(\mathbb{D})) \leq \mathbb{D}$. On the other hand

$$\bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} \mathcal{L}(\ell e\mathbb{D}_\pi) t^{\ell e} \subset \mathcal{R}(\mathbb{D});$$

indeed, if $e \in \mathbb{Z}_{\geq 0}$ and $f \in \mathcal{L}(\ell e\mathbb{D}_\pi)$ is non-zero, then $\operatorname{div}(f) + \ell e\mathbb{D}_{\pi_0} \geq 0$, and the same holds on every model $X_{\pi'}$ of X since pullback and pushforward preserve effectivity, hence $f \in \mathcal{R}(\mathbb{D})$.

Now let s_1, \dots, s_n be a set of generating sections of $\mathcal{O}(e\mathbb{D}_\pi)$, and s the canonical rational section of $\mathcal{O}(e\mathbb{D}_\pi)$ with $\operatorname{div}(s) = e\mathbb{D}_\pi$. Write $s_i = f_i s$. Since s_i is a global section, we have $e\mathbb{D}_\pi + \operatorname{div}(f_i) \geq 0$ and $f_i \in \mathcal{R}(\mathbb{D})_e$.

Since the sections s_i generate, for every prime divisor P on every modification $\pi' \in R(X)$ above π , there is an i such that

$$(3.8) \quad \operatorname{ord}_P(e\mathbb{D}_{\pi'} + \operatorname{div}(f_i)) = 0.$$

On the other hand, since $f_i \in \mathcal{R}(\mathbb{D})_e$ we see from the definition of $\mathfrak{b}\text{-div}(\mathcal{R}(\mathbb{D}))$ that

$$(3.9) \quad \operatorname{ord}_P(e \mathfrak{b}\text{-div}(\mathcal{R}(\mathbb{D})) + \operatorname{div}(f_i)) \geq 0.$$

Combining (3.8) and (3.9) we deduce

$$\mathfrak{b}\text{-div}(\mathcal{R}(\mathbb{D})) \geq \mathbb{D}. \quad \square$$

3.7. The volume of a b -divisor.

Definition 3.13. Let \mathbb{D} be a Weil b -divisor. The *volume* of \mathbb{D} is defined as

$$(3.10) \quad \operatorname{Vol}(\mathbb{D}) = \limsup_{\ell} \frac{\dim \mathcal{R}(\mathbb{D})_{\ell}}{\ell^d/d!}.$$

A Weil b -divisor \mathbb{D} is called *big* if $\operatorname{Vol}(\mathbb{D}) > 0$.

Remark 3.14. Since $\mathcal{R}(\mathbb{D})$ is a graded algebra of almost integral type, it follows from [19, Corollary 3.11] that the lim sup in (3.10) is in fact a lim for sufficiently divisible ℓ .

In case the b -divisor is Cartier, this definition agrees with the usual notion of the volume of a divisor (see e.g. [23, Definition 2.2.31]).

Lemma 3.15. *Let \mathbb{D} be a Weil b -divisor. If there is a big Cartier divisor B on some modification X_π of X such that $m\mathbb{D} \geq B$ for some $m > 0$, then \mathbb{D} is big.*

Proof. Since B is assumed to be big, we have $\text{Vol}(B) > 0$. On the other hand,

$$\dim \mathcal{R}(\mathbb{D})_{m\ell} \geq \dim \mathcal{R}(B)_\ell$$

by the assumption that $m\mathbb{D} \geq B$. Therefore,

$$\text{Vol}(\mathbb{D}) \geq \limsup_\ell \frac{\dim \mathcal{R}(\mathbb{D})_{m\ell}}{m^d \ell^d / d!} \geq \frac{1}{m^d} \limsup_\ell \frac{\dim \mathcal{R}(B)_\ell}{\ell^d / d!} = \frac{1}{m^d} \text{Vol}(B) > 0.$$

□

3.8. The case of a toroidal nef and big b-divisor.

Theorem 3.16. *Let (X_1, π_1, D) be a toroidal structure on X . Let \mathbb{D} be a nef and big b-divisor which is toroidal with respect to D . Then*

$$\text{b-div}(\mathcal{R}(\mathbb{D})) = \mathbb{D}.$$

The proof requires an intermediate lemma. We begin by constructing some auxiliary objects. By [3, Lemma 4.9] we know that there is a sequence $\{\mathbb{D}_i\}_{i \in \mathbb{N}}$ of toroidal \mathbb{Q} -Cartier b-divisors, generated by global sections and converging monotonically decreasing to \mathbb{D} . Moreover by the proof of [3, Lemma 5.12] we can pick a sequence of big toroidal \mathbb{Q} -Cartier b-divisors $\{\mathbb{B}_j\}_{j \in \mathbb{N}}$ with $\mathbb{B}_j \leq \mathbb{D}$ and $\text{Vol}(\mathbb{B}_j)$ converging to $\text{Vol}(\mathbb{D})$. This is where the toroidal condition is used.

By Fujita's approximation theorem [23, Theorem 11.4.4], for every $j > 0$ there exists a \mathbb{Q} -Cartier b-divisor \mathbb{A}_j generated by global sections and satisfying

$$\text{Vol}(\mathbb{A}_j) \geq \text{Vol}(\mathbb{B}_j) - \frac{1}{j} \quad \text{and} \quad \mathbb{A}_j \leq \mathbb{B}_j.$$

Thus $\text{Vol}(\mathbb{A}_j)$ also converges to $\text{Vol}(\mathbb{D})$.

The key technical result is

Lemma 3.17. *We have*

$$(3.11) \quad \sup_j \mathbb{A}_j = \mathbb{D},$$

where the supremum is computed componentwise.

Proof. We need to show that, for every $\pi \in R(X)$ and every prime divisor P in X_π , we have

$$\sup_j (\text{ord}_P(\mathbb{A}_j)) = \text{ord}_P(\mathbb{D}).$$

We proceed by contradiction; suppose this does *not* hold. This means that there exists a proper modification $\pi \in R(X)$, a prime divisor P on X_π and a positive number $\varepsilon > 0$ such that for all j ,

$$\text{ord}_P(\mathbb{A}_j) \leq \text{ord}_P(\mathbb{D}) - \varepsilon.$$

In what follows we use the theory of Okounkov bodies, for which we refer to [19, 24] for more details. Upgrade P to a complete flag

$$\mathcal{F}: \quad P = Y_1 \supset Y_2 \supset \cdots \supset Y_d$$

and for a graded linear series A denote by $O_{\mathcal{F}}(A)$ the Okounkov body of A on X associated to this flag. We briefly recall its construction. By an iterative procedure, one constructs for each $\ell \in \mathbb{N}$ a valuation map

$$\nu_{\mathcal{F}}: A_\ell \setminus \{0\} \longrightarrow \mathbb{Z}^d$$

by taking the order of vanishing along the given Y_i into account. So, if $f \in A_\ell$,

$$\nu_{\mathcal{F}}(f) = (\text{ord}_P f, *, \dots, *).$$

That is, the first component of $\nu_{\mathcal{F}}(f)$ is the order of f at P . This gives rise to a semigroup

$$\Gamma(A) = \{(\nu_{\mathcal{F}}(f), \ell) \mid f \in A_{\ell} \setminus \{0\}, \ell \in \mathbb{N}\} \subset \mathbb{Z}^d \times \mathbb{N}.$$

The Okounkov body of A with respect to \mathcal{F} is then given by

$$O_{\mathcal{F}}(A) = \overline{\text{cone}(\Gamma(A))} \cap (\mathbb{R}^d \times \{1\}).$$

It is a closed convex set of \mathbb{R}^d , and if A is of almost integral type, then it is bounded, hence compact [19, Theorem 2.30].

Let $\omega_P: \mathbb{R}^d \rightarrow \mathbb{R}$ be the projection onto the first variable. If $0 \neq f \in A_{\ell}$ is of degree ℓ and $x = \nu_{\mathcal{F}}(f)/\ell$ the corresponding point in the Okounkov body, then by construction one has

$$\omega_P(x) = \text{ord}_P(f)/\ell.$$

For a b-divisor \mathbb{E} we write $O_{\mathcal{F}}(\mathbb{E})$ for the Okounkov body $O_{\mathcal{F}}(\mathcal{R}(\mathbb{E}))$. From $\mathbb{A}_j \leq \mathbb{D} \leq \mathbb{D}_i$ we have $\mathcal{R}(\mathbb{A}_j) \subset \mathcal{R}(\mathbb{D}) \subset \mathcal{R}(\mathbb{D}_i)$ and hence

$$O_{\mathcal{F}}(\mathbb{A}_j) \subset O_{\mathcal{F}}(\mathbb{D}) \subset O_{\mathcal{F}}(\mathbb{D}_i)$$

for all natural numbers i, j . Since each \mathbb{D}_i is generated by global sections there exist $f_i \in H^0(X_i, \ell_i \mathbb{D}_i)$ for some ℓ_i such that

$$\text{ord}_P(f_i)/\ell_i = -\text{ord}_P \mathbb{D}_i \leq -\text{ord}_P(\mathbb{D}).$$

Therefore, there is a point $x_i \in O_{\mathcal{F}}(\mathbb{D}_i)$ with

$$\omega_P(x_i) \leq -\text{ord}_P(\mathbb{D}).$$

Since $O_{\mathcal{F}}(\mathbb{D}_1)$ is compact and $O_{\mathcal{F}}(\mathbb{D}_i) \subset O_{\mathcal{F}}(\mathbb{D}_1)$, the sequence $\{x_i\}_{i \in \mathbb{N}}$ has at least one accumulation point x . Moreover we claim that $\bigcap O_{\mathcal{F}}(\mathbb{D}_i) = O_{\mathcal{F}}(\mathbb{D})$. Indeed, we have $\bigcap O_{\mathcal{F}}(\mathbb{D}_i) \subset O_{\mathcal{F}}(\mathbb{D})$. On the other hand $\{O_{\mathcal{F}}(\mathbb{D}_i)\}_{i \in \mathbb{N}}$ form a decreasing (under inclusion) sequence of compact convex sets, hence their intersection $O = \bigcap_i O_{\mathcal{F}}(\mathbb{D}_i)$ is again a compact convex set. We have that

$$\text{Vol}(O) = \lim_i \text{Vol}(O_{\mathcal{F}}(\mathbb{D}_i)) = \lim_i \text{Vol}(\mathbb{D}_i) = \text{Vol}(\mathbb{D}) = \text{Vol}(O_{\mathcal{F}}(\mathbb{D})),$$

where the second and last equalities follow from [3, Theorem 5.13]. This proves the claim since two full-dimensional compact convex sets with equal volume and such that one is contained in the other have to agree. Now, the compactness of $O_{\mathcal{F}}(\mathbb{D})$ implies that the accumulation point x lies in $O_{\mathcal{F}}(\mathbb{D})$. In particular, there is a point $x \in O_{\mathcal{F}}(\mathbb{D})$ with $\omega_P(x) \leq -\text{ord}_P(\mathbb{D})$.

On the other hand, since $\text{ord}_P(\mathbb{A}_j) \leq \text{ord}_P(\mathbb{D}) - \varepsilon$ we have that

$$\emptyset \neq O_{\mathcal{F}}(\mathbb{A}_j) \subset \{x \in \mathbb{R}^d \mid \omega_P(x) \geq -\text{ord}_P(\mathbb{D}) + \varepsilon\}.$$

The set $O_{\mathcal{F}}(\mathbb{D})$ is convex, has non-zero volume, contains a point with $\omega_P(x) \leq -\text{ord}_P(\mathbb{D})$, and also contains a point y with $\omega_P(y) \geq -\text{ord}_P(\mathbb{D}) + \varepsilon$ (just choose any point of $O_{\mathcal{F}}(\mathbb{A}_j)$ for some j). Hence

$$\text{Vol}(O_{\mathcal{F}}(\mathbb{D}) \cap \{-\text{ord}_P(\mathbb{D}) \leq \omega_P \leq -\text{ord}_P(\mathbb{D}) + \varepsilon\}) =: \eta > 0.$$

This implies that $\text{Vol}(O_{\mathcal{F}}(\mathbb{A}_j)) \leq \text{Vol}(O_{\mathcal{F}}(\mathbb{D})) - \eta$, contradicting the fact that $\text{Vol}(O_{\mathcal{F}}(\mathbb{A}_j))$ converges to $\text{Vol}(O_{\mathcal{F}}(\mathbb{D}))$. \square

Proof of Theorem 3.16. From Lemma 3.8 and the inequalities $\mathbb{A}_j \leq \mathbb{D} \leq \mathbb{D}_i$ for any natural numbers i, j , we deduce

$$\text{b-div}(\mathcal{R}(\mathbb{A}_j)) \leq \text{b-div}(\mathcal{R}(\mathbb{D})) \leq \text{b-div}(\mathcal{R}(\mathbb{D}_i)).$$

By Lemma 3.8 and Proposition 3.12 we get

$$\mathbb{A}_j \leq \text{b-div}(\mathcal{R}(\mathbb{D})) \leq \mathbb{D}_i.$$

Invoking Lemma 3.17 and once more Lemma 3.8, we get

$$\mathbb{D} = \sup(\mathbb{A}_j) \leq \text{b-div}(\mathcal{R}(\mathbb{D})) \leq \mathbb{D}. \quad \square$$

Remark 3.18. The toroidal condition in Theorem 3.16 is necessary. Indeed, consider the b-divisor \mathbb{D} from [4, Appendix A]. Then \mathbb{D} is a nef and big b-divisor on \mathbb{P}^2 . It satisfies

$$\mathcal{R}(\mathbb{D}) = \mathcal{R}(2H - L),$$

where L and H are two lines in \mathbb{P}^2 as defined in *loc. cit.* Hence, by the construction we get

$$\mathbb{D} \neq 2H - L = \text{b-div}(\mathcal{R}(2H - L)) = \text{b-div}(\mathcal{R}(\mathbb{D})).$$

From Lemma 3.10 and Theorem 3.16 we obtain also the following compatibility in the case of toroidal psh metrics.

Corollary 3.19. *Let (L, h) be a line bundle with a toroidal psh metric with singularity divisor D , and s a non-zero rational section of L . If $\mathbb{D}(L, s, h)$ is nef and big and for every irreducible component D_i of D the condition $\text{ord}_{D_i}(\mathbb{D}(L, s, h)) > 0$ holds, then*

$$\text{b-div}(\mathcal{R}(L, s, h)) = \mathbb{D}(L, s, h).$$

Proof. By Lemma 3.9 and Corollary 3.11, for every $j > 1$ we have

$$\mathcal{R}(((1 - 1/j)\mathbb{D}(L, s, h)) \subset \mathcal{R}(L, s, h) \subset \mathcal{R}(\mathbb{D}(L, s, h)).$$

Since h is a toroidal psh metric, by Proposition 3.5 the b-divisor $\mathbb{D}(L, s, h)$ is toroidal with respect to a toroidal structure D' above D . Note that we need a toroidal structure above D in order to make $\text{div}(s)$ toroidal. By Lemma 3.8 and Theorem 3.16, we deduce

$$(1 - 1/j)\mathbb{D}(L, s, h) \leq \text{b-div}(\mathcal{R}(L, s, h)) \leq \mathbb{D}(L, s, h).$$

Since $(1 - 1/j)\mathbb{D}(L, s, h)$ converges to $\mathbb{D}(L, s, h)$ when $j \rightarrow \infty$, we obtain the corollary. \square

3.9. Criterion for not being finitely generated.

Lemma 3.20. *If A is a finitely generated graded linear series on X , then $\text{b-div}(A)$ is a \mathbb{Q} -Cartier b-divisor.*

Proof. Assume that A is generated by $f_1 t^{d_1}, \dots, f_n t^{d_n}$, with $f_i \in F = K(X)$.

Claim. If a b-divisor \mathbb{D} satisfies $d_i \mathbb{D} + \text{div}(f_i) \geq 0$, for $i = 1, \dots, n$, then $\ell \mathbb{D} + \text{div}(f) \geq 0$ for all $\ell \geq 0$ and all $f \in A_\ell$.

The claim follows from the compatibility of multiplicity with products and the ultrametric inequality for sums. Namely, if $f \in A_\ell$ and $f' \in A_{\ell'}$, then the compatibility of the valuation with products yields,

$$\left. \begin{array}{l} \ell \mathbb{D} + \text{div}(f) \geq 0 \\ \ell' \mathbb{D} + \text{div}(f') \geq 0 \end{array} \right\} \implies (\ell + \ell') \mathbb{D} + \text{div}(ff') \geq 0,$$

and if $f, f' \in A_\ell$ the ultrametric inequality implies

$$\left. \begin{array}{l} \ell \mathbb{D} + \text{div}(f) \geq 0 \\ \ell \mathbb{D} + \text{div}(f') \geq 0 \end{array} \right\} \implies \ell \mathbb{D} + \text{div}(f + f') \geq 0.$$

From the claim we deduce

$$\text{b-div}(A)_\pi = \inf \{ D \in \text{Div}_\mathbb{R}(X_\pi) \mid d_i D + \text{div}(f_i) \geq 0, i = 1, \dots, n \}.$$

Choose D_0 such that $d_i D_0 + \text{div}(f_i) \geq 0$. Let $e = \text{lcm}(d_1, \dots, d_n)$ and let \mathfrak{a} be the fractional ideal generated by f_i^{e/d_i} , $i = 1, \dots, n$. Since $f_i^{e/d_i} \in \mathcal{O}(eD_0)$, we obtain $\mathfrak{a} \subset \mathcal{O}(eD_0)$, so

$$\mathcal{I} := \mathfrak{a} \mathcal{O}_X(-eD_0) \subset \mathcal{O}_X$$

is a coherent ideal sheaf.

Let $\pi_0: X_{\pi_0} \rightarrow X$ be a proper modification such that $\pi^{-1}(\mathcal{I})\mathcal{O}_{X_{\pi_0}}$ is principal and equal to $\mathcal{O}(-D')$ for some effective integral divisor D' . Then

$$e(\mathfrak{b}\text{-div}(A)) = eD_0 - D',$$

and in particular $\mathfrak{b}\text{-div}(A)$ is \mathbb{Q} -Cartier. \square

4. SIEGEL–JACOBI FORMS

4.1. Basic definitions. We recall the definition of Siegel–Jacobi forms. For more details we refer to [22] and [43].

Let $g \geq 1$ be an integer. The real symplectic group $\text{Sp}(2g, \mathbb{R})$ is the group of real $2g \times 2g$ matrices of the form

$$(4.1) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

such that

$$A^t C = C^t A, \quad D^t B = B^t D, \quad A^t D = \text{Id}_g + C^t B,$$

where Id_g is the identity matrix of dimension g . There is an inclusion

$$\text{Sp}(2g, \mathbb{R}) \hookrightarrow \text{Sp}(2g + 2, \mathbb{R})$$

sending a matrix of the form (4.1) to

$$\begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For any commutative ring R let $H_R^{(g,1)}$ be the Heisenberg group

$$H_R^{(g,1)} = \{[(\lambda, \mu), x] \mid \lambda, \mu \in R^{(1,g)}, x \in R\},$$

where $R^{(1,g)}$ denotes the set of row vectors of size g and coefficients in R , with the composition law given by

$$[(\lambda, \mu), x] \circ [(\lambda', \mu'), x'] = [\lambda + \lambda', \mu + \mu', x + x' + \lambda\mu'^t - \mu\lambda'^t].$$

These are the same definitions as in [43] for the case $g = 1$. The real Heisenberg group $H_{\mathbb{R}}^{(g,1)}$ can be realized as the subgroup of $\text{Sp}(2g + 2, \mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} \text{Id}_g & 0 & 0 & \mu^t \\ \lambda & 1 & \mu & x \\ 0 & 0 & \text{Id}_g & -\lambda^t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The full Jacobi group $G_{\mathbb{R}}^{(g,1)} = \text{Sp}(2g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,1)}$ is the subgroup of $\text{Sp}(2g + 2, \mathbb{R})$ generated by $\text{Sp}(2g, \mathbb{R})$ and $H_{\mathbb{R}}^{(g,1)}$.

Let

$$\mathcal{H}_g = \{Z = X + iY \mid X, Y \in \text{Mat}_{g \times g}(\mathbb{R}), Z^t = Z, Y > 0\}$$

be the Siegel upper half space. The group $\text{Sp}(2g, \mathbb{R})$ acts transitively on \mathcal{H}_g , where for

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{R})$ and $Z \in \mathcal{H}_g$ the action is given by

$$Z \mapsto M\langle Z \rangle = (AZ + B)(CZ + D)^{-1}.$$

On the other hand, the group $G_{\mathbb{R}}^{(g,1)}$ acts transitively on $\mathcal{H}_g \times \mathbb{C}^{(1,g)}$ by the action

$$(M, (\lambda, \mu, x)) \cdot (Z, W) = (M\langle Z \rangle, (W + \lambda Z + \mu)(CZ + D)^{-1}).$$

Let $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ be a subgroup of finite index. We write $\tilde{\Gamma} = \Gamma \ltimes H_{\mathbb{Z}}^{(g,1)} \subset G_{\mathbb{Z}}^{(g,1)}$.

Recall that a subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ is called *neat* when for every $M \in \Gamma$, the subgroup of \mathbb{C}^\times generated by the eigenvalues of M is torsion free. If Γ is neat, then the quotient $\mathcal{A}(\Gamma) = \Gamma \backslash \mathcal{H}_g$ is a smooth complex manifold and the quotient $\mathcal{B}(\Gamma) = \tilde{\Gamma} \backslash \mathcal{H}_g \times \mathbb{C}^{(1,g)}$ is a fibration over $\mathcal{A}(\Gamma)$ by principally polarized abelian varieties.

Following [43] we now introduce automorphy factors for the group $G_{\mathbb{R}}^{(g,1)}$ in order to obtain interesting line bundles on the quotient $\mathcal{B}(\Gamma)$.

Let $\phi: \mathcal{H}_g \times \mathbb{C}^g \rightarrow \mathbb{C}$ be a holomorphic map. Let $\rho_k: \mathrm{GL}(g, \mathbb{C}) \rightarrow \mathbb{C}^\times$ be the representation given by $N \mapsto (\det N)^k$. For $M \in \mathrm{Sp}(2g, \mathbb{R})$, $\zeta = (\lambda, \mu, x) \in H_{\mathbb{R}}^{(g,1)}$ and $m \in \mathbb{N}_{\geq 0}$ define

$$(4.2) \quad (\phi|_{k,m} M)(Z, W) := \rho_k(CZ + D)^{-1} e^{-2\pi i m W(CZ + D)^{-1} C W^t} \phi(M\langle Z \rangle, W(CZ + D)^{-1})$$

and

$$(4.3) \quad (\phi|_{k,m} \zeta)(Z, W) := e^{2\pi i m (\lambda Z \lambda^t + 2\lambda W^t + (x + \mu \lambda^t))} \phi(Z, W + \lambda Z + \mu).$$

A matrix T is called *half integral* if $2T$ has integral entries and the diagonal entries of T are integral. Note that if T is symmetric then T is half integral if and only if the associated quadratic form is integral.

Definition 4.1. A holomorphic map $\phi: \mathcal{H}_g \times \mathbb{C}^g \rightarrow \mathbb{C}$ is called a *Siegel–Jacobi form of weight k and index m* for a subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ of finite index if the following conditions are satisfied:

- (1) $\phi|_{k,m} M = \phi$ for all $M \in \Gamma$.
- (2) $\phi|_{k,m} \zeta = \phi$ for all $\zeta \in H_{\mathbb{Z}}^{(g,1)}$.
- (3) For each $M \in \mathrm{Sp}(2g, \mathbb{Z})$ the function $\phi|_{k,m} M$ has a Fourier expansion of the form

$$(4.4) \quad (\phi|_{k,m} M)(Z, W) = \sum_{\substack{T=T^t \geq 0 \\ T \text{ half integral}}} \sum_{R \in \mathbb{Z}^{(g,1)}} c(T, R) e^{2\pi i / \lambda_\Gamma \mathrm{tr}(TZ)} e^{2\pi i WR}$$

for some suitable integer $0 < \lambda_\Gamma \in \mathbb{Z}$, and such that $c(T, R) \neq 0$ implies

$$\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & m \end{pmatrix} \geq 0.$$

A Siegel–Jacobi form ϕ is said to be a *cuspidal form* if

$$\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} R^t & m \end{pmatrix} > 0.$$

for any T, R with $c(T, R) \neq 0$.

We note that when $g \geq 2$, condition 3 is a consequence of conditions 1 and 2 due to the Koecher principle [43, Lemma 1.6].

Definition 4.2. The vector space of all Siegel–Jacobi forms of weight k and index m for Γ is denoted by $J_{k,m}(\Gamma)$ and the space of cuspidal forms is denoted by $J_{k,m}^{\mathrm{cusp}}(\Gamma)$.

The following lemma follows easily from the definitions.

Lemma 4.3. *If ϕ is a Siegel–Jacobi form of weight k and index m and ψ is a Siegel–Jacobi cuspidal form of weight k' and index m' , then $\phi\psi$ is a Siegel–Jacobi cuspidal form of weight $k + k'$ and index $m + m'$.*

The following result is [43, Theorem 1.5]

Lemma 4.4. *Let ϕ be a Siegel–Jacobi form of weight k and index m for a subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ of finite index and let $\lambda, \mu \in \mathbb{Q}^{(1,g)}$ be rational vectors. Then there is a finite index subgroup $\Gamma' \subset \mathrm{Sp}(2g, \mathbb{Z})$ that depends only on Γ , λ and μ such that the function*

$$f(Z) = e^{2\pi i m \lambda Z \lambda^t} \phi(Z, \lambda Z + \mu)$$

is a Siegel modular form of weight k for Γ' .

Example 4.5. Here we list some examples of Siegel–Jacobi forms which turn out to be useful for our purposes.

- (1) The only Siegel–Jacobi forms of weight 0 and index 0 are the constants (i.e. $J_{0,0}(\Gamma) = \mathbb{C}$). Moreover $J_{k,m}(\Gamma) = 0$ whenever $k < 0$ or $m < 0$. The first case follows from Lemma 4.4 using the Corollary to Proposition 1 in [21, Section 4] or the remark after Theorem 1 in [21, Section 8], while the second follows from the geometric description (4.7) of the line bundle of Siegel–Jacobi forms; indeed, if $m < 0$ the restriction to any fibre is strictly anti-effective.
- (2) Any Siegel modular form of weight k for Γ defines a Siegel–Jacobi form of weight k and index 0. A classical example is as follows (for details we refer to [43, Section 2]). For $k > g + 1$ even, the *Eisenstein series*

$$(4.5) \quad E_{g,k}(Z) = \sum_{M \in P_g \backslash \mathrm{Sp}(2g, \mathbb{Z})} \rho_k(CZ + D)^{-1}$$

is convergent and defines a Siegel modular form of weight k for the group $\mathrm{Sp}(2g, \mathbb{Z})$, in particular for any subgroup $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$. Hence it defines a Siegel–Jacobi form of weight k and index 0 for Γ . In equation (4.5), P_g is the subgroup

$$P_g = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}) \mid C = 0 \right\}$$

and M is written as in (4.1).

- (3) Similarly, *Poincaré series* can be used to produce non-zero Siegel cusp forms of any weight $k > 2g$ such that kg is even (see [21, Proposition 2 and its Corollary]). By pullback they produce Siegel–Jacobi cusp forms of index zero.
- (4) Eisenstein series can be generalized to produce Siegel–Jacobi modular forms of arbitrary index. Let $m > 0$ be an integer and $k > g + 2$ be an even integer. Write

$$H_\infty = \left\{ [(\lambda, \mu), x] \in H_{\mathbb{Z}}^{(g,1)} \mid \lambda = 0 \right\}.$$

Then the series

$$E_{g,k,m}(Z, W) = \sum_{M \in P_g \backslash \mathrm{Sp}(2g, \mathbb{Z})} \rho_k(CZ + D)^{-1} e^{-2\pi i m W (CZ + D)^{-1} C W^t} \sum_{\lambda \in H_\infty \backslash H_{\mathbb{Z}}^{(g,1)}} e^{2\pi i m (\lambda M \langle Z \rangle \lambda^t + 2\lambda (CZ + D)^{-1} W^t)}$$

is convergent and defines a Siegel–Jacobi modular form of weight k and index m . See [43, Theorem 2.1] for details.

Lemma 4.6. *If ϕ is a Siegel–Jacobi form of weight 0 and index $m \neq 0$, then $\phi = 0$.*

Proof. Let ϕ be such a Siegel–Jacobi form. For $Z \in \mathcal{H}_g$ and $\lambda, \mu \in \mathbb{R}^{(1,g)}$, write

$$f(Z, \lambda, \mu) = e^{2\pi i m \lambda Z \lambda^t} \phi(Z, \lambda Z + \mu).$$

By Lemma 4.4, whenever $\lambda, \mu \in \mathbb{Q}^{(1,g)}$ are fixed we obtain a Siegel modular form of weight zero for a certain finite index subgroup of $\mathrm{Sp}(2g, \mathbb{Z})$, which is necessarily constant. By continuity, for every $\lambda, \mu \in \mathbb{R}^{(1,g)}$ the function $f(Z, \lambda, \mu)$ is constant. Therefore, there is a function $g: \mathbb{R}^{(1,g)} \times \mathbb{R}^{(1,g)} \rightarrow \mathbb{C}$ such that

$$(4.6) \quad \phi(Z, \lambda Z + \mu) = e^{-2\pi i m \lambda Z \lambda^t} g(\lambda, \mu).$$

We now see that this is incompatible with the modular condition $\phi|_{k,m} M = \phi$ for all $M \in \Gamma$. Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,$$

then the modular condition and equation (4.6) imply that

$$e^{-2\pi i m (\lambda Z + \mu)(CZ + D)^{-1} C(\lambda Z + \mu)^t + \lambda' M(Z) \lambda'^t - \lambda Z \lambda^t} g(\lambda', \mu') = g(\lambda, \mu),$$

where

$$(\lambda', \mu') = (\lambda, \mu) M^{-1}.$$

When $m \neq 0$, $M \neq \mathrm{Id}$ and $\lambda \neq 0$, the exponential term actually depends on Z . As the function g does not depend on Z , we conclude that $g = 0$. \square

A standard consequence of the previous examples and Lemma 4.6 is the following (see also [16]).

Proposition 4.7. *The ring $J_{*,*}(\Gamma) := \bigoplus_{k,m} J_{k,m}(\Gamma)$ is not finitely generated.*

Proof. Assume that the ring $J_{*,*}(\Gamma)$ is finitely generated. Let $1, f_i, i = 1, \dots, r$ be a set of homogeneous generators with f_i non constant. Let k_i and m_i be the weight and index of f_i . Then by Example 4.5.1 and Lemma 4.6 we have $k_i > 0$ for $i = 1, \dots, r$. Thus the possible ratios m_i/k_i are bounded above. Therefore if f is a non-constant Siegel–Jacobi modular form of weight k and index m , then the ratio m/k is bounded. But in Example 4.5.4 we have a construction of non-zero Siegel–Jacobi modular forms of fixed weight and arbitrary index. \square

4.2. The line bundle of Siegel–Jacobi forms. As mentioned in the introduction, to the neat arithmetic group Γ we associate a fibration of principally polarized abelian varieties over a complex manifold

$$\pi: \mathcal{B}(\Gamma) \longrightarrow \mathcal{A}(\Gamma).$$

The transformations (4.2) and (4.3) define a cocycle for the group $\tilde{\Gamma}$. Therefore they determine a line bundle $L_{k,m}$ such that the Siegel–Jacobi forms of weight k and index m can be seen as global sections of this line bundle. We recall the geometric interpretation of the line bundle $L_{k,m}$.

Let $e: \mathcal{A}(\Gamma) \rightarrow \mathcal{B}(\Gamma)$ be the zero section. Let M be the line bundle on $\mathcal{A}(\Gamma)$ defined as

$$M = \det \left(e^* \Omega_{\mathcal{B}(\Gamma)/\mathcal{A}(\Gamma)}^1 \right).$$

The space \mathbb{C}^g comes equipped with canonical holomorphic coordinates (z_1, \dots, z_g) on \mathbb{C}^g . Then $dz_1 \wedge \dots \wedge dz_g$ is a multivalued section of M well defined up to a global constant. Note that this is a multivalued section because $dz_1 \wedge \dots \wedge dz_g$ is not invariant under the action of Γ . Let f be a Siegel modular form of weight k . Then one can verify that the symbol

$$f(dz_1 \wedge \dots \wedge dz_g)^{\otimes k}$$

is invariant under the action of Γ and therefore determines a section of the line bundle $M^{\otimes k}$. In this way we obtain an identification of $L_{k,0}$ with $\pi^* M^{\otimes k}$.

Since $\mathcal{B}(\Gamma)$ is a family of principally polarized abelian varieties it comes equipped with a biextension line bundle B . This line bundle is defined as follows. Let $\mathcal{B}(\Gamma)^\vee$ be the dual family of abelian varieties and P the Poincaré line bundle on $\mathcal{B}(\Gamma) \times_{\mathcal{A}(\Gamma)} \mathcal{B}(\Gamma)^\vee$. Let $\lambda: \mathcal{B}(\Gamma) \rightarrow \mathcal{B}(\Gamma)^\vee$ be the isomorphism defined by the polarization. Then

$$B = (\text{Id}, \lambda)^* P.$$

The line bundle B gives on each fibre twice the principal polarization. There is also an identification $L_{0,m}$ with $B^{\otimes m}$. Therefore, the line bundle of Siegel–Jacobi forms of weight k and index m is given by

$$(4.7) \quad L_{k,m} = \pi^* M^{\otimes k} \otimes B^{\otimes m}.$$

Lemma 4.8. *We have*

$$J_{k,m}(\Gamma) \subseteq H^0(\mathcal{B}(\Gamma), L_{k,m}),$$

with equality if $g \geq 2$.

Proof. This follows from Definition 4.1 and the remark afterwards. \square

4.3. The invariant metric. The aim of this subsection is to discuss the canonical invariant metric on the line bundle of Siegel–Jacobi forms.

For $Z \in \mathcal{H}_g$ we write $Z = X + iY$ with $X, Y \in \text{Mat}_{g \times g}(\mathbb{R})$ and for $W \in \mathbb{C}^{(1,g)}$ we write $W = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}^{(1,g)}$.

Definition 4.9. Let $\phi \in J_{k,m}(\Gamma)$ be a Siegel–Jacobi form. Then the *standard invariant norm* $h^{\text{inv}}(\phi)$ of ϕ is defined by

$$h^{\text{inv}}(\phi(Z, W))^2 = |\phi(Z, W)|^2 \rho_k(Y) e^{-4\pi m \beta Y^{-1} \beta^t}.$$

This quantity is readily checked to be $\tilde{\Gamma}$ -invariant. When Γ is neat it induces a smooth hermitian metric on $L_{k,m}(\Gamma)$.

The standard invariant metric of Definition 4.9 gives in particular metrics on M and B . These metrics agree with the classical Hodge metric on M and the canonical biextension metric on B . We refer to [12, Section 5] for a brief discussion of these classical metrics and a verification of this agreement.

Lemma 4.10. *The standard invariant metric h^{inv} on $L_{k,m}(\Gamma)$ is a psh (i.e., semipositive) metric.*

Proof. It is enough to show that $-\log \det(Y)$ and $\beta Y^{-1} \beta^t$ are psh functions on $\mathcal{H}_g \times \mathbb{C}^{(1,g)}$. A smooth function $f: \mathbb{C}^n \rightarrow \mathbb{R}$ of the form

$$f(z_1, \dots, z_n) = g(\text{Im}(z_1), \dots, \text{Im}(z_n))$$

is psh if and only if g is convex (see [13, I (5.13)]). Then the lemma follows from the Example log-determinant in [8, Section 3.1.5] and by [8, Example 3.4, Section 3.1.7]. \square

A useful result by Ziegler is the following (see [15, Proposition 1 and its Corollary]).

Proposition 4.11. *A Siegel–Jacobi form ϕ is a cusp form if and only if $h^{\text{inv}}(\phi)$ is bounded.*

5. TOROIDAL COMPACTIFICATIONS OF THE UNIVERSAL ABELIAN VARIETY

5.1. Toroidal compactifications. We briefly describe the theory of toroidal compactifications of $\mathcal{A}(\Gamma)$ and of $\mathcal{B}(\Gamma)$. For more details we refer to the book [17] by Faltings and Chai and Namikawa's work [27, 28]. Note that in [17] everything is worked out for the full modular group $\mathrm{Sp}(2g, \mathbb{Z})$ and the principal congruence subgroups $\Gamma(N)$. This is because the authors are mainly interested in integral models. If one is only interested in the theory over the complex numbers everything carries over to the general case of a commensurable subgroup Γ . On the other hand, to avoid having to deal with algebraic stacks we will deal only with the case when Γ is neat.

Let C_g be the (open) cone of symmetric positive definite real $g \times g$ matrices and \overline{C}_g the cone of symmetric semipositive real matrices with rational kernel. Let $\tilde{C}_g \subset \overline{C}_g \times \mathbb{R}^{(1,g)}$ be the cone

$$\tilde{C}_g = \{(\Omega, \beta) \in \overline{C}_g \times \mathbb{R}^{(1,g)} \mid \exists \alpha \in \mathbb{R}^{(1,g)}, \beta = \alpha\Omega\}.$$

In the reference [17], instead of matrices, the language of bilinear and linear forms is used. There, the space \tilde{C}_g is described as the set of pairs (b, l) , where b is a semidefinite symmetric bilinear form with rational radical and l is a linear form such that $b + 2l$ is bounded below. The gap between both descriptions is closed by the lemma below.

Lemma 5.1. *Let Ω be a real symmetric positive semidefinite matrix with rational kernel and β a row vector. Then the following statements are equivalent.*

- (1) *There is an $\alpha \in \mathbb{R}^{(1,g)}$ such that $\beta = \alpha\Omega$.*
- (2) *The set $x^t\Omega x + 2\beta x$, $x \in \mathbb{R}^{(g,1)}$ is bounded below.*

Proof. Since Ω is positive semidefinite, the function $x \mapsto x^t\Omega x + 2\beta x$ is convex. Therefore it is bounded below if and only if it has a stationary point. Taking derivatives and dividing by 2, we deduce that the function is bounded below if and only if the equation $x^t\Omega + \beta = 0$ has a solution. \square

We will denote by $\overline{C}_{g,\mathbb{Z}}$ the subset of half-integral matrices of \overline{C}_g . Furthermore we set $\tilde{C}_{g,\mathbb{Z}} = \tilde{C}_g \cap (\overline{C}_{g,\mathbb{Z}} \times \mathbb{Z}^{(1,g)})$.

It will be convenient to write down the elements of \tilde{C}_g as pairs of the form $(\Omega, \zeta\Omega)$ with $\zeta \in \mathbb{R}^{(1,g)}$. Such an element will belong to $\tilde{C}_{g,\mathbb{Z}}$ if and only if $\Omega \in \overline{C}_{g,\mathbb{Z}}$ and ζ has rational coefficients, but $\beta = \zeta\Omega$ has integral coefficients. This is just to be sure that β belongs to the image of Ω . But one has to be careful that (Ω, ζ) are not *affine* coordinates in the interior of \tilde{C}_g . In particular any reference to convexity is with respect to the affine structure given by the coordinates (Ω, β) .

Recall that $P_g \subset \mathrm{Sp}(2g, \mathbb{Z})$ denotes the subgroup of symplectic matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying $C = 0$. There is a group homomorphism

$$\begin{aligned} P_g &\longrightarrow \mathrm{GL}(g, \mathbb{Z}) \\ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} &\longmapsto A. \end{aligned}$$

We denote by $\overline{\Gamma}$ the image of $\Gamma \cap P_g$ in $\mathrm{GL}(g, \mathbb{Z})$. Similarly we denote by $\widetilde{\Gamma}$ the image of $\tilde{\Gamma} \cap P_{g+1}$ in $\mathrm{GL}(g+1, \mathbb{Z})$. Then $\overline{\Gamma}$ is contained in the subgroup of matrices of the form $\begin{pmatrix} A & 0 \\ \lambda & 1 \end{pmatrix}$ and is a semidirect product $\widetilde{\Gamma} = \overline{\Gamma} \ltimes \mathbb{Z}^{(1,g)}$. The group $\overline{\Gamma}$ acts on \overline{C}_g by the action

$$A \cdot \Omega = A\Omega A^t$$

and the group $\widetilde{\Gamma}$ acts on \widetilde{C}_g by the action

$$(A, \lambda) \cdot (\Omega, \beta) = (A\Omega A^t, (\beta + \lambda\Omega)A^t).$$

This action can be written also as

$$(5.1) \quad (A, \lambda) \cdot (\Omega, \zeta\Omega) = (A\Omega A^t, (\zeta + \lambda)A^{-1}A\Omega A^t).$$

Definition 5.2. An *admissible cone decomposition* of \overline{C}_g is a set Σ of cones in \overline{C}_g such that

- (1) Each $\sigma \in \Sigma$ is generated by a finite set of elements of $\overline{C}_{g,\mathbb{Z}}$ and contains no lines. In other words it is a rational polyhedral strictly convex cone.
- (2) If σ belongs to Σ each face of σ belongs to Σ .
- (3) If σ and τ belong to Σ their intersection is a common face.
- (4) The union of all the cones of Σ is \overline{C}_g .
- (5) The group $\overline{\Gamma}$ leaves Σ invariant with finitely many orbits.

Definition 5.3. Let Σ be an admissible cone decomposition of \overline{C}_g . An *admissible cone decomposition of \widetilde{C}_g over Σ* is a set of cones Π in \widetilde{C}_g such that

- (1) Each $\tau \in \Pi$ is generated by a finite set of elements of $\widetilde{C}_{g,\mathbb{Z}}$ and contains no lines.
- (2) If σ belongs to Π each face of σ belongs to Π .
- (3) If σ and τ belong to Π their intersection is a common face.
- (4) The union of all the cones of Π is \widetilde{C}_g .
- (5) The group $\widetilde{\Gamma}$ leaves Π invariant with finitely many orbits.
- (6) For each $\tau \in \Pi$, the projection of τ to \overline{C}_g is contained in a cone $\sigma \in \Sigma$.

We say that Σ (resp. Π) is *smooth* if every cone of Σ (resp. Π) is generated by part of a \mathbb{Z} -basis of the abelian group generated by $\overline{C}_{g,\mathbb{Z}}$ (resp. $\widetilde{C}_{g,\mathbb{Z}}$). We say that Π is *equidimensional* (over Σ) if for every cone τ the projection of τ to \overline{C}_g is a cone $\sigma \in \Sigma$.

Definition 5.4. Let Σ be an admissible cone decomposition for \overline{C}_g . An *admissible divisorial function* on Σ is a continuous $\overline{\Gamma}$ -invariant function $\phi: \overline{C}_g \rightarrow \mathbb{R}$ satisfying the following properties:

- (1) it is conical, in the sense that $\phi(\lambda x) = \lambda\phi(x)$ for all $x \in \overline{C}_g$ and $\lambda \in \mathbb{R}_{\geq 0}$;
- (2) it is linear on each cone σ of Σ ;
- (3) takes integral values on $\overline{C}_{g,\mathbb{Z}}$.

An admissible divisorial function is called *strictly anti-effective* if furthermore

- (4) $\phi(x) > 0$ for $x \neq 0$.

A strictly anti-effective divisorial function is called an *admissible polarization function* if, in addition, it satisfies

- (5) ϕ is concave;
- (6) it is strictly concave on Σ in the sense that, if τ is a cone on \overline{C}_g such that the restriction of ϕ is linear on τ then τ is contained in a cone σ of Σ . In other words, the maximal cones of Σ are the maximal cones of linearity of ϕ .

Remark 5.5. For a given admissible cone decomposition Σ it may happen that there are no admissible polarization functions on Σ . A cone decomposition that admits an admissible polarization function is called *projective*. As explained in [17, §V.5], every admissible cone decomposition admits a smooth projective refinement.

Definition 5.6. Let Σ be an admissible cone decomposition for \overline{C}_g and Π an admissible cone decomposition for \widetilde{C}_g over Σ . An *admissible divisorial function* on Π is a continuous $\widetilde{\Gamma}$ -invariant function $\phi: \widetilde{C}_g \rightarrow \mathbb{R}$ satisfying the following properties:

- (1) it is conical: $\phi(tx) = t\phi(x)$ for all $t \in \mathbb{R}_{\geq 0}$ and $x \in \tilde{C}_g$;
- (2) it takes rational values on $\tilde{C}_{g,\mathbb{Z}}$ with bounded denominators;
- (3) it is linear on each $\tau \in \Pi$;
- (4) for each $\lambda \in \mathbb{Z}^{(1,g)}$ and $q = (\Omega, \zeta\Omega) \in \tilde{C}_g$, the condition

$$(5.2) \quad \phi(q) - \phi(\lambda \cdot q) = \lambda\Omega\lambda^t + 2\zeta\Omega\lambda^t$$

holds. Recall that the action (5.1) gives $\lambda \cdot q = (\Omega, (\zeta + \lambda)\Omega)$.

An admissible divisorial function ϕ on Π is called an *admissible polarization function* if it also satisfies the conditions

- (5) ϕ is concave;
- (6) ϕ is strictly concave over each cone σ of Σ . That is, for each maximal cone τ over σ , there is a linear function φ_τ such that $\varphi_\tau(q) = \phi(q)$ for each $q \in \tau$, but $\varphi_\tau(q) > \phi(q)$ for each $q = (\Omega, \zeta\Omega) \notin \tau$, with $\Omega \in \sigma$.

Remark 5.7. For a given admissible cone decomposition Π , even smooth, it may be possible that there are no admissible polarization functions on Π . Nevertheless there is always a refinement Π' of Π such that there exists an admissible polarization function on Π' . As for Σ , the admissible cone decompositions that admit an admissible polarization function are called *projective*.

The theory of toroidal embeddings allows us to compactify the moduli spaces $\mathcal{A}(\Gamma)$ and $\mathcal{B}(\Gamma)$. The precise statement is as follows.

Theorem 5.8. *Let Σ be a projective admissible cone decomposition for \overline{C}_g and Π a projective admissible cone decomposition for \tilde{C}_g over Σ .*

- (1) *To the cone decomposition Σ there is attached a projective scheme $\overline{\mathcal{A}}(\Gamma)_\Sigma$ that contains $\mathcal{A}(\Gamma)$ as an open dense subset.*
- (2) *To the cone decomposition Π there is attached a projective scheme $\overline{\mathcal{B}}(\Gamma)_\Pi$ that contains $\mathcal{B}(\Gamma)$ as an open dense subset, and a projective morphism*

$$\pi_{\Sigma,\Pi}: \overline{\mathcal{B}}(\Gamma)_\Pi \rightarrow \overline{\mathcal{A}}(\Gamma)_\Sigma$$

that extends the canonical projection $\mathcal{B}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$.

- (3) *If Σ or Π are smooth, then the corresponding schemes $\overline{\mathcal{A}}(\Gamma)_\Sigma$ or $\overline{\mathcal{B}}(\Gamma)_\Pi$ are smooth. If Π is equidimensional over Σ , then $\pi_{\Sigma,\Pi}$ is equidimensional.*
- (4) *The space $\overline{\mathcal{A}}(\Gamma)_\Sigma$ admits a stratification by locally closed subschemes, indexed by the $\overline{\Gamma}$ -orbits of Σ*

$$\overline{\mathcal{A}}(\Gamma)_\Sigma = \bigcup_{\overline{\sigma} \in \Sigma/\overline{\Gamma}} \overline{\mathcal{A}}(\Gamma)_{\overline{\sigma}}.$$

The correspondence between cones and strata reverses dimensions and a stratum $\overline{\mathcal{A}}(\Gamma)_{\overline{\sigma}}$ lies in the closure of another stratum $\overline{\mathcal{A}}(\Gamma)_{\overline{\tau}}$ if and only if there are representatives σ and τ of $\overline{\sigma}$ and $\overline{\tau}$ such that τ is a face of σ .

- (5) *There is an analogous stratification of $\overline{\mathcal{B}}(\Gamma)_\Pi$ indexed by the $\overline{\Gamma}$ orbits of Π .*

Remark 5.9. The projectivity condition is used to show that formal versions of these moduli spaces are algebraizable.

5.2. Local coordinates. Assume now that Σ and Π are smooth. We want to describe local coordinates of $\overline{\mathcal{A}}(\Gamma)_\Sigma$ and $\overline{\mathcal{B}}(\Gamma)_\Pi$.

Let $\overline{\sigma}$ and $\overline{\tau}$ be orbits of cones in Σ and Π respectively. To $\overline{\sigma}$ corresponds a stratum $\overline{\mathcal{A}}(\Gamma)_{\overline{\sigma}}$ and to $\overline{\tau}$ a stratum $\overline{\mathcal{B}}(\Gamma)_{\overline{\tau}}$. We choose a point $x_{\overline{\sigma}} \in \overline{\mathcal{A}}(\Gamma)_{\overline{\sigma}}$ and a point $y_{\overline{\tau}} \in \overline{\mathcal{B}}(\Gamma)_{\overline{\tau}}$.

We now describe local coordinates in both spaces around $x_{\bar{\sigma}}$ and $y_{\bar{\tau}}$. These coordinates and the uniformization map depend on the choice of representatives σ and τ of the orbits.

We start with $x_{\bar{\sigma}}$. Write $G = g(g+1)/2$ for the dimension of $\mathcal{A}(\Gamma)$ and let $n = \dim \sigma$. For $r > 0$, we denote by $\Delta_r \subset \mathbb{C}$ the disk of radius r centered at 0. The chosen cone σ is generated by a set of symmetric half-integral semipositive matrices $\Omega_1, \dots, \Omega_n$ that are part of an integral basis of the lattice of symmetric half integral matrices. Then there exists a symmetric semidefinite matrix Ω_0 and symmetric matrices $\Omega_{n+1}, \dots, \Omega_G$, such that the set $\Omega_1, \dots, \Omega_G$ is a basis of the same lattice, and there are positive real numbers $0 < r_1, \dots, r_G < 1$ such that

$$U := \left\{ i\Omega_0 + \sum_j t_j \Omega_j \mid \begin{array}{l} \operatorname{Im}(t_j) > -\log r_j, \quad j \leq n, \\ |t_j| < r_j, \quad j > n. \end{array} \right\} \subset \mathcal{H}_g.$$

For U small enough there is a coordinate neighbourhood $V \subset \overline{\mathcal{A}}(\Gamma)_{\Sigma}$ centered at $x_{\bar{\sigma}}$ of the form $\Delta_{r_1} \times \dots \times \Delta_{r_G}$, such that the uniformization map $\mathcal{H}_g \rightarrow \mathcal{A}(\Gamma)$ sends U to V through the map

$$(t_1, \dots, t_G) \mapsto (z_1, \dots, z_G) = (e^{2\pi i t_1}, \dots, e^{2\pi i t_n}, t_{n+1}, \dots, t_G).$$

The situation for $y_{\bar{\tau}}$ is similar. Let $m = \dim \tau$. The chosen cone τ is generated by part of an integral basis $\{(\Omega'_1, \beta_1), \dots, (\Omega'_m, \beta_m)\}$, where the Ω'_j are half-integral semipositive symmetric matrices, and the vectors β_j are integral and of the form $\beta_j = \zeta_j \Omega_j$. There are pairs $(\Omega'_{m+1}, \beta_{m+1}), \dots, (\Omega'_{G+g}, \beta_{G+g})$, such that

$$\{(\Omega'_1, \beta_1), \dots, (\Omega'_{G+g}, \beta_{G+g})\}$$

is an integral basis of the lattice $\tilde{\mathcal{C}}_{g,\mathbb{Z}}$. There is also a pair (Ω'_0, β_0) satisfying $\beta_0 = \zeta_0 \Omega'_0$ and real numbers $0 < r_j < 1$, $j = 1, \dots, G+g$ such that

$$U' := \left\{ i(\Omega_0, \beta_0) + \sum_j s_j (\Omega_j, \beta_j) \mid \begin{array}{l} \operatorname{Im}(s_j) > -\log r_j, \quad j \leq s, \\ |s_j| < r_j, \quad j > s. \end{array} \right\} \subset \mathcal{H}_g \times \mathbb{C}^{(1,g)}.$$

Again, for U' small enough, there is a coordinate neighborhood V' centered at $y_{\bar{\tau}}$ such that the uniformization map sends U' to V' via the map

$$(5.3) \quad (s_1, \dots, s_{G+g}) \mapsto (w_1, \dots, w_{G+g}) = (e^{2\pi i s_1}, \dots, e^{2\pi i s_m}, s_{m+1}, \dots, s_{G+g}).$$

Assume now that σ and τ are maximal, so $n = G$ and $m = G+g$, that $\pi(\tau) \subset \sigma$ and that $\pi_{\Sigma, \Pi}(y_{\bar{\tau}}) = x_{\bar{\sigma}}$. We describe the map $\pi_{\Sigma, \Pi}$ in these coordinates. Since each Ω'_j , $j = 1, \dots, m$ is contained in the cone generated by the Ω_k 's, $k = 1, \dots, n$ and we are assuming that the cone decomposition is smooth, there are integer vectors $\underline{a}_j = (a_{j,1}, \dots, a_{j,n}) \in \mathbb{Z}_{\geq 0}^n$ such that

$$\Omega'_j = \sum_k a_{j,k} \Omega_k, \quad j = 1, \dots, m.$$

Then the map $\pi_{\Sigma, \Pi}$ is given in the w - and z -coordinates by

$$(w_1, \dots, w_{G+g}) \mapsto \left(\prod_{j=1}^{G+g} w_j^{a_{j,k}} \right)_{k=1, \dots, G}.$$

5.3. Extending the line bundle of Siegel–Jacobi forms. We next discuss the extension of the line bundles $L_{k,m}$ to the toroidal compactifications.

We start by recalling how to extend the line bundle of modular forms on the Siegel modular variety. This is related with the construction of the Satake–Baily–Borel (or minimal) compactification (which is in general not toroidal).

Let Σ be a projective admissible cone decomposition of \overline{C}_g . The universal abelian variety $\mathcal{B}(\Gamma)$ over $\mathcal{A}(\Gamma)$ can be uniquely extended to a semi-abelian variety over $\overline{\mathcal{A}}(\Gamma)_\Sigma$ that we denote $\overline{\mathcal{B}}(\Gamma)_\Sigma^0$. The zero section $e: \mathcal{A}(\Gamma) \rightarrow \mathcal{B}(\Gamma)$ extends to a section $\bar{e}: \overline{\mathcal{A}}(\Gamma)_\Sigma \rightarrow \overline{\mathcal{B}}(\Gamma)_\Sigma^0$. Therefore the line bundle $M = \det \left(e^* \Omega_{\mathcal{B}(\Gamma)/\mathcal{A}(\Gamma)}^1 \right)$ on $\mathcal{A}(\Gamma)$ can be extended canonically to the line bundle

$$\overline{M} = \det \left(\bar{e}^* \Omega_{\overline{\mathcal{B}}(\Gamma)_\Sigma^0/\overline{\mathcal{A}}(\Gamma)_\Sigma}^1 \right)$$

on $\overline{\mathcal{A}}(\Gamma)_\Sigma$ by [17, §V.1]. Moreover there is a $k > 0$ such that the line bundle $\overline{M}^{\otimes k}$ is globally generated [17, Chapter V, Proposition 2.1]. Here we are using that Γ is neat and in particular torsion-free.

Let R_Γ be the graded ring

$$R_\Gamma = \bigoplus_{k \geq 0} H^0 \left(\overline{\mathcal{A}}(\Gamma)_\Sigma, \overline{M}^{\otimes k} \right).$$

By [17, Chapter V, Theorem 2.3], the ring R_Γ is finitely generated and does not depend on the choice of Σ . Then

$$(5.4) \quad \mathcal{A}(\Gamma)^* = \text{Proj}(R_\Gamma)$$

is the Satake–Baily–Borel compactification of $\mathcal{A}(\Gamma)$, a projective complex variety. In particular this implies that $\mathcal{A}(\Gamma)$ is quasi-projective and M is an algebraic line bundle. Finally we have a canonical projection map

$$\overline{\mathcal{A}}(\Gamma)_\Sigma \longrightarrow \mathcal{A}(\Gamma)^*.$$

Lemma 5.10. *The standard invariant metric on M extends to a psh metric on \overline{M} .*

Proof. It follows from Lemma 4.10 that the standard invariant metric is psh on M . Then, by the theory of psh functions, we only have to show that for every point $y \in \overline{\mathcal{A}}(\Gamma)_\Sigma$ there exists $k \in \mathbb{Z}_{>0}$ and a local section s of $\overline{M}^{\otimes k}$ that generates $\overline{M}^{\otimes k}$ around y , such that the function $-\log \|s\|$ is bounded above locally around y . For simplicity we discuss only the case of a point corresponding to a maximal cone of Σ as the general case is only notationally more complex. Let $\sigma \in \Sigma$ be a cone of maximal dimension. Since σ is maximal, the stratum $\overline{\mathcal{B}}(\Gamma)_\sigma$ contains a single point y_σ . Let $U \subset \mathcal{H}_g$ and $y_\sigma \in V \subset \overline{\mathcal{A}}(\Gamma)_\Sigma$ be open sets as in the previous section.

Let (u_1, \dots, u_g) be linear coordinates on $\mathbb{C}^{(1,g)}$ defined by an integral basis. If V is small enough, the symbol $du_1 \wedge \dots \wedge du_g$ defines a generating section of M over $V \cap \mathcal{A}(\Gamma)$. As explained in [17, V §1] this symbol extends to a generating section of \overline{M} on V . Let $k \in \mathbb{Z}_{>0}$ and let $s = f(du_1 \wedge \dots \wedge du_g)^{\otimes k}$ be a section of $\overline{M}^{\otimes k}$ over V . Note that f lifts to $U \subset \mathcal{H}_g$ and can be interpreted there as a meromorphic Siegel modular form on \mathcal{H}_g of weight k . Then the section s can be extended to a local generating section of $\overline{M}^{\otimes k}$ around y_σ if and only if $-\log |f|$ is bounded in U (for U small enough). Assume therefore that $-\log |f|$ is bounded on U . Then

$$-\log \|s\| = -\log |f| - k \log \det \left(\sum \frac{-1}{2\pi} \log |z_j| \Omega_j \right)$$

is bounded above. Indeed, after shrinking U if necessary, we can assume that $2\pi \leq -\log |z_j|$ for all j . Since the Ω_j are positive semidefinite and since for two semidefinite matrices A, B the inequality $\det(A + B) \geq \det(A)$ holds, we deduce that

$$\det \left(\sum \frac{-1}{2\pi} \log |z_j| \Omega_j \right) \geq \det \left(\sum \Omega_j \right)$$

from which the boundedness above follows. \square

We next recall how the theory of toroidal compactifications allows us to extend line bundles from $\mathcal{A}(\Gamma)$ and $\mathcal{B}(\Gamma)$ to $\overline{\mathcal{A}}(\Gamma)_\Sigma$ and $\overline{\mathcal{B}}(\Gamma)_\Pi$.

Definition 5.11. To every divisorial function ϕ we associate a divisor D_ϕ as follows. Each irreducible component D_α of the boundary $\overline{\mathcal{A}}(\Gamma)_\Sigma \setminus \mathcal{A}(\Gamma)$ corresponds to a $\overline{\Gamma}$ -orbit of one-dimensional cones of Σ . Choose one element of this orbit σ_α . Let v_α be the primitive generator of $\sigma_\alpha \cap \overline{\mathcal{C}}_{g, \mathbb{Z}}$. Then we write

$$D_\phi = \sum_{\alpha} -\phi(v_\alpha) D_\alpha.$$

Since ϕ is $\overline{\Gamma}$ -invariant, this divisor is independent of the choice of σ_α .

Proposition 5.12. *The divisor D_ϕ has support contained in the boundary $\overline{\mathcal{A}}(\Gamma)_\Sigma \setminus \mathcal{A}(\Gamma)$. If the divisorial function is strictly anti-effective then $-D_\phi$ is strictly effective and the support of D_ϕ agrees with the whole boundary. If ϕ is an admissible polarization function, then $\mathcal{O}(D_\phi)$ is relatively ample with respect to the canonical projection*

$$\overline{\mathcal{A}}(\Gamma)_\Sigma \longrightarrow \mathcal{A}(\Gamma)^*.$$

Proof. The first two statements follow directly from the definition of D_ϕ and the last statement follows from [17, Theorem V.5.8] and the proof of [18, Proposition II.7.13]. \square

Also the next result follows from the theory of toroidal compactifications (see [17, Chapter VI, Theorem 1.13]).

Proposition 5.13. *Let Σ be an admissible cone decomposition for $\overline{\mathcal{C}}_g$ and Π an admissible cone decomposition for $\widetilde{\mathcal{C}}_g$ over Σ . Let ϕ be an admissible divisorial function on Π and let $m > 0$ be an integer such that $m\phi$ has integral values on $\widetilde{\mathcal{C}}_{g, \mathbb{Z}}$. Associated with $m\phi$ there is a line bundle $\overline{\mathcal{B}}_{m\phi}$ on $\overline{\mathcal{B}}(\Gamma)_\Pi$ such that its restriction to $\mathcal{B}(\Gamma)$ agrees with $B^{\otimes m}$. Moreover, if ϕ is an admissible polarization function then $\overline{\mathcal{B}}_{m\phi}$ is relatively ample with respect to the projection map*

$$\pi_{\Sigma, \Pi}: \overline{\mathcal{B}}(\Gamma)_\Pi \rightarrow \overline{\mathcal{A}}(\Gamma)_\Sigma.$$

In particular, if Π is projective, the latter map is projective.

Remark 5.14. Condition 4 in Definition 5.6 ensures that the restriction of $\overline{\mathcal{B}}_{m\phi}$ to $\mathcal{B}(\Gamma)$ agrees with $B^{\otimes m}$. In fact this condition implies that the restriction of $\overline{\mathcal{B}}_{m\phi}$ to $\mathcal{B}(\Gamma)$ satisfies the cocycle condition determining $B^{\otimes m}$.

Remark 5.15. Let Σ be an admissible cone decomposition for $\overline{\mathcal{C}}_g$ and Π an admissible cone decomposition for $\widetilde{\mathcal{C}}_g$ over Σ . We can combine admissible divisorial functions on Σ and Π . Let ϕ be an admissible divisorial function on Π and ψ an admissible divisorial function on Σ . By composing with the projection $\widetilde{\mathcal{C}}_g \rightarrow \overline{\mathcal{C}}_g$, the function ψ defines a function on $\widetilde{\mathcal{C}}_g$ which (by abuse of notation) we also denote ψ . Let $m > 0$ be an integer such that $m\phi$ has integral values on $\widetilde{\mathcal{C}}_{g, \mathbb{Z}}$. Then $m\phi + \psi$ is again an admissible divisorial function with integral values on $\widetilde{\mathcal{C}}_{g, \mathbb{Z}}$ and, by construction,

$$\overline{\mathcal{B}}_{m\phi+\psi} = \overline{\mathcal{B}}_{m\phi} \otimes (\pi_{\Sigma, \Pi})^* \mathcal{O}(D_\psi).$$

We next see a criterion for when a rational section of $\overline{B}_{m\phi}$ is holomorphic at a point of the boundary.

Lemma 5.16. *Let Σ and Π be as in Proposition 5.13. Let ϕ be an admissible divisorial function on Π and let $m > 0$ be an integer such that $m\phi$ has integral values on $\widetilde{C}_{g,\mathbb{Z}}$. Let f be a rational section of $\overline{B}_{m\phi} \otimes (\pi_{\Sigma,\Pi})^* \overline{M}^{\otimes k}$, that we view as a meromorphic Siegel–Jacobi form of weight k and index m . Let $\tau \in \Pi$ and $x \in \overline{\mathcal{B}}(\Gamma)_{\overline{\tau}}$. Let V' be a sufficiently small open coordinate neighborhood of x and $U' \subset \mathcal{H}_g \times \mathbb{C}^{(1,g)}$ an open set as in section 5.1. Then f extends to a holomorphic section around x (resp. a non-vanishing holomorphic section) if and only if the function*

$$-\log |f(Z, W)| - 2\pi m\phi(\operatorname{Im} Z, \operatorname{Im} W)$$

is bounded below (resp. bounded) in U' .

Proof. We start by recalling a few steps in the construction of $\overline{\mathcal{B}}(\Gamma)_{\Pi}$ and $\overline{B}_{m\phi}$ (see the proof of [17, Chapter VI, Theorem 1.13]). We describe the situation analytically over the complex numbers as that is enough for our purposes.

Over an open set V of $\overline{\mathcal{A}}(\Gamma)_{\Sigma}$ that contains the image of V' there is a complex manifold P . The lattice $\mathbb{Z}^{(1,g)}$ acts freely and discontinuously on P and

$$P/\mathbb{Z}^{(1,g)} = (\pi_{\Sigma,\Pi})^{-1}(V) \subset \overline{\mathcal{B}}(\Gamma)_{\Pi}.$$

The map $p: P \rightarrow (\pi_{\Sigma,\Pi})^{-1}(V)$ is étale and we can find an open subset $V'' \subset P$, that depends on the representative τ of $\overline{\tau}$, and that maps isomorphically to V' . Thus the holomorphic coordinates of V' also give us holomorphic coordinates of V'' . Moreover, the uniformization $U \rightarrow V'$ factors through V'' as the map $U \rightarrow V''$ is also given by formula (5.3). Then the preimage of $\overline{B}_{m\phi}$ in V'' is the sheaf $\mathcal{O}(D)$, where

$$D = \sum_j -m\phi(\Omega'_j, \beta_j) D_j, \quad D_j = \{w_j = 0\}.$$

As in the proof of Lemma 5.10 the symbol $du_1 \wedge \cdots \wedge du_g$ defines a generating section of $(\pi_{\Sigma,\Pi})^* \overline{M}$ over V' . Therefore $f(du_1 \wedge \cdots \wedge du_g)^{\otimes k}$ extends to a holomorphic section of $\overline{B}_{m\phi} \otimes (\pi_{\Sigma,\Pi})^* \overline{M}^{\otimes k}$ if and only if the function

$$g = f \prod_j w_j^{-m\phi(\Omega'_j, \beta_j)}$$

is holomorphic in V'' . Since g is meromorphic, it is holomorphic if and only if the function

$$\begin{aligned} -\log |g| &= -\log |f| + \sum_j m\phi(\Omega'_j, \beta_j) \log |w_j| \\ &= -\log |f| + \sum_j -2\pi m\phi(\Omega'_j, \beta_j) \operatorname{Im}(s_j) \\ &= -\log |f| - 2\pi m\phi \left(\sum_j \operatorname{Im}(s_j(\Omega'_j, \beta_j)) \right) \end{aligned}$$

is bounded below. This proves the lemma for holomorphic sections. Similarly g is holomorphic and non-vanishing if and only if $-\log |g|$ is bounded; this proves the second case. \square

Definition 5.17. An admissible divisorial function ϕ on Π is called *sufficiently negative* if

$$\phi(\Omega, \zeta\Omega) \leq -\zeta\Omega\zeta^t.$$

Remark 5.18. For each given smooth Π we can always find a sufficiently negative admissible divisorial function. Indeed, for every cone τ of Π take the linear function that agrees with $-\zeta\Omega\zeta^t$ on the one dimensional faces of τ . This function has rational values on $\tilde{C}_{g,\mathbb{Z}}$ with bounded denominators. Nevertheless, even if the function $(\Omega, \beta) \mapsto -\beta\Omega^{-1}\beta^t$ is concave, the function we have constructed is not necessarily concave. In particular it might not be a polarization.

The interest of the definition of *sufficiently negative* admissible divisorial functions lies in the following result.

Lemma 5.19. *Let Σ and Π be as in Proposition 5.13. Let ϕ be an admissible divisorial function on Π and let $m > 0$ be an integer such that $m\phi$ has integral values on $\tilde{C}_{g,\mathbb{Z}}$. If ϕ is sufficiently negative then the metric of Definition 4.9 on $B^{\otimes m}$ extends to a singular psh metric on $\overline{B}_{m\phi}$.*

Proof. As in the proof of Lemma 5.10, since we already know that the standard invariant metric is psh on $B^{\otimes m}$, we only have to show that, for every point $x \in \overline{\mathcal{B}}(\Gamma)_{\Pi}$ and every local section s of $\overline{B}_{m\phi}$ that generates $\overline{B}_{m\phi}$ around x , the function $-\log \|s\|$ is bounded above locally around x . Again we discuss only the case when $x = x_{\tau}$ corresponds to a maximal cone τ of Π . Let $U' \subset \mathcal{H}_g \times \mathbb{C}^{(1,g)}$ and $x_{\tau} \in V'$ be open coordinate sets as in Section 5.2. A rational section of $B^{\otimes m}$ determines a meromorphic Siegel–Jacobi form f of index m . By Lemma 5.16 it extends to a generating section s_f of $\overline{B}_{m\phi}$ around x_{τ} if and only if the function

$$-\log |f| - 2\pi m\phi$$

is bounded in U' . By the sufficient negativity of ϕ and the definition of the standard invariant metric in Definition 4.9 we deduce that

$$\begin{aligned} -\log \|s_f\| &= -\log |f(\Omega, \zeta\Omega)| + 2\pi m\zeta\Omega\zeta^t \\ &= -\log |f(\Omega, \zeta\Omega)| - 2\pi m\phi + 2\pi m\phi + 2\pi m\zeta\Omega\zeta^t \end{aligned}$$

is bounded above, hence extends to a psh function on V' . \square

Remark 5.20. When Π is smooth, we can choose the sufficiently negative function ϕ that agrees with $\zeta\Omega\zeta^t$ on the rays of Π and is linear on every cone. Then the obtained extension is the one considered by Lear [25]. In this case the standard invariant metric extends to a psh metric that is continuous up to a set of codimension at least 2.

Combining the previous results we see that the line bundle of Siegel–Jacobi forms $L_{k,m}$ can be extended to a toroidal compactification with the help of an admissible divisorial function.

Definition 5.21. Let Σ , Π and ϕ as in Proposition 5.13. Let $\ell > 0$ be an integer such that $\ell\phi$ has integral values on $\tilde{C}_{g,\mathbb{Z}}$. Then for every integer m divisible by ℓ and for every integer k we define the line bundle

$$\overline{L}_{k,m,\phi} = (\pi_{\Sigma,\Pi})^* \overline{M}^k \otimes \overline{B}_{m\phi}$$

on $\overline{\mathcal{B}}(\Gamma)_{\Pi}$.

Note that $m\phi$ is integer-valued, so that $\overline{B}_{m\phi}$ is well-defined. Clearly the restriction of $\overline{L}_{k,m,\phi}$ to $\mathcal{A}(\Gamma)$ agrees with $L_{k,m}$.

Remark 5.22. The extension depends on the choice of ϕ . Moreover, for m not divisible by ℓ we can only extend $L_{k,m}$ as a \mathbb{Q} -line bundle. We note that the notion of psh metric readily generalizes to the context of \mathbb{Q} -line bundles.

Combining Lemmas 5.10 and 5.19 we obtain a criterion for when the invariant metric on $L_{k,m}$ extends to a psh metric on $\overline{L}_{k,m,\phi}$.

Proposition 5.23. *Let Σ , Π , m , ϕ and k be as in Definition 5.21. Assume that ϕ is a sufficiently negative admissible divisorial function on Π . Then the standard invariant metric h^{inv} of $L_{k,m}$ of Definition 4.9 extends to a singular psh metric on $L_{k,m,\phi}$ that we denote by $\overline{h}^{\text{inv}}$.*

In a different direction one may ask when the extension $L_{k,m,\phi}$ of Definition 5.21 is ample.

Lemma 5.24. *Let Σ be a projective admissible cone decomposition for \overline{C}_g and Π a projective admissible cone decomposition for \tilde{C}_g over Σ . Let ϕ be an admissible polarization function on Π and $\ell_0 \geq 1$ an integer such that $\ell_0\phi$ has integral values on $\tilde{C}_{g,\mathbb{Z}}$. Then there exists an admissible polarization function ψ on Σ and for every $m > 0$ a number $k_0 \gg 0$ such that, for every $k \geq k_0$, the line bundle $\overline{L}_{\ell_0 k, \ell_0 m, \phi + \psi}$ is ample on $\overline{\mathcal{B}}(\Gamma)_\Pi$. Moreover we can choose ψ such that, for all $\ell > 0$ divisible by ℓ_0 ,*

$$(5.5) \quad H^0(\overline{\mathcal{B}}(\Gamma)_\Pi, \overline{L}_{\ell k, \ell m, \phi + \psi}) \subset J_{\ell k, \ell m}(\Gamma).$$

Proof. Let ψ_0 be a polarization function on Σ . By Proposition 5.12 the line bundle $\mathcal{O}(D_{\psi_0})$ is relatively ample for the map $\overline{\mathcal{A}}(\Gamma)_\Sigma \rightarrow \mathcal{A}(\Gamma)^*$. By Proposition 5.13 the line bundle $\overline{B}_{\ell_0\phi}$ is relatively ample with respect to the map

$$\pi_{\Sigma, \Pi}: \overline{\mathcal{B}}(\Gamma)_\Pi \longrightarrow \overline{\mathcal{A}}(\Gamma)_\Sigma.$$

Therefore we can find an integer $a > 0$ such that the line bundle $\overline{B}_{\ell_0\phi} \otimes (\pi_{\Sigma, \Pi})^* \mathcal{O}(a\ell_0 D_{\psi_0})$ is relatively ample with respect to the map

$$\overline{\mathcal{B}}(\Gamma)_\Pi \longrightarrow \mathcal{A}(\Gamma)^*.$$

Writing $\psi = a\psi_0$, following Remark 5.15 we have $\overline{B}_{\ell_0\phi} \otimes (\pi_{\Sigma, \Pi})^* \mathcal{O}(a\ell_0 D_{\psi_0}) = \overline{B}_{\ell_0(\phi + \psi)}$.

By the definition of $\mathcal{A}(\Gamma)^*$ in (5.4) the line bundle \overline{M} is ample on $\mathcal{A}(\Gamma)^*$. Therefore there exists k_1 such that

$$\overline{L}_{k_1, \ell_0, (\phi + \psi)} = \overline{B}_{\ell_0(\phi + \psi)} \otimes (\pi_{\Sigma, \Pi})^* \overline{M}^{\otimes k_1}$$

is ample in $\overline{\mathcal{B}}(\Gamma)_\Pi$. Therefore it is enough to choose $k_0 = mk_1$.

When $g \geq 2$ the final statement follows from the Koecher principle as the restriction map

$$H^0(\overline{\mathcal{B}}(\Gamma)_\Pi, \overline{L}_{\ell k, \ell m, \phi + \psi}) \longrightarrow H^0(\mathcal{B}(\Gamma), L_{\ell k, \ell m})$$

is injective. The case $g = 1$ needs more care as we do not have the Koecher principle. By Lemma 5.25 below we only need to choose ψ such that

$$(5.6) \quad \phi + \psi > -\zeta\Omega\zeta^t.$$

Since ϕ and $-\zeta\Omega\zeta^t$ satisfy the same transformation rule with respect to $\overline{\Gamma}$, it is only necessary to impose equation (5.6) in a finite number of cones. Since both functions are conic it is enough to impose the condition on a finite number of simplices; in particular on a compact set. Since the polarization function ψ_0 is strictly positive, it is enough to choose the number a so that $\phi + a\psi_0 = \phi + \psi \geq -\zeta\Omega\zeta^t$ in this compact set. Therefore, we see that condition (5.6) can be attained for a suitable choice of ψ . \square

Lemma 5.25. *Assume the hypotheses of Lemma 5.24. If $g = 1$ and $\phi(\Omega, \zeta\Omega) \geq -\zeta\Omega\zeta^t$ then*

$$H^0(\overline{\mathcal{B}}(\Gamma)_\Pi, \overline{L}_{\ell k, \ell m, \phi}) \subset J_{\ell k, \ell m}(\Gamma).$$

Proof. Let $f \in H^0(\overline{\mathcal{B}}(\Gamma)_\Pi, \overline{L}_{\ell k, \ell m, \phi})$ and assume that $f \notin J_{\ell k, \ell m}(\Gamma)$. This means that, in the Fourier expansion (4.4) there is a non-zero coefficient $c(T, R)$, with $4mT/\lambda_\Gamma - R^2 < 0$. Recall that, since $g = 1$, T is a positive integer and R is an integer. Therefore there is a real number q such that

$$T/\lambda_\Gamma + Rq + mq^2 < 0.$$

We can choose the real number q to be transcendental so there is no possible relation $T + Rq = T' + R'q$ for $T \neq T'$ and $R \neq R'$. Then the rate of growth of $|f|$ along the ray $t \cdot (i, iq)$, $t > 0$ has to be at least the rate of growth of

$$e^{-2\pi/\lambda_\Gamma t(T/\lambda_\Gamma + Rq)} > e^{2\pi mq^2 t}.$$

That is,

$$-\log |f((it, iqt))| < -2\pi mq^2 t + K$$

for some constant K . On the other hand, since we are assuming that f extends to a holomorphic section, Lemma 5.16 implies that

$$2\pi m\phi((1, q))t \leq -\log |f(t(i, iq))| + K'$$

for a constant K' . Taking the limit when t goes to ∞ , we obtain

$$\phi((1, q)) < -q^2$$

contradicting the hypothesis as $-\zeta\Omega\zeta^t(1, q) = -q^2$. \square

Remark 5.26. As seen in the proof of Lemma 5.24, when $g \geq 2$ the inclusion (5.5) is always satisfied, even when $\phi + \psi$ is just a divisorial function and not necessarily a polarization. By contrast, when $g = 1$ the condition (5.6) is sufficient to have the inclusion (5.5), but that condition is incompatible with being sufficiently negative. So, it is not clear in the case $g = 1$ that one can attain at the same time the inclusion (5.5) and that the invariant metric extends to a psh metric.

5.4. The standard invariant metric is toroidal.

Proposition 5.27. *Let Σ , Π and ϕ be as in Proposition 5.13. Assume that ϕ is sufficiently negative. Let $\ell > 0$ be an integer such that $\ell\phi$ is integral. Then, for every k and every m divisible by ℓ , the singular psh metric $\overline{h}^{\text{inv}}$ on $\overline{L}_{k, m, \phi}$ is toroidal.*

Proof. Again, we prove this only in a neighborhood of a point $x_\tau \in \overline{\mathcal{B}}(\Gamma)_\Pi$ corresponding to a maximal cone τ of Π . We use the coordinate neighborhood V' of x_τ as described at the end of Section 5.1. Let $G = g(g+1)/2$. For $i = 1, \dots, G+g$ we write

$$u_i = \text{Im}(s_i) = \frac{-1}{2\pi} \log |w_i|,$$

where (w_1, \dots, w_{G+g}) are the coordinates of V' . Recall also that the cone τ is generated by the points $(\Omega'_i, \zeta_i \Omega'_i)$, $i = 1, \dots, G+g$.

Let f be a meromorphic Siegel–Jacobi form that defines a rational section s_f of $\overline{L}_{k, m, \phi}$ that is a generating local section on V' . This means that on the set U' the function

$$-\log |f| - 2\pi m\phi$$

is bounded (see Lemma 5.16). Therefore

$$\begin{aligned}
-\log \|s_f\| &= -\log |f| - \frac{k}{2} \log \det \sum_{i=1}^{G+g} u_i \Omega'_i \\
&\quad + 2\pi m \left(\sum_{i=1}^{G+g} u_i \zeta_i \Omega'_i \right) \left(\sum_{i=1}^{G+g} u_i \Omega'_i \right)^{-1} \left(\sum_{i=1}^{G+g} u_i \zeta_i \Omega'_i \right)^t \\
&= \gamma - \frac{k}{2} \log \det \left(\sum_{i=1}^{G+g} u_i \Omega'_i \right) + 2\pi m \phi(u_1, \dots, u_{G+g}) \\
&\quad + 2\pi m \left(\sum_{i=1}^{G+g} u_i \zeta_i \Omega'_i \right) \left(\sum_{i=1}^{G+g} u_i \Omega'_i \right)^{-1} \left(\sum_{i=1}^{G+g} u_i \zeta_i \Omega'_i \right)^t,
\end{aligned}$$

where γ is bounded. We already know that ϕ is linear on the cone τ and that the other two functions appearing in the last equation are convex, as seen in Lemma 4.10. Moreover, since ϕ is assumed to be sufficiently negative, we know that

$$\phi(u_1, \dots, u_{G+g}) + \left(\sum_{i=1}^{G+g} u_i \zeta_i \Omega'_i \right) \left(\sum_{i=1}^{G+g} u_i \Omega'_i \right)^{-1} \left(\sum_{i=1}^{G+g} u_i \zeta_i \Omega'_i \right)^t$$

is bounded above. Thus it only remains to show that the function

$$\varphi_1(u_1, \dots, u_{G+g}) = -\log \det \left(\sum_{i=1}^{G+g} u_i \Omega'_i \right)$$

is bounded above and Lipschitz continuous in U' , and that the function

$$\varphi_2(u_1, \dots, u_{G+g}) = \left(\sum_{i=1}^{G+g} u_i \zeta_i \Omega'_i \right) \left(\sum_{i=1}^{G+g} u_i \Omega'_i \right)^{-1} \left(\sum_{i=1}^{G+g} u_i \zeta_i \Omega'_i \right)^t$$

is Lipschitz continuous on U' .

With respect to φ_1 , we can assume that the $u_i \geq M$ for some constant M . If A is a positive definite matrix and B is a semidefinite positive matrix then $\det(A+B) \geq \det(A)$, hence

$$\varphi_1(u_1, \dots, u_{G+g}) \leq \varphi_1(M, \dots, M),$$

and $\varphi_1(u_1, \dots, u_{G+g})$ is bounded above. To prove that it is Lipschitz continuous, following [8, A 4.1] we first compute

$$\frac{\partial \varphi_1}{\partial u_i} = -\operatorname{tr} \left(\left(\sum_{j=1}^{G+g} u_j \Omega'_j \right)^{-1} \Omega'_i \right).$$

Then, applying Lemma 5.28 below to

$$A = \sum_j M \Omega_j, \quad B = \sum_j (u_j - M) \Omega_j, \quad \text{and } C = \Omega_i,$$

we deduce that

$$0 \geq \frac{\partial \varphi_1}{\partial u_i}(u_1, \dots, u_{G+g}) \geq \frac{\partial \varphi_1}{\partial u_i}(M, \dots, M).$$

Hence the function φ_1 is Lipschitz continuous in U' .

Regarding φ_2 , by [9, Theorem 3.2.2] we know that φ_2 is continuous and bounded on the open simplex $\sum_i u_i = 1$, $u_i > 0$. Since it is homogeneous of degree 1, it is Lipschitz continuous on the open quadrant $\mathbb{R}_{>0}^{G+g}$ that contains U' . Therefore it is Lipschitz continuous on U' . \square

Lemma 5.28. *Let A be a real positive definite symmetric matrix of dimension r and B, C real positive semi-definite symmetric matrices of the same dimension. Then*

$$0 \leq \operatorname{tr}((A+B)^{-1}C) \leq \operatorname{tr}(A^{-1}C).$$

Proof. Let $A^{1/2}$ be the symmetric positive definite square root of A . Then

$$\begin{aligned} \operatorname{tr}((A+B)^{-1}C) &= \operatorname{tr}\left(\left(\operatorname{Id} + A^{-1/2}BA^{-1/2}\right)^{-1}A^{-1/2}CA^{-1/2}\right) \text{ and} \\ \operatorname{tr}(A^{-1}C) &= \operatorname{tr}(A^{-1/2}CA^{-1/2}). \end{aligned}$$

Thus it is enough to prove the statement for $A = \operatorname{Id}$. Write now

$$B = X^{-1}DX,$$

with D a diagonal matrix with non-negative entries and X an orthogonal matrix, i.e. $X^{-1} = X^t$. Since

$$\operatorname{tr}((\operatorname{Id}+B)^{-1}C) = \operatorname{tr}((\operatorname{Id}+D)^{-1}XCX^{-1}) \quad \text{and} \quad \operatorname{tr}(XCX^{-1}) = \operatorname{tr}(C),$$

we are reduced to the case when B is a diagonal matrix with non-negative entries. This case is an easy verification, as C being positive semi-definite implies that its diagonal entries are all non-negative. \square

We now assume the hypothesis of Proposition 5.23. Fix a rational section s of $\overline{L}_{k,m,\phi}$, and let $h = \overline{h}^{\operatorname{inv}}$ be the psh metric on $\overline{L}_{k,m,\phi}$ induced by the standard invariant metric. As in Section 3.3 we denote by $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ the b-divisor associated to s and h .

Let $(\overline{\mathcal{B}}(\Gamma)_{\Pi})_{\pi}$ be a model of $\overline{\mathcal{B}}(\Gamma)_{\Pi}$ on which the pullback E of the union of $\operatorname{div}(s)$ and the boundary of $\overline{\mathcal{B}}(\Gamma)_{\Pi}$ has simple normal crossings. Then we can view $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ as a b-divisor on $(\overline{\mathcal{B}}(\Gamma)_{\Pi})_{\pi}$.

Corollary 5.29. *The b-divisor $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is toroidal with respect to E .*

Proof. By Proposition 5.27 the metric h is toroidal with respect to the boundary of $\overline{\mathcal{B}}(\Gamma)_{\Pi}$. Hence, by Proposition 3.5, it follows that $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h) - \operatorname{div}(s)$ is toroidal with respect to the boundary of $\overline{\mathcal{B}}(\Gamma)_{\Pi}$, and hence with respect to E . Since $\operatorname{div}(s)$ is also toroidal with respect to E we deduce the result. \square

6. PROOF OF THE MAIN RESULT

We continue with the notation and assumptions of the previous section. We assume from now on that Σ is a smooth and projective admissible cone decomposition for \overline{C}_g , and Π is a smooth and projective admissible cone decomposition for \widetilde{C}_g over Σ .

6.1. A b-divisor which is not Cartier. In view of Corollary 5.29, in order to determine $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h) - \operatorname{div}(s)$ it is enough to compute Lelong numbers along toroidal divisors. Let τ be a maximal cone of Π and x_{τ} the corresponding point. Let $G = g(g+1)/2$ and let ζ_j, Ω'_j , $j = 1, \dots, G+g$ be local coordinates as in Section 5.2. Let $u = (\Omega_0, \zeta\Omega_0) \in \widetilde{C}_{g,\mathbb{Z}}$ be a primitive vector in the interior of the cone τ . Let Π' be an admissible cone decomposition subdividing Π , such that the ray generated by u is a ray of Π' . Let P_u be the irreducible divisor of $\overline{\mathcal{B}}(\Gamma)_{\Pi'}$ corresponding to the ray generated by u .

Lemma 6.1. *The Lelong number of h at P_u is given by*

$$\nu(h, P_u) = -m\phi(\Omega_0, \zeta\Omega_0) - m\zeta\Omega_0\zeta^t.$$

In particular it does not depend on the subdivision Π' .

Proof. Since the metric of \overline{M} is good in the sense of [26], it has zero Lelong numbers everywhere (see [4, Example 2.34]). So it is sufficient to treat the case $k = 0$. Let s be a rational section of $\overline{B}_{m\phi}$ that is generating on a neighborhood of x_τ . As in the proof of Lemma 5.19, s corresponds to a meromorphic Siegel–Jacobi form f of index m such that $-\log|f| - 2\pi m\phi$ is bounded in U' . Since u belongs to the interior of τ and is integral, there are positive integers a_1, \dots, a_{G+g} such that

$$u = (\Omega_0, \zeta\Omega_0) = \sum_{j=1}^{G+g} a_j(\Omega'_j, \zeta_j\Omega'_j).$$

Let $z = (z_1, \dots, z_{G+g})$ be a point of V' and consider the curve

$$\beta(t) = (z_1 t^{a_1}, \dots, z_{G+g} t^{a_{G+g}}), \quad |t| \leq 1.$$

For a general point z , the strict transform of the curve β in $\overline{\mathcal{B}}(\Gamma)_{\Pi'}$ goes through a general point of P_u .

By the explicit description of the standard invariant metric we have

$$\begin{aligned} -\log \|s(\beta(t))\| &= -\log |f(\beta(t))| + 2\pi m \frac{-1}{2\pi} \log |t| \zeta\Omega_0\zeta^t + C \\ &= \eta(t) + 2\pi m\phi \left(\frac{-1}{2\pi} \log |t|u \right) + 2\pi m \frac{-1}{2\pi} \log |t| \zeta\Omega_0\zeta^t + C \\ &= \eta(t) - m (\phi(u) + \zeta\Omega_0\zeta^t) \log |t| + C, \end{aligned}$$

where C is a constant depending on the point z and where the function

$$\eta(t) := -\log |f(\beta(t))| - 2\pi m\phi \left(\frac{-1}{2\pi} \log |t|u \right)$$

is bounded. Therefore

$$(6.1) \quad \lim_{t \rightarrow 0} \frac{|\eta(t) + C|}{-\log |t|} = 0.$$

Thus by Lemma 2.23 the Lelong number $\nu(h, P_u)$ is given by

$$\nu(h, P_u) = -m (\phi(u) + \zeta\Omega_0\zeta^t),$$

proving the lemma. \square

Remark 6.2. We note that in the proof above, instead of relying on the fact that good metrics have zero Lelong numbers to reduce to the case $k = 0$, we can prove directly the result for arbitrary $k \geq 0$. Namely, in this case the singularities of the metric of $\overline{M}^{\otimes k}$ can be absorbed into the function η . Then the function η will no longer be bounded but will have a growth of the shape $K \log(-\log |t|)$ when t goes to zero. The estimate (6.1) would still be true and we could finish the proof in the same way.

Since $(\Omega_0, \zeta\Omega_0)$ belongs to the interior of the cone, the matrix Ω_0 is positive definite, in particular invertible. Writing $\beta = \zeta\Omega$ (recall that the affine coordinates of \tilde{C}_g are (Ω, β)) we have

$$\zeta\Omega\zeta^t = \beta\Omega^{-1}\beta^t.$$

The function

$$(\Omega, \beta) \mapsto \beta\Omega^{-1}\beta^t$$

is a smooth convex function with non-zero Hessian, hence is not piecewise linear. On the other hand, the function ϕ is linear. Therefore the function that to each primitive vector v in the maximal cone τ associates the value $\nu(h, P_v)$ is not the restriction of a piecewise linear function on τ .

Corollary 6.3. *The b -divisor $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is not Cartier.*

Proof. Assume that it is a Cartier b -divisor. Replacing $\pi: X_\pi \rightarrow \overline{\mathcal{B}}(\Gamma)_\Pi$ by a finer toroidal modification we may assume $D = \mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is realized as a \mathbb{Q} -divisor on X_π . Moreover we can assume that the union of the support of D and the preimage of the boundary divisor of $\overline{\mathcal{B}}(\Gamma)_\Pi$ is contained in a simple normal crossings divisor E .

Again, the theory of toroidal compactifications [20] assigns to (X_π, E) and D a conical rational complex Δ and a piecewise linear function φ_D such that each ray ρ of Δ corresponds to an irreducible component E_ρ of E and the value of φ_D at the primitive generator of ρ is $-\text{ord}_{E_\rho} D$. Moreover, each primitive vector v contained in a cone of Δ corresponds to an irreducible exceptional divisor in some toroidal modification of (X_π, E) . Similarly, associated to the toroidal embedding $\mathcal{B}(\Gamma) \subset \overline{\mathcal{B}}(\Gamma)_\Pi$ there is a rational conical complex Λ such that

$$\Lambda = \Pi/\overline{\Gamma}.$$

Note that, by the conditions on the function ϕ , the function $-m\phi(\Omega, \zeta\Omega) - m\zeta\Omega\zeta^t$ is invariant under the action of $\overline{\Gamma}$. Hence it descends to a continuous conical function g on Λ . As discussed above the function g is not piecewise linear in any maximal cone of Λ .

Since the preimage of the boundary component is contained in E , there is a retraction map $r: \Delta \rightarrow \Lambda$ such that for each cone σ of Δ there is a cone τ of Λ with $r(\sigma) \subset \tau$. Choose a cone σ of Δ of maximal dimension such that $r(\sigma)$ is also maximal. The line bundle $\overline{L}_{k,m,\phi}$ and the section s define a Cartier divisor D_0 that we may assume has support on E because we can always enlarge E . To the divisor D_0 corresponds a piecewise linear function φ_{D_0} . Since $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is defined using Lelong numbers, Lemma 6.1 shows that on the cone σ ,

$$\varphi_D = \varphi_{D_0} + g.$$

Since φ_{D_0} is piecewise linear but g is not, this contradicts the piecewise linearity of φ_D . We conclude that $\mathbb{D}(\overline{L}_{k,m}, s, h)$ is not Cartier. \square

6.2. Graded linear series and b -divisors associated to Siegel–Jacobi forms. Let $U = \mathcal{B}(\Gamma)$ be the universal family of abelian varieties with level Γ . Let $F = K(U)$ denote the field of rational functions on U . Let $k, m \geq 0$ be integers and fix a rational section of $L_{k,m}$, that is, a meromorphic Siegel–Jacobi form s of weight k , index m and level Γ . We denote

$$\begin{aligned} \mathcal{J}_{k,m}(\Gamma, s)_\ell &= \{f \in F^\times \mid fs^\ell \in J_{\ell k, \ell m}(\Gamma)\} \cup \{0\}, \\ \mathcal{J}_{k,m}(\Gamma, s) &= \bigoplus_{\ell} \mathcal{J}_{k,m}(\Gamma, s)_\ell t^\ell \subset F[t], \\ \mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)_\ell &= \{f \in F^\times \mid fs^\ell \in J_{\ell k, \ell m}^{\text{cusp}}(\Gamma)\} \cup \{0\}, \\ \mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s) &= \bigoplus_{\ell} \mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)_\ell t^\ell \subset F[t]. \end{aligned}$$

Both $\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)$ and $\mathcal{J}_{k,m}(\Gamma, s)$ are graded linear series.

The first observation to make is that the above graded linear series are non-trivial as long as $k > 0$.

Lemma 6.4. *Let $k > 0$ and $m \geq 0$. There is an $\ell > 0$ such that $\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)_\ell \neq \{0\}$.*

Proof. Let $\ell > 0$ be big enough so that we can write

$$\ell k = k_1 + k_2$$

with $k_1 > 2g$, $k_2 > g + 2$ both even. As recalled in Example 4.5.3, using Poincaré series one can produce a non-zero cusp form φ_1 of weight k_1 and index 0. By Example 4.5.4 there is a non-zero Siegel–Jacobi form φ_2 of weight k_2 and index ℓm . Then by Lemma 4.3 we have $\varphi_1 \varphi_2$ a non-zero cusp form of weight ℓk and index ℓm and the non-zero function $\varphi_1 \varphi_2 / s^\ell$ belongs to $\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)_\ell$. \square

We next choose Σ and Π admissible cone decompositions as in Proposition 5.13. Let ϕ be a sufficiently negative admissible divisorial function on Π (which exists by Remark 5.18). Assume that m is divisible enough so that $m\phi$ has integral values on $\tilde{C}_{g,\mathbb{Z}}$. To ease notation we write $X = \overline{\mathcal{B}}(\Gamma)_\Pi$.

Let $\overline{L}_{k,m,\phi}$ be the extension of $L_{k,m}$ on X determined by the divisorial function ϕ . We can view the meromorphic Siegel–Jacobi form s as a rational section of $\overline{L}_{k,m,\phi}$. Let $h = \overline{h}^{\text{inv}}$ be the psh metric of $\overline{L}_{k,m,\phi}$ obtained by Proposition 5.23. Following Section 3.1 we have an associated graded linear series $\mathcal{R}(\overline{L}_{k,m,\phi}, s, h)$.

Lemma 6.5. *The graded linear series $\mathcal{R}(\overline{L}_{k,m,\phi}, s, h)$ and $\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)$ are equal.*

Proof. Let $\ell \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{R}(\overline{L}_{k,m,\phi}, s, h)_\ell$ is the set of $f \in F$ such that $f s^\ell$ is a meromorphic Siegel–Jacobi form, holomorphic on $\mathcal{H}_g \times \mathbb{C}^{(1,g)}$ and such that $\|f s^\ell\|$ is bounded. By Proposition 4.11 the latter set is exactly the set of $f \in F$ such that $f s^\ell$ is a Siegel–Jacobi cusp form. \square

To the graded linear series $\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)$ and $\mathcal{J}_{k,m}(\Gamma, s)$ we can associate the b-divisors $\text{b-div}(\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s))$ and $\text{b-div}(\mathcal{J}_{k,m}(\Gamma, s))$.

Lemma 6.6. *The equality*

$$\text{b-div}(\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)) = \text{b-div}(\mathcal{J}_{k,m}(\Gamma, s))$$

holds.

Proof. Since $\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s) \subset \mathcal{J}_{k,m}(\Gamma, s)$, by Lemma 3.8 we have that

$$\text{b-div}(\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)) \leq \text{b-div}(\mathcal{J}_{k,m}(\Gamma, s)).$$

To prove the converse inequality let $\pi \in R(X)$ and let P be a prime divisor of X_π . Set

$$r = \text{ord}_P \text{b-div}(\mathcal{J}_{k,m}(\Gamma, s)) \quad \text{and} \quad r_0 = \text{ord}_P \text{b-div}(\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)).$$

Also choose a non-zero $f_0 \in \mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)_{\ell_0}$ for some ℓ_0 (this exists by Lemma 6.4), and let $\varepsilon > 0$. The number r can be characterized as

$$r = \sup\{(1/\ell) \text{ord}_P f \mid \ell \geq 0, f \in \mathcal{J}_{k,m}(\Gamma, s)_\ell\}.$$

Hence we can find $\ell \gg 0$ and $f \in \mathcal{J}_{k,m}(\Gamma, s)_\ell$ satisfying the conditions

$$\frac{\text{ord}_P f}{\ell} \geq r - \varepsilon, \quad \ell_0/\ell \leq \varepsilon \quad \text{and} \quad \frac{\text{ord}_P f_0}{\ell} \geq -\varepsilon.$$

Note that to achieve the second and third condition we only need to make ℓ big enough. Since $f f_0$ is a cusp form by Lemma 4.3, we have

$$\frac{\text{ord}_P f + \text{ord}_P f_0}{\ell + \ell_0} \leq r_0.$$

Together with the above conditions, this implies

$$\ell(r - \varepsilon) \leq \text{ord}_P f \leq r_0(\ell + \ell_0) - \text{ord}_P f_0$$

and hence

$$r \leq r_0(1 + \varepsilon) + 2\varepsilon.$$

As $\varepsilon > 0$ can be chosen arbitrarily we deduce that $r \leq r_0$. This completes the proof. \square

Lemma 6.7. *For $k, m \geq 0$, the b-divisor $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is nef.*

Proof. By Proposition 5.23 the metric h is psh. The result then follows from Proposition 3.4. \square

Lemma 6.8. *For $k, m > 0$, the b-divisor $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is big.*

Proof. By Lemma 5.24, after choosing projective refinements Σ_0 and Π_0 of Σ and Π we can find numbers m_0, k_0 and polarization functions ϕ_0 and ψ_0 satisfying that $\overline{L}_{k_0,m_0,\phi_0+\psi_0}$ is ample and

$$H^0(\overline{\mathcal{B}}(\Gamma)_{\Pi_0}, \overline{L}_{\ell k_0, \ell m_0, \phi_0 + \psi_0}) \subset J_{\ell k_0, \ell m_0}(\Gamma).$$

After taking a multiple we can also assume that $\overline{L}_{m_0, k_0, \phi_0 + \psi_0}$ is very ample, hence generated by global sections. Let $r > 0$ such that

$$rk > k_0, \quad rm > m_0, \quad J_{rk-k_0, rm-m_0}^{\text{cusp}}(\Gamma) \neq \{0\}.$$

Such an $r > 0$ exists by Lemma 6.4. Let $0 \neq \varphi \in J_{rk-k_0, rm-m_0}^{\text{cusp}}(\Gamma)$ be a cusp form and let s_0 be the rational section of $\overline{L}_{k_0, m_0, \phi_0 + \psi_0}$ such that $s^r = s_0 \varphi$.

If f is a rational function such that $f s_0^\ell \in H^0(\overline{\mathcal{B}}(\Gamma)_{\Pi_0}, \overline{L}_{\ell m_0, \ell k_0, \phi_0 + \psi_0})$, then by Lemmas 5.24 and 4.3 we obtain that $f(s_0 \varphi)^\ell \in J_{\ell rk, \ell rm}^{\text{cusp}}(\Gamma)$. Therefore there is an inclusion of graded linear series

$$\mathcal{R}(\text{div}(s_0)) \subset \mathcal{J}_{rk, rm}^{\text{cusp}}(\Gamma, s^r).$$

Lemmas 3.9 and 6.5 yield the inclusions of graded linear series

$$\mathcal{R}(\text{div}(s_0)) \subset \mathcal{R}(\overline{L}_{rk, rm, \phi}, s^r, h^r) \subset \mathcal{R}(\mathbb{D}(\overline{L}_{rk, rm, \phi}, s^r, h^r)).$$

Since $\text{div}(s_0)$ is very ample, it is also big and generated by global sections. Applying now Proposition 3.12 and Lemma 3.8.3 we get

$$\text{div}(s_0) \leq r \mathbb{D}(\overline{L}_{k,m,\phi}, s, h).$$

By Lemma 3.15 the b-divisor $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is big. \square

Theorem 6.9. *Let $k, m > 0$. Then the graded algebra $\bigoplus_\ell J_{\ell k, \ell m}(\Gamma)$ is not finitely generated.*

Proof. Assume that $\bigoplus_\ell J_{\ell k, \ell m}(\Gamma)$ is finitely generated. Choose Σ, Π and ϕ as before. Choose s a meromorphic Siegel–Jacobi form of weight k and index m . Then the graded algebra $\bigoplus_\ell J_{\ell k, \ell m}(\Gamma)$ is isomorphic to the graded linear series $\mathcal{J}_{k,m}(\Gamma, s)$. We see that this graded linear series is finitely generated as an algebra. Hence by Lemma 3.20 the b-divisor $\text{b-div}(\mathcal{J}_{k,m}(\Gamma, s))$ is a Cartier b-divisor. Since $\text{b-div}(\mathcal{J}_{\ell k, \ell m}(\Gamma, s)) = \ell \text{b-div}(\mathcal{J}_{k,m}(\Gamma, s))$, the first one is Cartier if and only if the second one is Cartier. Therefore to achieve a contradiction we may assume that m is divisible enough so that $m\phi$ has integral values on $\tilde{C}_{g,\mathbb{Z}}$. By Lemmas 6.5 and 6.6 we know that

$$\text{b-div}(\mathcal{J}_{k,m}(\Gamma, s)) = \text{b-div}(\mathcal{R}(\overline{L}_{k,m,\phi}, s, h)).$$

We will now show that $\text{b-div}(\mathcal{R}(\overline{L}_{k,m,\phi}, s, h))$ is not Cartier. To this end we are allowed to replace the pair (k, m) by a suitable multiple and to change the section s .

Let D be a singularity divisor of h . As seen in the proof of Lemma 6.8, the linear series $\mathcal{R}(\overline{L}_{k,m,\phi}, s, h)$ contains an ample linear series. This implies that after replacing k and m by appropriate multiples we can change the section s so that the condition $\text{ord}_{D_i} \mathbb{D}(\overline{L}_{k,m,\phi}, s, h) > 0$ holds for all irreducible components of D .

By Lemmas 6.8 and 6.7 and Corollary 5.29 the b-divisor $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is big, nef, and toroidal. By Corollary 3.19 we deduce that

$$\text{b-div}(\mathcal{R}(\overline{L}_{k,m,\phi}, s, h)) = \mathbb{D}(\overline{L}_{k,m,\phi}, s, h).$$

By Corollary 6.3, we know that $\mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ is not a Cartier b-divisor. This proves the result. \square

Since Runge works with bounded (rather than fixed) ratio between the weight and the index, we must work slightly more to show that Theorem 6.9 disproves Runge’s claim in [31, Theorem 5.5].

Let J be a bigraded ring, graded over $\mathbb{Z}_{\geq 0}^2$. If $f \in J_{k,m}$ is (bi)homogeneous with $k \neq 0$ then we define the *relative index* of f as $r(f) = m/k$. Given $n \in \mathbb{Q}_{\geq 0}$, we define

$$(6.2) \quad J_n = \bigoplus_{(k,m):m=kn} J_{m,k} \quad \text{and} \quad J_{\leq n} = \bigoplus_{(k,m):m \leq kn} J_{m,k}.$$

Proposition 6.10. *Fix $n \in \mathbb{Q}_{>0}$, and suppose that $J_{\leq n}$ is finitely generated as an algebra over $J_{0,0}$. Then J_n is finitely generated as an algebra over $J_{0,0}$.*

Proof. Given a finite set of elements $a_i \in J_{m_i, k_i}$, note that

$$(6.3) \quad \min_i r(a_i) \leq r\left(\prod_i a_i\right) \leq \max_i r(a_i),$$

with equalities if and only if all $r(a_i)$ are equal.

Now let a_1, \dots, a_l be generators for $J_{\leq n}$ as $J_{0,0}$ -algebra; we may assume each a_i is bihomogeneous, say of degree (m_i, k_i) . Then we claim that J_n is generated by exactly those a_i with $r(a_i) = n$; in particular, J_n is finitely generated.

To prove our claim, let $f \in J_n$, say $f \in J_{m,k}$, and write $f = \sum_I \lambda_I a^I$ with $I \in \mathbb{Z}_{\geq 0}^l$, $\lambda_I \in J_{0,0}$, and $a^I := \prod_i a_i^{I_i}$. Then each a^I is bihomogeneous, hence if $\lambda_I \neq 0$ then $a^I \in J_{m,k}$, in particular $r(a^I) = r(f)$. But then by (6.3) we know each of the a_i with $I_i \neq 0$ has $r(a_i) = r(f) = n$. \square

Remark 6.11. As was mentioned in the introduction, Theorem 6.9 disproves [31, Theorem 5.5]. We can trace back the oversight in Runge’s proof to the proof of Theorem 5.1 in *loc. cit.* Here, the author states that the space of Siegel–Jacobi forms can be seen as the set of global sections of a natural invertible sheaf on a compactification of the universal abelian scheme living over the Satake–Baily–Borel compactification of the moduli space of principally polarized abelian varieties of level Γ . Let us be more precise. Consider the Satake–Baily–Borel compactification $\mathcal{A}(\Gamma)^*$ of $\mathcal{A}(\Gamma)$. Then the canonical fibration morphism $\pi: \mathcal{B}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$ extends to a compactification $\overline{\pi}: \mathcal{B}(\Gamma)^* \rightarrow \mathcal{A}(\Gamma)^*$, where $\mathcal{B}(\Gamma)^*$ is constructed as the BiProj of a bigraded ring R contained in the ring of Siegel–Jacobi forms.

The core of Runge’s argument leading to Theorem 5.1 of *loc. cit.* is that for every Siegel–Jacobi form in R , the limit when one approaches a point on the boundary of $\mathcal{B}(\Gamma)^*$ only depends on g parameters. Then it is claimed that this implies that all the fibres of the projection map $\mathcal{B}(\Gamma)^* \rightarrow \mathcal{A}(\Gamma)^*$ have dimension g .

The problem with this argument is that Siegel–Jacobi forms are sections of a line bundle and we are looking at the completion of the projective embedding defined by this line bundle. In this situation, if all the Siegel–Jacobi forms converge to zero simultaneously when approaching a point at the boundary, the dimension of the fibre may be bigger than g .

7. THE ASYMPTOTIC DIMENSION OF SPACES OF SIEGEL–JACOBI FORMS

The volume of a graded linear series \mathcal{R} on a variety X of dimension n is the non-negative real number given by

$$\text{Vol}(\mathcal{R}) = \limsup_{k \rightarrow \infty} \frac{\dim(\mathcal{R}_k)}{k^n/n!}.$$

In particular, given a b-divisor \mathbb{D} on X , the volume of \mathbb{D} can be expressed as $\text{Vol}(\mathbb{D}) = \text{Vol}(\mathcal{R}(\mathbb{D}))$, see Definition 3.13.

We recall that by [11, Theorem 3.2] any nef b-divisor \mathbb{D} on X has a well-defined degree \mathbb{D}^n in $\mathbb{R}_{\geq 0}$.

Lemma 7.1. *Assume \mathbb{D} is a big and nef toroidal b-divisor on X . Then we have the Hilbert-Samuel formula*

$$\text{Vol } \mathbb{D} = \mathbb{D}^n.$$

Proof. See [3, Theorem 5.13]. □

Lemma 7.2. *The function Vol is continuous on the space of big and nef toroidal b-divisors on X .*

Proof. See [3, Corollary 5.15]. □

As before let $h = \bar{h}^{\text{inv}}$ be the psh metric of $\bar{L}_{k,m,\phi}$ obtained by Proposition 5.23. Let s be a non-zero rational section of $L_{k,m}$.

Theorem 7.3. *Let $k, m > 0$ and let $\mathbb{D}_{k,m} = \mathbb{D}(\bar{L}_{k,m,\phi}, s, h)$, $\mathcal{J}_{k,m}(\Gamma, s)$ and $\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)$ be as in Section 6.2. Let $n = g(g+1)/2 + g$. Then the following sequence of equalities is satisfied.*

$$\text{Vol}(\mathcal{J}_{k,m}(\Gamma, s)) = \text{Vol}(\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)) = \text{Vol}(\mathbb{D}_{k,m}) = \mathbb{D}_{k,m}^n.$$

Proof. We know by Lemmas 6.8 and 6.7 and Corollary 5.29 that the b-divisor $\mathbb{D}_{k,m}$ is nef, big, and toroidal. Let D be a singularity divisor of h . As in the proof of Theorem 6.9 we may assume that the condition $\text{ord}_{D_i} \mathbb{D}_{k,m} > 0$ holds for all irreducible components D_i of D . We let $\mathbb{D}_j = (1 - 1/j)\mathbb{D}_{k,m}$. The sequence $\{\mathbb{D}_j\}_j$ forms a sequence of nef and big toroidal b-divisors converging to $\mathbb{D}_{k,m}$. Again by Lemma 3.9 and Corollary 3.11 we have

$$\mathcal{R}(\mathbb{D}_j) \subset \mathcal{R}(\bar{L}_{k,m,\phi}, s, h) \subset \mathcal{R}(\mathbb{D}_{k,m}).$$

Taking limits, using the continuity statement in Lemma 7.2, and using Lemma 6.5 we obtain

$$(7.1) \quad \text{Vol}(\mathbb{D}_{k,m}) = \lim_{j \rightarrow \infty} \text{Vol}(\mathbb{D}_j) = \text{Vol}(\mathcal{R}(\bar{L}_{k,m,\phi}, s, h)) = \text{Vol}(\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s)).$$

On the other hand, we have

$$(7.2) \quad \mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s) \subset \mathcal{J}_{k,m}(\Gamma, s) \subset \mathcal{R}(\text{b-div}(\mathcal{J}_{k,m}(\Gamma, s))),$$

where the last inclusion follows from Lemma 3.8. Moreover, as in the proof of Theorem 6.9 we have

$$\mathcal{R}(\text{b-div}(\mathcal{J}_{k,m}(\Gamma, s))) = \mathcal{R}(\text{b-div}(\mathcal{J}_{k,m}^{\text{cusp}}(\Gamma, s))) = \mathcal{R}(\bar{L}_{k,m,\phi}, s, h) = \mathcal{R}(\mathbb{D}_{k,m}).$$

Taking volumes in (7.2) and using (7.1) we get

$$(7.3) \quad \text{Vol}(\mathcal{J}_{k,m}(\Gamma, s)) = \text{Vol}(\mathbb{D}_{k,m}).$$

Finally, since $\mathbb{D}_{k,m}$ is toroidal, by Lemma 7.1 we have

$$(7.4) \quad \text{Vol}(\mathbb{D}_{k,m}) = \mathbb{D}_{k,m}^n.$$

Combining (7.1), (7.3) and (7.4) we obtain the result. □

Corollary 7.4. *We have*

$$(7.5) \quad \limsup_{\ell \rightarrow \infty} \frac{\dim J_{\ell k, \ell m}(\Gamma)}{\ell^n/n!} = \limsup_{\ell \rightarrow \infty} \frac{\dim J_{\ell k, \ell m}^{\text{cusp}}(\Gamma)}{\ell^n/n!} = \mathbb{D}_{k,m}^n.$$

Therefore, to obtain an asymptotic estimate of the growth of $\dim J_{\ell k, \ell m}(\Gamma)$ and $\dim J_{\ell k, \ell m}^{\text{cusp}}(\Gamma)$ we are reduced to computing $\mathbb{D}_{k,m}^n$.

Remark 7.5. The \limsup in (7.5) is actually a \lim for sufficiently divisible ℓ as it is the volume of a graded linear series of almost integral type (see Remark 3.14).

Our next task is to compute the degree $\mathbb{D}_{k,m}^n$. We recall that $\mathbb{D}_{k,m} = \mathbb{D}(\overline{L}_{k,m,\phi}, s, h)$ and that (see Proposition 5.27) the metric h is toroidal with respect to the boundary divisor $\overline{\mathcal{B}}(\Gamma)_{\Pi} \setminus \mathcal{B}(\Gamma)$ for any admissible cone decomposition Π . Then by [4, Theorem 5.20] we have that

$$\mathbb{D}_{k,m}^n = \int_{\overline{\mathcal{B}}(\Gamma)_{\Pi}} \langle c_1(\overline{L}_{k,m,\phi}, h)^n \rangle = \int_{\mathcal{B}(\Gamma)} c_1(L_{k,m}, h)^n.$$

Here the integral in the middle is the so called *non-pluripolar volume*, which agrees with the integral on the right hand side because the metric h is smooth on $\mathcal{B}(\Gamma)$.

We let h_B resp. h_M denote the canonical metrics on the line bundles M and B , see Definition 4.9 and the remarks immediately thereafter. As equation (4.7) also gives the canonical invariant metric h on $L_{k,m}$, by transport of structure we may use multi-linearity to deduce that

$$\int_{\mathcal{B}(\Gamma)} c_1(L_{k,m}, h)^n = \sum_{r=1}^n \binom{n}{r} \int_{\mathcal{B}(\Gamma)} m^r k^{n-r} c_1(B, h_B)^r \pi^* c_1(M, h_M)^{n-r}.$$

All the integrals are finite by [4, Remark 2.23].

Recall that here $n = \dim \mathcal{B}(\Gamma) = g + (g+1)g/2$. Let $[2]: \mathcal{B}(\Gamma) \rightarrow \mathcal{B}(\Gamma)$ be the map “multiplication by 2” fiber by fiber. Then $[2]$ is a finite map of degree 2^{2g} . Moreover, since B is symmetric and is rigidified along the origin, we have a canonical isomorphism

$$[2]^* B \simeq B^{\otimes 4}.$$

We have

$$[2]^* c_1(B, h_B) = 4c_1(B, h_B),$$

i.e., the isomorphism is compatible with the metric (see [14, Lemma 2.6], for instance).

Therefore

$$\begin{aligned} \int_{\mathcal{B}(\Gamma)} c_1(B, h_B)^r \pi^* c_1(M, h_M)^{n-r} &= \frac{1}{2^{2g}} \int_{\mathcal{B}(\Gamma)} [2]^* c_1(B, h_B)^r [2]^* \pi^* c_1(M, h_M)^{n-r} \\ &= \frac{1}{2^{2g}} \int_{\mathcal{B}(\Gamma)} 2^{2r} c_1(B, h_B)^r \pi^* c_1(M, h_M)^{n-r} = \frac{2^{2r}}{2^{2g}} \int_{\mathcal{B}(\Gamma)} c_1(B, h_B)^r \pi^* c_1(M, h_M)^{n-r}. \end{aligned}$$

Hence this integral is zero unless $r = g$. So

$$\int_{\mathcal{B}(\Gamma)} c_1(L_{k,m}, h)^n = \binom{n}{g} m^g k^{\frac{g(g+1)}{2}} \int_{\mathcal{B}(\Gamma)} c_1(B, h_B)^g \pi^* c_1(M, h_M)^{\frac{g(g+1)}{2}}.$$

Let A be a fibre of the map $\mathcal{B}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$. It is an abelian variety of dimension g . By the projection formula

$$\int_{\mathcal{B}(\Gamma)} c_1(B, h_B)^g \pi^* c_1(M, h_M)^{\frac{g(g+1)}{2}} = \deg(B|_A) \int_{\mathcal{A}(\Gamma)} c_1(M, h_M)^{\frac{g(g+1)}{2}}.$$

Since $B|_A$ is twice the principal polarization,

$$\deg(B|_A) = 2^g g!.$$

We write $\Gamma_0 = \mathrm{Sp}(2g, \mathbb{Z})$. Then

$$\int_{\mathcal{A}(\Gamma)} c_1(M, h_M)^{\frac{g(g+1)}{2}} = [\Gamma_0 : \Gamma] \int_{\mathcal{A}_g} c_1(M, h_M)^{\frac{g(g+1)}{2}},$$

where the second integral is an orbifold integral. By the formula after [36, Conjecture 8.3], writing $G = g(g+1)/2$ (so $n = g + G$) gives

$$\int_{\mathcal{A}_g} c_1(M, h_M)^{\frac{g(g+1)}{2}} = (-1)^G \frac{G!}{2^g} \prod_{k=1}^g \frac{\zeta(1-2k)}{(2k-1)!}.$$

Summing up we obtain

$$\mathbb{D}_{k,m}^n = (-1)^G m^g k^G (G+g)! [\Gamma_0 : \Gamma] \prod_{k=1}^g \frac{\zeta(1-2k)}{(2k-1)!}.$$

By Corollary 7.4 we obtain the asymptotic growth of the dimension of the spaces of Siegel–Jacobi forms and of cusp Siegel–Jacobi forms explicitly, recovering a formula already implicit in Tai’s work [34].

Corollary 7.6. *The asymptotic growth of the dimension of the spaces $J_{\ell k, \ell m}(\Gamma)$ and $J_{\ell k, \ell m}^{\mathrm{cusp}}(\Gamma)$ when ℓ goes to infinity is given by the following formulae:*

$$\begin{aligned} \limsup_{\ell \rightarrow \infty} \frac{\dim J_{\ell k, \ell m}(\Gamma)}{\ell^n / n!} &= \limsup_{\ell \rightarrow \infty} \frac{\dim J_{\ell k, \ell m}^{\mathrm{cusp}}(\Gamma)}{\ell^n / n!} \\ &= (-1)^G n! m^g k^G [\Gamma_0 : \Gamma] \prod_{k=1}^g \frac{\zeta(1-2k)}{(2k-1)!} \\ &= (-1)^n n! m^g k^G 2^{G-g} [\Gamma_0 : \Gamma] \prod_{k=1}^g \frac{(k-1)! B_{2k}}{(2k)!} \\ &= V_g \cdot n! m^g k^G 2^{-G-1} \pi^{-G} [\Gamma_0 : \Gamma], \end{aligned}$$

where $B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k)$ are the Bernoulli numbers and

$$V_g = (-1)^n 2^{g^2+1} \pi^G \prod_{k=1}^g \frac{(k-1)! B_{2k}}{(2k)!}$$

is the symplectic volume of \mathcal{A}_g computed by Siegel in [33, Section VIII].

By Remark 7.5 the lim sup above is actually a lim for sufficiently divisible ℓ .

Remark 7.7. The above formulas can also be obtained by combining the formulas in the proofs of Propositions 2.1 and 2.5 of Tai’s work [34].

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