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QCD IN THE INFRARED WITH EXACT ANGULAR INTEGRATIONS

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In a previous paper we have shown that in quantum chromodynamics the gluon propagator vanishes in the infrared limit, while the ghost propagator is more singular than a simple pole. These results were obtained after angular averaging, but here we go beyond this approximation and perform an exact calculation of the angular integrals. The powers of the infrared behavior of the propagators are changed substantially. We find the very intriguing result that the gluon propagator vanishes in the infrared exactly like p^2, whilst the ghost propagator is exactly as singular as $1/p^4$. We also find that the value of the infrared fixed point of the QCD coupling is much decreased: it is now equal to $4\pi/3$.

Following a recent study by von Smekal *et al.*,[1] we analyzed in Ref. 2 the coupled Dyson–Schwinger equations for the gluon and ghost form factors F and G. The approximations were twofold: first the vertices were taken bare, and second angular averaging was introduced. Here we seek to remove the deficiency of the angular averaging. On the one hand, the results might be regarded simply as quantitative adjustments to the calculations, on the other hand, they are far from negligible. The numerical value of the infrared fixed point is reduced by a factor of almost three; and the finding that the gluon propagator has a simple zero, while the ghost propagator has double poles, might perhaps be deemed a qualitatively new result.

As an improvement on the approximation used in Ref. 2, we now solve the coupled integral equations for the gluon and ghost propagators with an exact treatment of the angular integrals. Although the angular averaging, the so-called y-max approximation,[a] is good in the ultraviolet region where the form factors run logarithmically, we will see that it is a crude approximation in the infrared region where the form factors exhibit power behavior. The approximation apparently leads to a difficulty in the ghost equation, as the asymmetry in the treatment of the gluon and ghost momenta in the loop gives an ambiguous result. In our previous study

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[a] The y-max approximation of Ref. 2 amounts to replacing $F(z)$ by $F(\max(x, y))$ in Eq. (1), and $G(z)$ by $G(\max(x, y))$ in Eqs. (2) and (11).

we used the ghost and gluon momenta for the radial and angular integrations respectively. However, if we exchange these momenta, we no longer find a consistent solution to the integral equation. We will show later why the former is a better approximation than the latter, thus motivating the choice made in Ref. 2. Furthermore we will show that both choices are equivalent if we treat the angular integrals exactly though this equivalence is by no means trivial.

With the radial integration over the ghost momentum, the ghost form factor satisfies

$$\frac{1}{G(x)} = \tilde{Z}_3 - \frac{6\lambda}{\pi}\int_0^{\Lambda^2} dy\, yG(y)\int_0^{\pi} d\theta \sin^4\theta \frac{F(z)}{z^2}\,, \tag{1}$$

whereas with the radial integration over the gluon momentum, we have instead

$$\frac{1}{G(x)} = \tilde{Z}_3 - \frac{6\lambda}{\pi}\int_0^{\Lambda^2} dy\, F(y)\int_0^{\pi} d\theta \sin^4\theta \frac{G(z)}{z}\,, \tag{2}$$

where $\lambda = g^2/16\pi^2$, g being the strong coupling constant, and where $z = x + y - 2\sqrt{xy}\cos\theta$. Here we have used the fact that $\tilde{Z}_1$, the ghost–gluon vertex renormalization constant, is equal to unity in the Landau gauge.

We will show that

$$F(x) = Ax^{2\kappa}\,, \qquad G(x) = Bx^{-\kappa} \tag{3}$$

are solutions of Eqs. (1) and (2), yielding identical conditions on λAB^2, but that this equivalence *does not hold* in the y-max approximation.

Substituting expression (3) into Eq. (1) and evaluating the angular integral in terms of the hypergeometric function[3] ${}_2F_1$ using Eq. (A.1), we obtain

$$\begin{aligned}\frac{x^\kappa}{B} &= \tilde{Z}_3 - \frac{9\lambda AB}{4}\int_0^{\Lambda^2} dy\, y^{-\kappa+1} y_>^{2\kappa-2}\, {}_2F_1\left(-2\kappa+2, -2\kappa; 3; \frac{y_<}{y_>}\right)\\ &= \tilde{Z}_3 - \frac{9\lambda AB}{4}\left\{x^{2\kappa-2}\int_0^{x} dy\, y^{-\kappa+1}\, {}_2F_1\left(-2\kappa+2, -2\kappa; 3; \frac{y}{x}\right)\right.\\ &\qquad \left. + \int_x^{\Lambda^2} dy\, y^{\kappa-1}\, {}_2F_1\left(-2\kappa+2, -2\kappa; 3; \frac{x}{y}\right)\right\}.\end{aligned}$$

Set $t = y/x$ and $t = x/y$ in the infrared and ultraviolet integrations, respectively and take $\Lambda^2 \to \infty$:

$$\frac{x^\kappa}{B} = -\frac{9\lambda AB}{4}x^\kappa \int_0^1 dt(t^{-\kappa+1} + t^{-\kappa-1})\,{}_2F_1(-2\kappa+2, -2\kappa; 3; t)\,. \tag{4}$$

Note that we have dropped $\tilde{Z}_3$. This is in accordance with the standard regularization procedure: the integral in Eq. (1) is convergent if $\mathrm{Re}\,\kappa < 0$, whereas a subtraction is necessary if $0 \leq \kappa < 1$. By identifying $\tilde{Z}_3$ with this subtraction constant, we ensure that $G(x)$ is defined by analytic continuation in κ beyond $\mathrm{Re}\,\kappa = 0$. This continuation is made explicit in terms of the generalized hypergeometric function[3] ${}_3F_2$ (see Eq. (A.2)). After matching the coefficients of x^κ we obtain

$$\frac{1}{\lambda AB^2} = -\frac{9}{4}\left[\frac{1}{2-\kappa}\,{}_3F_2(-2\kappa+2,-2\kappa,2-\kappa;3,3-\kappa;1)\right.$$
$$\left. -\frac{1}{\kappa}\,{}_3F_2(-2\kappa+2,-2\kappa,-\kappa;3,1-\kappa;1)\right]. \tag{5}$$

The y-max approximation that we made in Ref. 2 amounts to replacing κ by zero in the hypergeometric function in Eq. (4). The result was

$$\frac{1}{\lambda AB^2} = -\frac{9}{4}\left[\frac{1}{2-\kappa} - \frac{1}{\kappa}\right]. \tag{6}$$

Next we make a similar analysis of Eq. (2):

$$\frac{x^\kappa}{B} = \tilde{Z}_3 - \frac{9\lambda AB}{4}\int_0^{\Lambda^2} dy\, y^{2\kappa} y_>^{-\kappa-1}\,{}_2F_1\left(\kappa+1,\kappa-1;3;\frac{y_<}{y_>}\right). \tag{7}$$

After defining $t = y/x$ in the infrared integration and $t = x/y$ in the ultraviolet integration, taking $\Lambda^2 \to \infty$ and eliminating $\tilde{Z}_3$ as before, we obtain

$$\frac{x^\kappa}{B} = -\frac{9\lambda AB}{4}x^\kappa \int_0^1 dt(t^{2\kappa} + t^{-\kappa-1})\,{}_2F_1(\kappa+1,\kappa-1;3;t)\,. \tag{8}$$

Substituting Eq. (A.2) in Eq. (8) and equating the coefficients of x^κ we have

$$\frac{1}{\lambda AB^2} = -\frac{9}{4}\left[\frac{1}{2\kappa+1}\,{}_3F_2(\kappa+1,\kappa-1,2\kappa+1;3,2\kappa+2;1)\right.$$
$$\left. -\frac{1}{\kappa}\,{}_3F_2(\kappa+1,\kappa-1,-\kappa;3,1-\kappa;1)\right]. \tag{9}$$

The y-max approximation of this expression is once more obtained by replacing κ by zero in the hypergeometric function in Eq. (8), yielding

$$\frac{1}{\lambda AB^2} = -\frac{9}{4}\left[\frac{1}{2\kappa+1} - \frac{1}{3(2\kappa+2)} - \frac{1}{\kappa} + \frac{1}{3(\kappa-1)}\right]. \tag{10}$$

This approximation is quite different from that of Eq. (6) illustrating the precariousness of the y-max approximation.

However, the exact forms Eqs. (5) and (9) are identical, as we verified by using Mathematica to check the numerical equivalence over a domain of κ values.

We now consider the gluon equation, and will show that it is also solved by a pure power behavior if we keep only the ghost loop. Even in the presence of the gluon loop, when the power behavior is no longer an exact solution of the equation, it will still represent the correct leading order infrared asymptotic behavior. Omitting then the gluon loop, we write the equation for the gluon form factor as follows:

$$\frac{1}{F(x)} = Z_3 + \frac{2\lambda}{\pi}\int_0^{\Lambda^2}\frac{dy}{x}G(y)\int_0^{\pi} d\theta \sin^2\theta M(x,y,z)G(z)\,, \tag{11}$$

where

$$M(x,y,z) = \left(\frac{x+y}{2} - \frac{y^2}{x}\right)\frac{1}{z} + \frac{1}{2} + \frac{2y}{x} - \frac{z}{x}\,.$$

We now substitute the solution (3) into Eq. (11), obtaining

$$\begin{aligned}\frac{x^{-2\kappa}}{A} &= Z_3 + \frac{2\lambda B^2}{\pi}\int_0^{\Lambda^2}\frac{dy}{x}y^{-\kappa}\int_0^{\pi} d\theta\sin^2\theta\left[\left(\frac{x+y}{2} - \frac{y^2}{x}\right)z^{-\kappa-1}\right.\\ &\quad \left. + \left(\frac{1}{2} + \frac{2y}{x}\right)z^{-\kappa} - \frac{z^{-\kappa+1}}{x}\right].\end{aligned}$$

We evaluate the angular integral with the help of Eq. (A.1):

$$\begin{aligned}\frac{x^{-2\kappa}}{A} &= Z_3 + \lambda B^2\int_0^{\Lambda^2}\frac{dy}{x}y^{-\kappa}\left[\left(\frac{x+y}{2} - \frac{y^2}{x}\right)y_>^{-\kappa-1}\,{}_2F_1\left(\kappa+1,\kappa;2;\frac{y_<}{y_>}\right)\right.\\ &\quad \left. + \left(\frac{1}{2} + \frac{2y}{x}\right)y_>^{-\kappa}\,{}_2F_1\left(\kappa,\kappa-1;2;\frac{y_<}{y_>}\right) - \frac{y_>^{-\kappa+1}}{x}\,{}_2F_1\left(\kappa-1,\kappa-2;2;\frac{y_<}{y_>}\right)\right].\end{aligned}$$

We split the integration region and define $t = y/x$ in the infrared integration and $t = x/y$ in the ultraviolet integration, as before, and finally implicitly implement the analytic regularization, taking $\Lambda^2 \to \infty$. After some re-arrangement, we obtain

$$\begin{aligned}\frac{x^{-2\kappa}}{A} &= \lambda B^2 x^{-2\kappa}\int_0^1 dt\left\{\left[\frac{1}{2}(t^{-\kappa} + t^{-\kappa+1} + t^{2\kappa-1} + t^{2\kappa-2})\right.\right.\\ &\quad \left. - (t^{-\kappa+2} + t^{2\kappa-3})\right]{}_2F_1(\kappa+1,\kappa;2;t)\\ &\quad + \left[\frac{1}{2}(t^{-\kappa} + t^{2\kappa-2}) + 2(t^{-\kappa+1} + t^{2\kappa-3})\right]{}_2F_1(\kappa,\kappa-1;2;t)\\ &\quad \left. - [t^{-\kappa} + t^{2\kappa-3}]{}_2F_1(\kappa-1,\kappa-2;2;t)\right\}.\end{aligned} \tag{12}$$

Substituting Eq. (A.2) into Eq. (12) and equating coefficients of x^κ, we find

$$\begin{aligned}
&\frac{1}{\lambda AB^2}\\
&\quad= \frac{1}{2(1-\kappa)}\,{}_3F_2(\kappa+1,\kappa,1-\kappa;2,2-\kappa;1)+\frac{1}{2(2-\kappa)}\,{}_3F_2(\kappa+1,\kappa,2-\kappa;2,3-\kappa;1)\\
&\qquad+\frac{1}{4\kappa}\,{}_3F_2(\kappa+1,\kappa,2\kappa;2,2\kappa+1;1)+\frac{1}{2(2\kappa-1)}\,{}_3F_2(\kappa+1,\kappa,2\kappa-1;2,2\kappa;1)\\
&\qquad-\frac{1}{3-\kappa}\,{}_3F_2(\kappa+1,\kappa,3-\kappa;2,4-\kappa;1)-\frac{1}{2(\kappa-1)}\,{}_3F_2(\kappa+1,\kappa,2\kappa-2;2,2\kappa-1;1)\\
&\qquad+\frac{1}{2(1-\kappa)}\,{}_3F_2(\kappa,\kappa-1,1-\kappa;2,2-\kappa;1)+\frac{1}{2(2\kappa-1)}\,{}_3F_2(\kappa,\kappa-1,2\kappa-1;2,2\kappa;1)\\
&\qquad+\frac{2}{2-\kappa}\,{}_3F_2(\kappa,\kappa-1,2-\kappa;2,3-\kappa;1)+\frac{1}{\kappa-1}\,{}_3F_2(\kappa,\kappa-1,2\kappa-2;2,2\kappa-1;1)\\
&\qquad-\frac{1}{1-\kappa}\,{}_3F_2(\kappa-1,\kappa-2,1-\kappa;2,2-\kappa;1)\\
&\qquad-\frac{1}{2(\kappa-1)}\,{}_3F_2(\kappa-1,\kappa-2,2\kappa-2;2,2\kappa-1;1)\,.
\end{aligned}\tag{13}$$

Some of these generalized hypergeometric functions are in fact infinite for certain values of κ; they should be understood by replacing the final argument "1" in each generalized hypergeometric function by τ, and then by taking the limit $\tau \to 1$ of the sum of all the terms. Since this amounts to replacing the upper integration limit in Eq. (12) by τ, the procedure is clearly legitimate.

Finally, to demonstrate that Eq. (3) is really a solution of the coupled integral equations, for a particular value of κ, we equate the right-hand side of Eq. (5) or Eq. (9) with that of Eq. (13) and solve for κ. We used the routine FindRoot of Mathematica and discovered that, in the limit

$$\kappa \to 1\,,$$

both expressions become equal. This limit has been checked analytically in Appendix B. We find

$$\lambda AB^2 = \frac{1}{3}$$

which means that the infrared fixed point for the gauge invariant running coupling, $\alpha(p^2) = 4\pi\lambda\tilde{Z}_1^2 F(p^2)G^2(p^2)$, has the value

$$\alpha(0) = \frac{4\pi}{3} \approx 4.19$$

which is quite different from the value we found using the y-max approximation. There we found $\kappa \approx 0.77$ and $\alpha(0) \approx 11.47$, nearly three times the exact result.

Our findings for the propagators in the infrared region can be summarized by the following formulas:

gluon propagator in Landau gauge

$$D^{ab}_{\mu\nu}(p) \sim -\delta^{ab}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right]p^2\,, \tag{14}$$

ghost propagator in Landau gauge

$$G^{ab}(p) \sim -\delta^{ab}\frac{1}{p^4}\,. \tag{15}$$

A. Angular and Radial Integrations

Throughout we used the angular integration formula (see Eq. (3.665.2) of Ref. 4)

$$\int_0^\pi d\theta \sin^{2r}\theta z^n = B\left(r+\frac{1}{2},\frac{1}{2}\right) y_>^n\, {}_2F_1\left(-n,-n-r;r+1;\frac{y_<}{y_>}\right)\,, \tag{A.1}$$

where $z = x + y - 2\sqrt{xy}\cos\theta$, $y_> = \max(x,y)$ and $y_< = \min(x,y)$.

The radial integrations of hypergeometric functions[3] are evaluated as follows:

$$\begin{aligned}\int_0^1 dt\, t^\nu\, {}_2F_1(a,b;c;t) &= \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!}\int_0^1 dt\, t^{\nu+n} \\ &= \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!}\frac{1}{\nu+n+1} \\ &= \frac{1}{\nu+1}\, {}_3F_2(a,b,\nu+1;c,\nu+2;1)\,. \end{aligned}\tag{A.2}$$

B. Limit of λAB^2 for $\kappa \to 1$

We check analytically that the limit of λAB^2 for $\kappa \to 1$ is identical for Eqs. (5), (9) and (13). Starting from Eq. (5) we find

$$\begin{aligned}\frac{1}{\lambda AB^2} &= -\frac{9}{4}\left\{{}_3F_2(0,-2,1;3,2;1) - \lim_{\kappa\to 1}\, {}_3F_2(2-2\kappa,-2,-1;3,1-\kappa;1)\right\} \\ &= -\frac{9}{4}\left[1 - \left(1+\frac{4}{3}\right)\right] = 3\end{aligned}$$

and from Eq. (9)

$$\begin{aligned}\frac{1}{\lambda AB^2} &= -\frac{9}{4}\left\{\frac{1}{3}\, {}_3F_2(2,0,3;3,4;1) - \lim_{\kappa\to 1}\, {}_3F_2(2,\kappa-1,-1;3,1-\kappa;1)\right\} \\ &= -\frac{9}{4}\left[\frac{1}{3} - \left(1+\frac{2}{3}\right)\right] = 3\,.\end{aligned}$$

We make an analogous check for the gluon equation. To circumvent convergence problems, we replace the argument of the generalized hypergeometric functions by τ, and take the limit $\tau \to 1$ at the end. From Eq. (13),

$$
\begin{aligned}
\frac{1}{\lambda AB^2} = \lim_{\tau\to1}\lim_{\kappa\to1}\Bigg\{ & \frac{1}{2(1-\kappa)}\,{}_3F_2(2,1,1-\kappa;2,1;\tau) + {}_3F_2(2,1,1;2,2;\tau) \\
& - \frac{1}{4}\,{}_3F_2(2,1,2;2,3;\tau) - \frac{1}{2(\kappa-1)}\,{}_3F_2(2,1,2\kappa-2;2,1;\tau) \\
& + \frac{1}{2(1-\kappa)}\,{}_3F_2(1,\kappa-1,1-\kappa;2,1;\tau) + \frac{5}{2}\,{}_3F_2(1,0,1;2,2;\tau) \\
& + \frac{1}{\kappa-1}\,{}_3F_2(1,\kappa-1,2\kappa-2;2,1;\tau) - \frac{1}{1-\kappa}\,{}_3F_2(\kappa-1,-1,1-\kappa;2,1;\tau) \\
& - \frac{1}{2(\kappa-1)}\,{}_3F_2(\kappa-1,-1,2\kappa-2;2,1;\tau)\Bigg\} \\
= \lim_{\tau\to1}\lim_{\kappa\to1}\Bigg\{ & \frac{1}{2(1-\kappa)}\sum_{n=1}^{\infty}\frac{(1-\kappa)_n + (2\kappa-2)_n}{n!}\tau^n + {}_2F_1(1,1;2;\tau) \\
& - \frac{1}{4}\,{}_2F_1(1,2;3;\tau)\Bigg\} + \frac{5}{2} \\
= \lim_{\tau\to1} & \left\{\sum_{n=1}^{\infty}\left(-\frac{1}{2n} + \frac{1}{n+1} - \frac{1}{2(2+n)}\right)\tau^n\right\} + \frac{13}{4} \\
= 3 + {} & \lim_{\tau\to1}\left\{\sum_{n=3}^{\infty}\left(-\frac{1}{2n} + \frac{1}{n} - \frac{1}{2n}\right)\tau^n\right\} \\
= 3\,. &
\end{aligned}
$$

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References

1. L. von Smekal, A. Hauck and R. Alkofer, *Phys. Rev. Lett.* **79**, 3591 (1997); hep-ph/9707327.
2. D. Atkinson and J. C. R. Bloch, hep-ph/9712459, submitted to *Phys. Rev. D.*

3. See, for instance, Bateman Manuscript Project, *Higher Transcendental Functions*, ed. A. Erdélyi (McGraw-Hill, 1953), Vol. I.
4. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 1994), 5th edition.