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Reasoning about uncertain conditionals

Abstract. There is a long tradition in formal epistemology and in the psychology of reasoning to investigate indicative conditionals. In psychology, the propositional calculus was taken for granted to be the normative standard of reference. Experimental tasks, evaluation of the participants' responses and psychological model building, were inspired by the semantics of the material conditional. Recent empirical work on indicative conditionals focuses on uncertainty. Consequently, the normative standard of reference has changed.

I argue why neither logic nor standard probability theory provide appropriate rationality norms for uncertain conditionals. I advocate coherence based probability logic as an appropriate framework for investigating uncertain conditionals. Detailed proofs of the probabilistic non-informativeness of a *paradox of the material conditional* illustrate the approach from a formal point of view. I survey selected data on human reasoning about uncertain conditionals which additionally support the plausibility of the approach from an empirical point of view.

Keywords: coherence, conditionals, formal epistemology, paradoxes of the material conditional, probability logic, psychology of reasoning

Introduction and overview

There is a long tradition in formal epistemology and in the psychology of reasoning to investigate indicative conditionals.[1] In the psychology of reasoning, the propositional calculus was taken for granted to be the correct normative standard of reference for investigating conditionals [14]. The chosen normative standard of reference guided the construction of psychological theories like a roadmap [37]. Proof-theoretic semantics, for example, stimulated the emergence of two prominent psychological theories of reasoning: Rips' theory of *mental rules* [47] and Braine and O'Brien's theory of *mental logic* [6]. Both are rule-based. Likewise, model-theoretic semantics influenced Johnson-Laird's psychological *mental models* theory [27].

Not only psychological theories but also the experimental paradigms for investigating human conditional reasoning were strongly influenced by the

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[1]Indicative conditionals are of the form "If A, then C", where the antecedent A and the consequent C are propositions.

A	C	$A \supset C$	$C\|A$
true	true	true	true
true	false	false	false
false	true	true	void
false	false	true	void

Table 1. Truth tables for the material conditional ($A \supset C$) and the conditional event ($C|A$).

propositional calculus. Typical examples are the well-known Wason selection task [50] and the truth table task paradigm (e.g., [13]). Deviations between the response patterns in the Wason selection task and the semantics of the material conditional ($A \supset C$; cf. Table 1) interpretation of indicative conditionals challenged logic-based conceptions of the rationality of human conditional reasoning. Moreover, an important empirical finding in the truth table task paradigm was already reported by Johnson-Laird and Tagard in 1969. The authors observed that about 80% of the participants evaluated a conditional "If A then C" as true if $A \wedge C$, false if $A \wedge \neg C$ but irrelevant if $\neg A$ ([29, p. 370], for more recent studies see [2, 46]). This response pattern was labeled "defective truth table" by Wason [50]. The negative connotation associated to "defective" signals the divergence of this response pattern from the truth table pattern of the material conditional: The $\neg A \wedge C$ and $\neg A \wedge \neg C$ cases are not irrelevant. They rather make $A \supset C$ true. Thus, propositional logic also provided rationality criteria for evaluating human inference: Logical validity and the interpretation of indicative conditionals as material conditionals were the most important rationality criteria for assessing human reasoning performance. The participant's interpretation of indicative conditionals was classified as "rational" if, and only if the participant's interpretation is consistent with the semantics of the material conditional.

For about ten years now, empirical work on indicative conditionals focuses on uncertainty. Consequently, the normative standard of reference has changed: More and more psychological studies of reasoning adopted probabilistic approaches as rationality frameworks (e.g., [3, 15, 25, 30, 34, 35, 39, 40, 45]). Many studies provided new evidence for the psychological plausibility of the conditional event interpretation of indicative conditionals. Thus, there is nothing "defective" about the response pattern which Wason called "defective truth table": The responses are consistent with the semantics of the conditional event (see Table 1).

My paper is structured as follows: In Section 1 I argue why neither logic nor standard probability theory provide appropriate rationality norms for uncertain conditionals. I advocate coherence based probability logic as an appropriate framework for investigating uncertain conditionals. After presenting the framework, I illustrate—from a formal point of view—how the coherence approach works by detailed proofs of the probabilistic non-informativeness of a paradox of the material conditional (Section 2). Section 3 surveys selected recent data on human reasoning about uncertain conditionals which additionally support the plausibility of the approach from an empirical point of view.

1. The rationality framework

Human reasoning about conditionals has to deal with uncertain, incomplete and defeasible information. Propositional logic, however, is a language for reasoning under certainty but not for reasoning under uncertainty. Moreover, the consequence relation of propositional logic is monotonic: Adding premises to a logically valid argument can only increase but not decrease the set of entailed conclusions. This makes retracting conclusions in the light of new premises impossible. Monotonicity of propositional logic is also reflected in *premise strengthening* ($A \supset C$ logically implies $(A \wedge B) \supset C$), which allows for adding formulas in the antecedent of a conditional. This has many counterintuitive instantiations and is one of the paradoxes of the material conditional. These paradoxes, as noted by Lewis in 1912 [33], show that the material conditional is in most cases inappropriate for formalizing everyday life conditionals. All these reasons against the application of the propositional calculus as a rationality framework are of theoretical nature. I discuss selected empirical data that speak for coherence based probability logic as an appropriate rationality framework for reasoning about uncertain conditionals in Section 3. The next sections introduce the proposed rationality framework.

1.1. Coherence based probability theory

The coherence approach to probability goes back to de Finetti [10, 11]. More recent work includes, e.g., [4, 8, 16, 21, 32, 49]. A formal characterization of coherence is given in Appendix A. The following paragraphs focus on the underlying philosophical intuitions of coherence and on how the coherence approach differs from standard approaches to probability (like Kolmogorov's approach to probability [31]).

"Coherence" refers here to a foundation of probability theory.[2] Specifically, coherence follows the tradition of subjective probability theory where probabilities are conceived as *degrees of belief* and not as ontological objective quantities. The probability function is defined on an *arbitrary* family of conditional events, which is another key feature of coherence based probability theory. Therefore, the assumption of a complete algebra—which is made in standard approaches to probability—is not required in the context of coherence.

Conditional probability, $P(B|A)$, is a *primitive* notion.[3] The probability value is assigned *directly* to the conditional event, $B|A$ (cf. Table 1), as a whole. This is another difference between the coherence approach and standard approaches to probability. In the latter approaches, conditional probability is defined by the fraction of the joint and the marginal probability,

$$P(B|A) \quad =_{\text{def.}} \quad \frac{P(A \wedge B)}{P(A)}, \text{ if } P(A) > 0\,.$$

$P(A) > 0$ is a necessary requirement here, otherwise $P(B|A)$ is undefined since divisions by zero are undefined.

Coherence, however, does not require positive probabilities of the conditioning events: for the evaluation of $P(B|A)$, A is *assumed* to be true but the probability of A does not play any role in the evaluation. Thus, the conditioning probability $P(A)$ can be positive as well as equal to zero. If (conditioning) probabilities equal to zero are available, they are exploited to reduce the complexity in the probabilistic inference. One way to find out whether sets of probabilistic assessments are coherent requires solving systems of equations: zero probabilities reduce the complexity of this task substantially (see Section 2).

In the coherence approach, the probability of a tautology must be equal to one: $P(\top) = 1$. The converse, however, does not hold: $P(A) = 1$ does not imply that A is a tautology. Approaches that reserve probability one

[2]This meaning of "coherence" should not be confused with the epistemological problem of characterizing formally "how sentences hang together", which is also denoted by "coherence" (see, e.g., [12]).

[3]As pointed out by the editors, also Popper functions use conditional probability as a primitive notion (see [26] for an introduction). The relationship between Popper functions and coherence is discussed in [9]. In a nutshell, to compare both approaches, the coherence approach needs to be extended by the possibility to conditionalize on $\bot$. The only extension compatible with Popper functions is $P^{\star}(A|\bot) = 1$. Then, the conditional probability $P^{\star}(\cdot|\cdot)$ is a Popper function. However, the converse does not hold: If $P(A) = 0$, the corresponding Popper function $P^{\star}(\cdot|A)$ is not necessarily a probability [9, Theorem 1].

(1)	$P(A) = x$	A
(2)	$P(B\|A) = y$	If A, then B
(3)	$xy \leq P(B) \leq xy + 1 - x$	B

Table 2. Probabilistic and non-probabilistic version of the *modus ponens*.

and zero for the extreme cases tautology and contradiction, respectively, cannot assign probability one to a contingent A. Psychologically, probability one seems appropriate for formalizing strong subjective convictions. I am, for example, happy to assign probability one to my conviction that the United States of America have currently more inhabitants than the Vatican City State, which is of course not a tautology. Reserving probability zero and one to logical truth and falsehood is an unnecessary restriction. By the way, probability one can be updated in the framework of coherence [8, see, e.g., Section 11.6]. Moreover, since both $P(\text{hypothesis}) = 0$ and $P(\text{hypothesis} \,|\, \text{evidence}) > 0$ are consistent in the coherence approach (see Section 2 below), it can also deal with the zero-prior problem, which is well-known in the philosophy of induction.

The next section combines the coherence approach to probability with logic.

1.2. Coherence based probability logic

The fundamental problem of non-probabilistic logics consists in determining if a conclusion $\mathcal{C}$ is entailed by a premise set $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n\}$. In contrast, probabilistic logics attach probabilities to the premises and the inference problem is to determine what (set of) probabilities should be attached to the conclusion [22, 24]. In coherence based probability logic the inference problem consists in determining the tightest *coherent* probability bounds on the conclusion [8, 18, 20, 42]. The coherent lower (l) and upper (u) probability bounds on a conclusion $\mathcal{C}$ are *tight* if and only if they are the best possible coherent probability bounds on $\mathcal{C}$, i. e., there is no coherent probability assessment of $\mathcal{C}$ less than l or greater than u. If the tightest coherent probability bounds on the conclusion are constrained by the premises, then the argument is *probabilistically informative.* If the conclusion probability coincides with the unit interval for all probability assessments of the premises, then the argument is *probabilistically non-informative* [41].

For an example of a probabilistically informative argument form consider the probabilistic version of the *modus ponens* in Table 2. Here, the conditional premise (2) is interpreted as a conditional probability. If the conditional premise is interpreted as the probability of the material conditional ($P(A \supset C)$), then the coherent probability of the conclusion is of course a different one, namely at least $\max\{0, x + y - 1\}$ and at most y [23, p. 203f]. Thus, the choice of the interpretation of the premise material is crucial for probability-logical analyses of commonsense arguments.

Moreover, it is important to distinguish clearly between the premise set and the conclusion. The following situation allows for at least two different interpretations:

Consider someone is thinking: If A then C. Will not-A, if not-C?

What this person has in mind can be interpreted as an instance of *modus tollens* (From *If A then C* and $\neg C$ infer $\neg A$) or as a *contraposition* (From *If A then C* infer *If* $\neg C$ *then* $\neg A$). While both arguments are valid in propositional logic, they differ substantially in probability logic [42]. *Modus tollens* is probabilistically informative:

$$\frac{\begin{array}{c} P(C|A) = x \\ P(\neg C) = y \end{array}}{\left.\begin{array}{rl} \max\left\{\dfrac{1-x-y}{1-x}, \dfrac{x+y-1}{x}\right\}, & \text{if } 0 < x < 1; \\ 1-y, & \text{if } x = 0; \\ y, & \text{if } x = 1. \end{array}\right\} \leq P(\neg A) \leq 1}$$

Contraposition, however, is probabilistically non-informative:

$$\frac{P(C|A) = x}{0 \leq P(\neg A|\neg C) \leq 1}$$

Usually, standard approaches to probability and the coherence approach yield the same lower and upper conclusion probabilities. However, there are philosophically interesting cases where the approaches diverge and standard approaches are at least incomplete. As an example, consider the following argument:

(P) C, therefore *If* A*, then* C.

If the conditional in argument (P) is formalized as a material conditional, then (P) is logically valid in the propositional calculus. However, it is easy to find natural language instances of (P) where the premise C is true but the conclusion *If* A, *then* C is false. Therefore, (P) is one of the paradoxes of the material conditional.

Intuitively—assuming only a degree of belief in C—it seems counterintuitive to infer anything about the degree of belief in the conditional *If* A, *then* C (if A and B are contingent[4] and nothing is known about the relationship between A and C). This intuition is captured by coherence-based probability logic, because (P) is probabilistically non-informative:

(P') $\quad P(C) = x$, therefore $0 \leq P(C|A) \leq 1$ is coherent.

This blocks the paradox.

In the context of standard approaches to probability, however, (P') is not probabilistically non-informative: If $P(C) = 1$, then $P(C|A) = 1$.[5] This is counterintuitive: the conclusion probability should not suddenly jump to one if $P(C) = 1$, whereas in all other premise assessments ($0 < P(C) < 1$) the tightest lower and upper probability bounds on the conclusion are zero and one, respectively. Thus, standard approaches to probability do not capture adequately (P).

The next section gives a detailed proof of (P'), explains an alternative geometric proof procedure, and illustrates how the coherence approach works.

2. Detailed coherence proof

The proof of (P') uses the second statement of Appendix A. By the theorem of coherent extensions of conditional probability the assessment $P(C) = x$ propagates to $0 \leq y' \leq P(C|A) \leq y'' \leq 1$. Firstly, I demonstrate that y' is equal to zero if $P(C) = 1$. Secondly, I show that y'' is equal to one if $P(C) = 1$. As both $P(C|A) = 0$ and $P(C|A) = 1$ are coherent, if $P(C) = x$, this completes the proof that for all probability values x: If $P(C) = x$, then $P(C|A) \in [0, 1]$ is coherent. Ω denotes the *certain event*[6] and $P(A) = P(A|\Omega)$, $\forall A$.

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[4]The requirement for contingency excludes probabilistically informative but trivial relationships between the premise and the conclusion, like the following ones: $P(C|\top) = P(C)$, $P(\bot|A) = P(\bot) = 0$, and $P(\top|A) = P(\top) = 1$ are coherent for contingent A and C.

[5]If $P(C) = 1$, then $P(C|A) = \frac{P(A \wedge C)}{P(A)} = \frac{P(A)}{P(A)}$. Thus, $P(C|A) = 1$ if $P(A) > 0$, otherwise $P(C|A)$ is undefined. Likewise, if $P(C) = 0$, then $P(C|A) = 0$ or $P(C|A)$ is undefined. In Adams' approach, $P(C|A) = 1$ is assumed by default if $P(A) = 0$ [1, p. 57]. This is incoherent since it implies $P(C|A) + P(\neg C|A) = 2$, which should equal to one.

[6]The certain event is equivalent to the disjunction of all n atoms, $\Omega =_{\text{def.}} A_1 \vee \cdots \vee A_n$.

2.1. $P(C|A)$ may be equal to zero if $P(C) = 1$

The problem: Is the following assessment on the list of conditional events $\mathscr{C} = \{C|\Omega, C|A\}$ coherent?

$$\begin{array}{ll} \text{(C1)} & P(C|\Omega) = 1 \\ \text{(C2)} & P(C|A) = 0 \end{array}$$

The set of the atoms $\mathscr{A}_0 = \{A_1, \ldots, A_4\}$ is generated by the following list of atoms A_i:

$$\begin{array}{ccccc} A_1 & \Leftrightarrow & A & \wedge & C \\ A_2 & \Leftrightarrow & A & \wedge & \neg C \\ A_3 & \Leftrightarrow & \neg A & \wedge & C \\ A_4 & \Leftrightarrow & \neg A & \wedge & \neg C \end{array}$$

x_i^α denotes the (unknown) probability value of the atom A_i. The (unconditional) probability function that assigns x_i^α to A_i is denoted by $P_\alpha(A_i) = x_i^\alpha$. The index α indicates that the probability function and the probability value are always relative to the respective system (S_α) in the sequence of the systems. The first system (S_0) is the following:

$$(S_0)\left\{\begin{array}{ll} (1) & (x_1^0 + x_3^0) = P(C|\Omega)(x_1^0 + x_2^0 + x_3^0 + x_4^0) \\ (2) & x_1^0 = P(C|A)(x_1^0 + x_2^0) \\ (3) & x_1^0 + x_2^0 + x_3^0 + x_4^0 = 1 \\ (4) & \forall i(x_i^0 \geq 0) \end{array}\right.$$

In the next steps the information given in (S_0) is transformed such that the probability values of the atoms x_i^0 are equal to zero.

$$\begin{array}{lll} (5) & x_2^0 + x_4^0 = 0 & (1), \text{(C1)} \\ (6) & x_1^0 = 0 & (2), \text{(C2)} \\ (7) & x_3^0 = 1 & (3), (4-6) \end{array}$$

$P_0(C|\Omega) = 1$ is satisfied, since $P_0(C|\Omega) = \frac{x_1^0+x_3^0}{x_1^0+x_2^0+x_3^0+x_4^0} = \frac{0+1}{0+0+1+0} = 1$. Since x_3^0 is not necessarily equal to zero, equation (1) can be deleted. The next systems is constructed as follows:[7]

$$(S_1)\left\{\begin{array}{ll} (1') & x_1^1 = P(C|A)(x_1^1 + x_2^1) \\ (2') & x_1^1 + x_2^1 = 1 \\ (3') & \forall i(x_i^1 \geq 0) \end{array}\right.$$

$$\begin{array}{lll} (4') & x_1^1 = 0 & (1'), \text{(C1)} \\ (5') & x_2^1 = 1 & (2'), (3', 4') \end{array}$$

[7]The condition "*if* $\sum_{A_r \subseteq H_i} x_r^{\alpha-1} = 0,\ \alpha \geq 1$" is not satisfied because step (7) states that $x_3^0 = 1$ (see the second statement of the characterization theorem in Appendix A).

$P_1(C|A) = 0$ is satisfied, since $P_1(C|A) = \frac{x_1^1}{x_1^1+x_2^1} = \frac{0}{0+1} = 0$. Therefore, the assessments (C1) and (C2) on $\mathscr{C}$ are coherent.

2.2. $P(C|A)$ may be equal to one if $P(C) = 1$

Is the following assessment on the list of conditional events $\mathscr{C} = \{C|\Omega, C|A\}$ coherent?

$$\begin{array}{ll} \text{(C1)} & P(C|\Omega) = 1 \\ \text{(C2)} & P(C|A) = 1 \end{array}$$

The set of the atoms $\mathscr{A}_0 = \{A_1, \ldots, A_4\}$ is generated by the following list of atoms A_i:

$$\begin{array}{cccrcr} A_1 & \Leftrightarrow & A & \wedge & C \\ A_2 & \Leftrightarrow & A & \wedge & \neg C \\ A_3 & \Leftrightarrow & \neg A & \wedge & C \\ A_4 & \Leftrightarrow & \neg A & \wedge & \neg C \end{array}$$

The system (S_0) is:

$$(S_0) \left\{ \begin{array}{ll} (1) & (x_1^0 + x_3^0) = P(C|\Omega)(x_1^0 + x_2^0 + x_3^0 + x_4^0) \\ (2) & x_1^0 = P(C|A)(x_1^0 + x_2^0) \\ (3) & x_1^0 + x_2^0 + x_3^0 + x_4^0 = 1 \\ (4) & \forall i (x_i^0 \geq 0) \end{array} \right.$$

$$\begin{array}{lll} (5) & x_2^0 + x_4^0 = 0 & (1), \text{(C1)} \\ (6) & x_1^0 + x_3^0 = 1 & (3), (4-6) \end{array}$$

A solution for the system (S_0) is $P_0(x_1^0) = P_0(x_3^0) = .5$ and $P_0(x_2^0) = P_0(x_4^0) = 0$. Then, $P_0(C|\Omega) = 1$ is satisfied, since $P_0(C|\Omega) = \frac{x_1^0+x_3^0}{x_1^0+x_2^0+x_3^0+x_4^0} = \frac{.5+.5}{.5+0+.5+0} = 1$. Moreover, $P_0(C|A) = 1$ is satisfied, since $P_0(C|A) = \frac{x_1^1}{x_1^1+x_2^1} = \frac{.5}{.5+0} = 1$. Therefore, the assessments (C1) and (C2) on $\mathscr{C}$ are coherent. In this proof equation (2) is irrelevant. This completes the coherence proof of (P').

Geometrical proof There is also an alternative (geometrical) procedure to prove (P'), which is based on coherence as well (see, e.g., [17, 19]). In a nutshell, the four atomic events A_i of sections 2.1 and 2.2 can be represented as points in a coordinate system (see Figure 1). These points represent the truth values of $C|\Omega$ and $C|A$, where "true", "false" and "void" are denoted by "1", "0" and by the corresponding conditional probability value, respectively. The atom A_1 (i.e, $A \wedge C$), for example, is located at $(1, 1)$, since

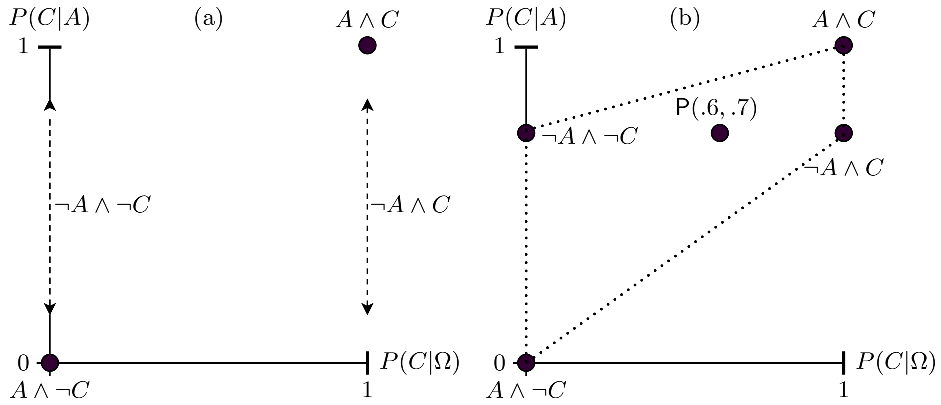

Figure 1. Geometric representation of (P'). Bullets denote the locations of the atoms A_1 and A_2, which are represented by the points $(1, 1)$ and $(0, 0)$, respectively. A_3 and A_4 are located at $(1, P(C|A))$ and $(0, P(C|A))$, respectively. Dashed lines in graph (a) indicate that the point $(1, P(C|A))$ representing A_3 may be located anywhere between $(1, 0)$ and $(1, 1)$ (including $(1, 0)$ and $(1, 1)$); similarly, the point $(0, P(C|A))$ representing A_4 may be located anywhere between $(0, 0)$ and $(0, 1)$. Graph (b) shows a concrete example: $P(C|\Omega) = .6$ and $P(C|A) = .7$ is coherent, since the point $\mathsf{P} = (.6, .7)$ belongs to the convex hull I (indicated by the dotted lines) of the points $(1, 1)$, $(0, 0)$, $(1, .7)$, $(0, .7)$.

A_1 makes both, $C|\Omega$ and $C|A$, true. Then the assessments $P(C|\Omega) = x$ and $P(C|A) = y$ are coherent if and only if $0 \leq x \leq 1, 0 \leq y \leq 1$ and the point $\mathsf{P} = (x, y)$ belongs to the convex hull I of the points $(1, 1)$, $(0, 0)$, $(1, y)$, $(0, y)$. As can be seen in Figure 1, for every $0 \leq x \leq 1$, $0 \leq y \leq 1$, the point P belongs to the convex hull. Then, the set of coherent assessments (x, y) on the pair $\{C|\Omega, C|A\}$ is the unit square. Thus, for any value of $P(C|\Omega)$, $0 \leq P(C|A) \leq 1$ is coherent.

3. Empirical evidence

In the previous sections I advocated theoretical reasons for using coherence-based probability logic as a rationality framework for uncertain conditionals. In this section I summarize selected key findings of how people reason about uncertain conditionals. The empirical evidence serves as an additional quality criterion—beyond formal strength—for the proposed approach.

The following example of a cover story illustrates the kind of tasks which were used in three paper-and-pencil experiments to investigate how people reason about (P') [38, 44]:

Katrin works in a factory that produces playing cards. She is responsible for what

is printed on the cards.
On each card, there is a **shape** (triangle, square, . . .) of a certain **color** (green, blue, . . .), like:

- green triangle, green square, green circle, . . .
- blue triangle, blue square, . . .
- red triangle, . . .

Imagine that a card got stuck in the printing machine. Katrin cannot see what is printed on this card. Since Katrin did observe the card production during the whole day, she is

A 90% certain: There is a **square** on this card.

Considering A*, how certain can Katrin be that the following sentence holds?*

If there is a **red** shape on this card, then there is a **square** on this card.

Here, line A denotes an instance of the premise of (P') and the sentence in the box denotes the respective conditional of the conclusion. Communicating an informative degree of belief in the premise could pragmatically invite the participant to respond by an informative degree of belief in the conclusion. To avoid this possible pragmatic influence a two-step response mode was installed, as follows:

Considering A, can Katrin infer—at all—how certain she can be, that the sentence in the box holds?

- ☐ NO, Katrin cannot infer her certainty, since everything between 0% and 100% is possible.
- ☐ YES, Katrin can infer her certainty.

In case you ticked YES, *please fill in*
Katrin can be certain **from** at least ___% **to** at most ___%, that the sentence in the box is true.

Variations on this task were investigated in three experiments [38, 44]. The degrees of belief in the premise were presented in terms of percentages ("60%", "70%", and "90%") or in terms of verbal descriptions ("pretty sure", "absolutely certain"). In all experiments, (P') tasks were investigated among other reasoning tasks. All participants were tested individually under careful experimental conditions.

The clear majority of participants responded by non-informative responses (see Table 3). The participants' understanding of the probabilistic non-informativeness of (P') explains—in purely semantical terms—why (P) is not endorsed. In the present context, the task item with the certain

minimum	*maximum*	*# of (P') tasks*	*study*
60%	60%	1	[38, Experiment 2, $n = 40$]
63%	81%	5	[44, Experiment 1, $n = 16$]
68%	79%	2	[44, Experiment 2, $n = 19$]

Table 3. Lowest and highest percentages of non-informative responses, number of (P') tasks, observed in three samples.

premise deserves special attention. The degree of belief in the premise was formulated verbally by "absolutely certain", which corresponds to the highest degree of belief in the premise, $P(C) = 1$. This is the interesting situation described above, where standard approaches and the coherence approach to probability diverge: whereas the former predicts $P(C|A) = 1$, the latter predicts $0 \leq P(C|A) \leq 1$. 69% of the responses in this task corroborate the coherence approach: the clear majority of the participants responded that one cannot infer an informative probability interval of the conclusion [44, Experiment 1, $n = 16$].

How are conditionals negated? The material conditional $A \supset C$ can be negated by a wide scope reading ($\neg(A \supset C)$) or by a narrow scope reading ($A \supset \neg C$). The conditional event $C|A$ is negated by the narrow scope reading ($\neg C|A$). Recent experiments on Aristotle's theses corroborate the conditional probability hypothesis and that people negate conditionals by the narrow scope reading of negating conditionals [38].

Further empirical data suggest that the inference from "A or B" to "If not A, then B" is judged to be strong if the disjunction is justified non-constructively, and weak if it is justified constructively [36]. This effect can be explained within the framework of coherence by suitable measures of "constructivity" and "closeness" [19]. As coherence requires that $0 \leq P(C|A) \leq P(\neg A \vee C)$, "strong" means here that the conditional probability is "close" to the disjunction probability.

Coherence based probability logic also received further strong empirical support by a series of experiments on other examples of the paradoxes of the material conditional: (i) inferring *If A, then C* from $\neg A$ [44] and (ii) premise strengthening: from *If A, then C* infer *If $A \wedge B$, then C* [43]. Again, the clear majority of participants responded by non-informative intervals, which is predicted by coherence based probability logic. Empirical studies on Gilio's coherence semantics of the basic nonmonotonic reasoning System P [18] provide further empirical support: The clear majority of people infer probabilistically informative and coherent responses in tasks that map the nonmonotonic argument forms of System P [40] but—as predicted—non-informative probabilities in monotonic argument forms [43]. This validates basic rationality norms for nonmonotonic reasoning which govern how conclusions should be retracted in the light of new evidence. But do people actually retract conclusions?

This was demonstrated experimentally by the suppression task paradigm [7, 48]. The data show that previously endorsed inferences can be suppressed by adding conditionals to the premise sets. Most people, for example, infer "Katrin studies late in the library" from the two premises "If Katrin has an essay to write, then Katrin studies late in the library" (K1) and "Katrin has an essay to write" (K2). If the conditional "If the library is open, then Katrin studies late in the library" (K3) is added to the premise set, modus ponens would be still applicable, but most people retract the initial conclusion. The additional premise K3 generates an enthymeme which, if made explicit, formally blocks the application of the modus ponens to K1 and K2.

Concluding remarks

In this paper, I advocated coherence based probability logic as a formally strong and empirically endorsed rationality framework for uncertain conditionals. As an illustrative example, I explained why standard approaches to probability are unable to express the probabilistic non-informativeness of (P) and I demonstrated how the coherence approach deals formally adequately with the problem. As an additional quality criterion for coherence based probability logic, which is independent of formal criteria, I surveyed recent empirical work on how people reason about uncertain conditionals. The data matches the predictions of the proposed approach.

While recent (non-probabilistic) psychological approaches use pragmatic principles to deal with the paradoxes of the material conditional [5, 28], I showed how the coherence approach allows for dealing with uncertainty and how it explains the paradoxes in purely semantical terms. Thus, the coherence approach allows for a more unified theory of conditionals. There are, however, cases where the coherence approach should be enriched by relevance and/or pragmatic principles. Consider, for example, the following disjunction introduction inference:

(DI) $P(C|A) = x$, therefore $x \leq P(C \vee B|A) \leq 1$ is coherent.

(DI) is a trivial theorem of probability theory: a disjunction (whether conditionalized on a contingent A or on Ω) is at least as probable as one of its disjuncts. However, under some natural language instances (DI) appears counterintuitive, like the following one:

> (DI') *If the letter is taken to the post office, then the letter will arrive at address C.* Therefore, *If the letter is taken to the post office, then the letter will arrive at address C or the moon is made of blue cheese.*

Here, the conclusion seems absurd whereas in countries of reliable postal services, the premise is highly probable. Future work is needed to investigate to what extent adding relevance and/or pragmatic principles on a meta-level to coherence based probability logic could block counterintuitive arguments like (DI').

Finally, as noted by Coletti & Scozzafava [8, p. 96f], the coherence approach allows for revising *plain belief* expressed by probability one, $P(C) = 1$. The proofs of argument form (P') in Section 2 show that even if an agent holds $P(C) = 1$, it is perfectly coherent to assign appropriate degrees of belief less or equal to one to the conditional event $C|A$, where "A" denotes some new evidence: then $P(C|A)$ expresses the revised degree of belief in C *given* the evidence A. As an example consider the initial belief state (B1) and the revised (or currently hypothetical) one (B2):

(B1) P(Bill Gates is a billionaire) $= 1$
(B2) P(Bill Gates is a billionaire | Bill Gates is bankrupt) $= 0$

This is not possible in standard approaches to probability where (B2) must either obtain probability one or is undefined. In the framework of coherence, however, it is perfectly coherent to revise (B1) by (B2) or to hold beliefs (B1) and (B2) simultaneously.

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A. Characterization theorem of coherence

This section reproduces the characterization theorem of coherence [8, Theorem 4, p. 81]:

> Let $\mathscr{C}$ be an *arbitrary* family of conditional events and consider, for every $n \in \mathbb{N}$, a *finite* subfamily
>
> $$\mathscr{F} = \{E_1|H_1, \ldots, E_n|H_n\} \subseteq \mathscr{C}\,;$$
>
> we denote by $\mathscr{A}$ the set of *atoms* A_r generated by the (unconditional) events $E_1, H_1, \ldots, E_n, H_n$ and by $\mathscr{G}$ the *algebra* spanned by them. For an *assessment* on $\mathscr{C}$ given by a real function P, the following three statements are equivalent:
>
> 1. P is a *coherent* conditional probability on $\mathscr{C}$;
> 2. for every $n \in \mathbb{N}$ and for every *finite* subset $\mathscr{F} \subseteq \mathscr{C}$ there exists a sequence of *compatible* systems, with unknowns $x_r^\alpha \geq 0$,
>
> $$(S_\alpha)\left\{\begin{array}{rcl} \sum_{r}\limits_{A_r \subseteq E_i \wedge H_i} x_r^\alpha & = & P(E_i|H_i) \sum_{r}\limits_{A_r \subseteq H_i} x_r^\alpha\,, \\ \left[\text{if } \sum_{r}\limits_{A_r \subseteq H_i} x_r^{\alpha-1} = 0,\ \alpha \geq 1\right] & & (i = 1, 2, \ldots, n) \\ \sum_{r}\limits_{A_r \subseteq H_0^\alpha} x_r^\alpha & = & 1 \end{array}\right.$$

with $\alpha = 0, 1, 2, \ldots, k \leq n$, where $H_0^0 = H_0 = H_1 \vee \ldots \vee H_n$ and H_0^α denotes, for $\alpha \geq 1$, the union of the H_i's such that $\sum_{\substack{r \\ A_r \subseteq H_i}} x_r^{\alpha-1} = 0$;

3. for every $n \in \mathbb{N}$ and for every *finite* subset $\mathscr{F} \subseteq \mathscr{C}$ there exists (at least) a *class* of (coherent) probabilities $\{P_0^{\mathscr{F}}, P_1^{\mathscr{F}}, \ldots, P_h^{\mathscr{F}}\}$, each probability $P_\alpha^{\mathscr{F}}$ being defined on a suitable subset $\mathscr{A}_\alpha \subseteq \mathscr{A}_0$ (with $\mathscr{A}_{\alpha'} \subseteq \mathscr{A}_{\alpha''}$ for $\alpha' > \alpha''$ and $P_{\alpha''}^{\mathscr{F}}(A_r) = 0$ if $A_r \in \mathscr{A}_{\alpha'}$) such that for every $G \in \mathscr{G}, G \neq \emptyset$, there is a unique $P_\alpha^{\mathscr{F}}$, with

$$\sum_{\substack{r \\ A_r \subseteq G}} P_\alpha^{\mathscr{F}}(A_r) \quad > \quad 0\,; \tag{I}$$

moreover, for every $E_i|H_i \in \mathscr{F}$ there exists a unique $P_\beta^{\mathscr{F}}$ satisfying (I) with $G = H_i$ and $\alpha = \beta$, and $P(E_i|H_i)$ is *represented* in the form

$$P(E_i|H_i) \quad = \quad \frac{\sum_{\substack{r \\ A_r \subseteq E_i \wedge H_i}} P_\beta^{\mathscr{F}}(A_r)}{\sum_{\substack{r \\ A_r \subseteq H_i}} P_\beta^{\mathscr{F}}(A_r)}\,. \tag{II}$$