LIPSCHITZ NORMAL EMBEDDING AMONG SUPERISOLATED SINGULARITIES

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ABSTRACT. Any germ of a complex analytic space is equipped with two natural metrics: the outer metric induced by the hermitian metric of the ambient space and the inner metric, which is the associated riemannian metric on the germ. A complex analytic germ is said Lipschitz normally embedded (LNE) if its outer and inner metrics are bilipschitz equivalent. LNE seems to be fairly rare among surface singularities; the only known LNE surface germs outside the trivial case (straight cones) are the minimal singularities. In this paper, we show that a superisolated hypersurface singularity is LNE if and only if its projectivized tangent cone has only ordinary singularities. This provides an infinite family of LNE singularities which is radically different from the class of minimal singularities.

1. INTRODUCTION

If $(X, 0)$ is a germ of a complex variety, then any embedding $\phi\colon (X, 0) \hookrightarrow (\mathbb{C}^n, 0)$ determines two metrics on $(X, 0)$: the outer metric

$$d_o(x, y) := \|\phi(x) - \phi(y)\| \qquad (\text{i.e., distance in } \mathbb{C}^n)$$

and the inner metric

$$d_i(x, y) := \inf\{\text{length}(\phi \circ \gamma) : \gamma \text{ is a rectifiable path in } X \text{ from } x \text{ to } y\},$$

using the riemannian metric on $X \smallsetminus \{0\}$ induced by the hermitian metric of $\mathbb{C}^n$. For all $x, y \in X$, $\|x - y\| \leq d_i(x, y)$.

Definition 1.1. A germ of a complex normal variety $(X, 0)$ is *Lipschitz normally embedded* (LNE) if the identity map of $(X, 0)$ is a bilipschitz homeomorphism between inner and outer metrics, i.e., there exists a neighborhood U of 0 in X and a constant $K \geq 1$ such that for all $x, y \in U$

$$d_i(x, y) \leq K d_o(x, y).$$

Lipschitz Normal Embedding (LNE) is a very active research area with many recent results, e.g., by Birbrair, Fernandes, Kerner, Mendes, Neumann, Nuño-Ballesteros, Pedersen, Pichon, Ruas, Sampaio ([2], [6], [8], [12], [16]), including a characterization of LNE for semialgebraic sets ([1]) and a characterization of LNE for complex surfaces ([15]).

If $(X, 0)$ is a curve germ then it is in fact bilipschitz equivalent to the metric cone over its link with respect to the inner metric, while the data of its Lipschitz outer geometry is equivalent to that of the embedded topology of a generic plane

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projection (see [17], [5], [13]). Therefore, an irreducible complex curve is LNE if and only if it is smooth. This is no longer true in higher dimension.

For example, if $f_d \in \mathbb{C}[x, y, z]$ is a homogeneous polynomial of degree $d \geq 2$ such that the associated projective curve $\{f_d = 0\} \subset \mathbb{P}^2$ is smooth, then it is easy to prove that the hypersurface in $\mathbb{C}^3$ with isolated singularity defined by the equation $f_d(x, y, z) = 0$, i.e. the straight cone over the projective curve $f_d = 0$, is LNE. A natural question is then to characterize all the isolated surface singularities which are LNE.

It turns out that it is difficult to find examples of complex isolated singularities with dimension ≥ 2 which are LNE but which are not conical. In [16], it is shown that such examples exist: every minimal surface singularity is LNE. This was the first known infinite class of LNE isolated surface singularities which are not conical.

In the present paper, we prove that the class of superisolated singulaties in $(\mathbb{C}^3, 0)$ contains an infinite amount of non metrically conical isolated singularities which are LNE and we describe them precisely (Theorem 1.4). This class of examples is very different from the class of minimal singularities. Indeed, in general, a minimal singularity has (by definition) a large embedding dimension. Among them, the only hypersurfaces in $(\mathbb{C}^3, 0)$ are the singularities $A_k, k \geq 1$.

Definition 1.2. A hypersurface singularity $(X, 0) \subset (\mathbb{C}^3, 0)$ is *superisolated* if it is given by an equation

$$f_d(x, y, z) + f_{d+1}(x, y, z) + f_{d+2}(x, y, z) + \ldots = 0,$$

where $d \geq 2$, for all $k \geq d$, f_k is a homogeneous polynomial of degree k and the projective curve $\{f_{d+1} = 0\} \subset \mathbb{P}^2$ does not contain any singular point of the projectivized tangent cone $C_0X = \{[x : y : z] \in \mathbb{P}^2 : f_d(x, y, z) = 0\}$. In particular, the curve C_0X is reduced.

The embedded topology of a superisolated singularity is completely determined by the combinatorial type of the projective curve $f_d = 0$, i.e. by the topology of a small tubular neighbourhood of $f_d = 0$ in $\mathbb{P}^2$. Then, when considering only embedded topology, we can assume that the equation is simply $f_d(x, y, z) + l^{d+1} = 0$ where $l(x, y, z) = ax + by + cz$ is a generic linear form.

In this paper, we consider the outer Lipschitz geometry of superisolated surface singularities with equations of the form:

$$f_d(x, y, z) + f_{d+1}(x, y, z) = 0,$$

i.e., $f_{d+k} = 0$ for all $k \geq 2$.

Definition 1.3. A reduced complex curve singularity (C, p) is *ordinary* if it consists of r smooth curve germs having r distinct tangent lines at p.

Theorem 1.4. *A superisolated singularity* $(X, 0) \subset (\mathbb{C}^3, 0)$ *with equation* $f_d + f_{d+1} = 0$ *is LNE if and only if its projectivized tangent cone* C_0X *is smooth or has only ordinary singularities.*

Outline of the proof. The "only if" direction of the theorem follows directly from [6, Theorem 2.6], stating that tangent cones of LNE sets are themselves LNE. For the other implication, which takes up the majority of the paper, we make essential use of the Test Curve Criterion [15, Theorem 3.8] which is a characterization of LNE for normal complex surface singularities. We recall its precise statement in Section 3. To give a rough idea, the test curve criterion is based on the arc criterion

for LNE of Birbrair and Mendes [1] which says that one can check if a semialgebraic germ $(X, 0)$ is LNE by testing LNE along each pair of real analytic arcs approaching 0. This arc criterion is difficult to use effectively since the amount of pairs of curves is incommensurable. The test curve criterion states that LNE of a complex surface singularity $(X, 0)$ is implied by only certain curves in X satisfying the condition. The strength of the criterion lies in the small number of curves that need to be checked. Essential to the criterion are generic projections $\ell : (X, 0) \to (\mathbb{C}^2, 0)$, and their polar curves. Under a fixed generic projection, the polar curves are mapped to a family of curve germs in $(\mathbb{C}^2, 0)$ which allow for a simultaneous resolution. Now, the test curves $\gamma \subset \mathbb{C}^2$, whose lifts under the projection ℓ have to be tested for LNE, are defined in terms of a resolution tree for the family of projected polars (details in Section 3). With this in mind, the proof is given in two main steps: first we give an explicit description of the aforementioned resolution tree for a superisolated surface singularity with ordinary projectivized tangent cone; see Proposition 4.1 in Section 4. In Section 5, we use this description to further simplify the curve testing procedure, using the notion of *inner* and *outer contact exponents* for parametrized test curves (see Lemma 5.1 and 5.2). This finally allows us to explicitly test LNE for all test curves, using concrete local parametrizations of the test curves.

Notice that superisolated singularities were already used in the context of Lipschitz geometry of singularities to provide examples of surface singularities in $(\mathbb{C}^3, 0)$ having same outer Lipschitz geometry but different embedded topological type (see [14]).

2. EXAMPLES

In view of Theorem 1.4, it becomes straightforward to explicitly construct superisolated surface singularities in $\mathbb{C}^3$ that are LNE. It suffices to consider hypersurfaces of the form $f_d + f_{d+1} = 0$, where f_d is carefully chosen to have only ordinary singularities. We illustrate this below with two infinite families of LNE superisolated singularities and one example of a superisolated singularity which fails to be LNE. We choose here f_{d+1} to be the power of a linear form, which allows us to check easily that the corresponding singularities are superisolated.

Example 2.1 ($C_0 X$ smooth curve in $\mathbb{P}^2$). Let $f_d \in \mathbb{C}[x, y, z]$ be a homogeneous polynomial of degree d such that the projective curve $\{f_d = 0\} \subset \mathbb{P}^2$ is smooth. Let $l = ax + by + cz$ be any linear form. Then the singularity $(X, 0) \subset (\mathbb{C}^3, 0)$ defined by the equation

$$f_d + l^{d+1} = 0$$

is LNE. Indeed, since the projectivized tangent cone $C_0 X = \{f_d = 0\}$ does not have any singularities, the condition of Theorem 1.4 on the singularities of $C_0 X$ is vacuously satisfied.

Remark 2.2. In fact, every hypersurface germ $(X, 0) \subset (\mathbb{C}^n, 0)$ with nonsingular projectivized tangent cone $C_0 X$ is LNE. Indeed, assume that X has equation $f(z_1, \ldots, z_n) = 0$ and set $f = f_d + f_{d+1} + \ldots$. Since $C_0 X = \{f_d = 0\}$ is nonsingular, by Taken's theorem [19], $(X, 0)$ is C^1-equivalent (hence bi-Lipschitz equivalent) to

$\{f_d = 0\}$. Since the smooth germ $\{f_d = 0\}$ is LNE, this implies that $(X, 0)$ is LNE. We thank one of the referees for this remark.

Example 2.3 ($C_0 X$ generic intersection of smooth curves)**.** Let $f_{d_1}, \ldots, f_{d_r} \in \mathbb{C}[x, y, z]$ be homogeneous polynomials of respective degrees $d_1, \ldots, d_r$ such that the r associated projective curves are smooth, pairwise transverse and without triple intersection points. Let $d = d_1 + \ldots + d_r$. Take any linear form l whose zero locus avoids the singular points of the curve $\{f_{d_1} \cdot \ldots \cdot f_{d_r} = 0\}$. Then the superisolated singularity $(X, 0)$ defined by the equation

$$f_{d_1} \cdot \ldots \cdot f_{d_r} + l^{d+1} = 0$$

is LNE, since $C_0 X = \{f_{d_1} \cdot \ldots \cdot f_{d_r} = 0\}$ has only ordinary singularities.

Example 2.4 ($C_0 X$ cuspidal curve)**.** For every generic linear form l, the hypersurface $(X, 0) \subset (\mathbb{C}^3, 0)$ defined by the equation

$$zx^2 + y^3 + l^4 = 0$$

is superisolated, but not LNE: the projectivized tangent cone $C_0 X \subset \mathbb{P}^2$ has a singularity at $[0 : 0 : 1]$ which is a cusp, hence not ordinary.

3. The Test Curve Criterion for LNE of a surface singularity

Before stating the Test Curve Criterion [15, Theorem 3.8], we need to introduce some material.

3.1. **Generic projections.** Let $\mathcal{D}$ be a $(n-2)$-plane in $\mathbb{C}^n$ and let $\ell_{\mathcal{D}} \colon \mathbb{C}^n \to \mathbb{C}^2$ be the linear projection with kernel $\mathcal{D}$. Suppose $(C, 0) \subset (\mathbb{C}^n, 0)$ is a complex curve germ. There exists an open dense subset $\Omega_C \subset \mathbf{G}(n-2, \mathbb{C}^n)$ such that for $\mathcal{D} \in \Omega_C$, $\mathcal{D}$ contains no limit of bisecant lines to the curve C (see [21]). The projection $\ell_{\mathcal{D}}$ is said to be *generic for C* if $\mathcal{D} \in \Omega_C$.

Let now $(X, 0) \subset (\mathbb{C}^n, 0)$ be a normal surface singularity. We restrict ourselves to those $\mathcal{D}$ in the Grassmannian $\mathbf{G}(n-2, \mathbb{C}^n)$ such that the restriction $\ell_{\mathcal{D}}|_{(X,0)} \colon (X, 0) \to (\mathbb{C}^2, 0)$ is finite. The *polar curve* $\Pi_{\mathcal{D}}$ of $(X, 0)$ for the direction $\mathcal{D}$ is the closure in $(X, 0)$ of the singular locus of the restriction of $\ell_{\mathcal{D}}$ to $X \smallsetminus \{0\}$. The *discriminant curve* $\Delta_{\mathcal{D}} \subset (\mathbb{C}^2, 0)$ is the image $\ell_{\mathcal{D}}(\Pi_{\mathcal{D}})$ of the polar curve $\Pi_{\mathcal{D}}$.

Denote by $\check{\mathbb{P}}^{n-1}$ the dual of the projective space $\mathbb{P}^{n-1}$, so an element of $\check{\mathbb{P}}^{n-1}$ is a hyperplane H corresponding to the kernel of a nonzero linear projection $\mathbb{C}^n \to \mathbb{C}$. Let $C(X)$ be the conormal space of X, i.e., the closure in $X \times \check{\mathbb{P}}^{n-1}$ of the set $\{(x, H) : x \in X, \ T_x X \subset H\}$. Let $\kappa \colon C(X) \to X$ be the conormal morphism, defined as the projection on the first factor, and let $\lambda \colon C(X) \to \check{\mathbb{P}}^{n-1}$ be the projection on the second factor. Each $\mathcal{D} \in \mathbf{G}(n-2, \mathbb{C}^n)$ can be identified with the pencil of hyperplanes in $\mathbb{C}^n$ which contain $\mathcal{D}$, i.e., with a line in $\check{\mathbb{P}}^{n-1}$. Using this identification, the polar curve can be expressed as $\Pi_{\mathcal{D}} = \kappa(\lambda^{-1}(\mathcal{D}))$.

Now, consider the tautological bundle $E \subset \check{\mathbb{P}}^{n-1} \times \mathbf{G}(n-2, \mathbb{C}^n)$ of lines of $\check{\mathbb{P}}^{n-1}$ and the intersection $\mathcal{P} = (X \times E) \cap (C(X) \times \mathbf{G}(n-2, \mathbb{C}^n)) \subset X \times \check{\mathbb{P}}^{n-1} \times \mathbf{G}(n-2, \mathbb{C}^n)$. Let $\nu \colon \mathcal{P} \to \mathbf{G}(n-2, \mathbb{C}^n)$ be the projection on the third factor, so that for all $\mathcal{D} \in \mathbf{G}(n-2, \mathbb{C}^n)$, $\Pi_{\mathcal{D}}$ is the image of $\nu^{-1}(\mathcal{D})$ by the projection $X \times \check{\mathbb{P}}^{n-1} \times \mathbf{G}(n-2, \mathbb{C}^n) \to X$ on the first factor.

Proposition 3.1. *An open dense subset $\Omega \subset \mathbf{G}(n-2, \mathbb{C}^n)$ exists such that:*

(i) *for each $\mathcal{D} \in \Omega$, $\lambda^{-1}(\mathcal{D}) \cap \kappa^{-1}(0)$ consists of a single point and the restriction $\nu\colon (\nu^{-1}(\Omega))_{red} \to \Omega$ is a flat analytic family of reduced curves with analytic section the intersection with $\kappa^{-1}(0)$.*

(ii) *for each $\mathcal{D} \in \Omega$, the restrictions $\kappa\colon \lambda^{-1}(\mathcal{D}) \to \Pi_\mathcal{D}$ and $\ell_\mathcal{D}\colon \Pi_\mathcal{D} \to \Delta_\mathcal{D}$ are finite birational morphisms and the second morphism is a generic projection of the polar curve $\Pi_\mathcal{D}$.*

(iii) *the family $(\Delta_\mathcal{D})_{\mathcal{D} \in \Omega}$ is equisingular in terms of strong simultaneous resolution.*

Condition (i) follows from Kleiman's transversality theorem [11]. The first part of (ii) follows from the proof of Théorème 2.2.4 in [9] and the fact that $\ell_\mathcal{D}\colon \Pi_\mathcal{D} \to \Delta_\mathcal{D}$ is a generic projection of the curve $\Pi_\mathcal{D}$ is [21, Lemme-clé V 1.2.2 (ii)]. Condition (iii) is [20, Section 4, Proposition 2].

Throughout the rest of the paper, we fix an open dense subset $\Omega \subset \mathbf{G}(n-2, \mathbb{C}^n)$ satisfying the three conditions (i), (ii) and (iii) stated above in Proposition 3.1.

Definition 3.2. The projection $\ell_\mathcal{D}\colon \mathbb{C}^n \to \mathbb{C}^2$ is *generic* for $(X, 0)$ if $\mathcal{D} \in \Omega$.

3.2. **Test curves.** Let $\ell\colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection, let Π be its polar curve and let $\Delta = \ell(\Pi)$ be its discriminant curve. Denote by $\rho'_\ell\colon Y_\ell \to \mathbb{C}^2$ the minimal composition of blow-ups of points starting with the blow-up of the origin which resolves the base points of the family of projections of generic polar curves $(\ell(\Pi_\mathcal{D}))_{\mathcal{D} \in \Omega}$.

Definition 3.3. A Δ-*curve* is an irreducible component of the exceptional curve $(\rho'_\ell)^{-1}(0)$ intersecting the strict transform of Δ.

Let us blow-up all the intersection points between two Δ-curves. We denote by $\sigma\colon Z_\ell \to Y_\ell$ and $\rho_\ell = \rho'_\ell \circ \sigma\colon Z_\ell \to \mathbb{C}^2$ the resulting morphisms (if no Δ-curves intersect, $\rho_\ell = \rho'_\ell$). The resolution graph of ρ_ℓ does not depend on ℓ. We denote it by T for the rest of the paper.

Definition 3.4. A Δ-*node* of T is a vertex (j) of T which represents a Δ-curve.

Let $E \subset Y$ be a complex curve in a complex surface Y and let $E_1, \ldots, E_n$ be the irreducible components of E. We say *curvette* of E_i for any smooth curve germ (β, p) in Y, where p is a point of E_i which is a smooth point of Y and E and such that β and E_i intersect transversely.

If G is a graph, we will denote by $V(G)$ its set of vertices.

Definition 3.5. Let T' be the subtree of T defined as the union of all the simple paths in T joining the root vertex to Δ-nodes (so the complement $T \smallsetminus T'$ consists of strings of valency 2 vertices ended by a valency 1 vertex).

For $(i) \in V(T)$, let C_i be the irreducible component of $\rho_\ell^{-1}(0)$ represented by (i), so we have $\rho_\ell^{-1}(0) = \bigcup_{(i) \in V(T)} C_i$. Let $(i) \in V(T)$. We call *test curve at (i)* (of ℓ) any complex curve germ $(\gamma, 0) \subset (\mathbb{C}^2, 0)$ such that

(1) $(i) \in V(T')$;
(2) the strict transform γ^* by ρ_ℓ is a curvette of C_i intersecting C_i at a smooth point of $\rho_\ell^{-1}(0)$;
(3) $\gamma^* \cap \Delta^* = \emptyset$.

3.3. **Principal components.** Let us first recall the definition of Nash modification.

Let $\lambda \colon X \smallsetminus \{0\} \to \mathbf{G}(2, \mathbb{C}^n)$ be the map which sends $x \in X \smallsetminus \{0\}$ to the tangent plane $T_x X$. The closure NX of the graph of λ in $X \times \mathbf{G}(2, \mathbb{C}^n)$ is a reduced analytic surface. By definition, the *Nash modification* of $(X, 0)$ is the induced morphism $\mathscr{N} \colon NX \to X$. A morphism $f \colon Y \to X$ factors through Nash modification if and only if it has no base points for the family of polar curves ([7, Section 2], [18, Part III, Theorem 1.2]).

Definition 3.6. Let $\pi_0 \colon X_0 \to X$ be the minimal good resolution of X which factors through both the Nash modification $\mathscr{N} \colon NX \to X$ of $(X, 0)$ and the blow-up of the maximal ideal of $(X, 0)$ and let G_0 be its resolution graph. For each vertex (v) of G_0 we denote by E_v the corresponding irreducible component of $\pi_0^{-1}(0)$. A vertex (v) of G_0 such that E_v is an irreducible component of the blow-up of the maximal ideal (resp. an exceptional curve of the Nash transform) is called an $\mathcal{L}$-node (resp. a $\mathcal{P}$-node) of G_0.

Definition 3.7. Consider the graph G_0' of G_0 defined as the union of all simple paths in G_0 connecting pairs of vertices among $\mathcal{L}$- and $\mathcal{P}$-nodes. Let $\ell \colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection. Let γ be a test curve for ℓ. A component $\widehat{\gamma}$ of $\ell^{-1}(\gamma)$ is called *principal* if its strict transform by π_0 either is a curvette of a component E_v with $v \in V(G_0')$ or intersects $\pi_0^{-1}(0)$ at an intersection between two exceptional curves E_v and $E_{v'}$ such that both v and v' are in $V(G_0')$.

3.4. **Inner and outer contact exponents.** We now define the outer and inner contacts between two complex curves on a complex surface germ. We use the "big-Theta" asymptotic notation of Bachmann-Landau: given two function germs $f, g \colon ([0, \infty), 0) \to ([0, \infty), 0)$ we say f is *big-Theta* of g and we write $f(t) = \Theta(g(t))$ if there exist real numbers $\eta > 0$ and $K \geq 1$ such that for all t with $f(t) \leq \eta$, $\frac{1}{K} g(t) \leq f(t) \leq K g(t)$.

Let $\mathbb{S}_\epsilon^{2n-1} = \{x \in \mathbb{C}^n \colon \|x\|_{\mathbb{C}^n} = \epsilon\}$. Let $(\gamma_1, 0)$ and $(\gamma_2, 0)$ be two germs of complex curves inside $(X, 0)$. Let $q_{out} = q_{out}(\gamma_1, \gamma_2)$ and $q_{inn} = q_{inn}(\gamma_1, \gamma_2)$ be the two rational numbers ≥ 1 defined by

$$d_o(\gamma_1 \cap \mathbb{S}_\epsilon^{2n-1}, \gamma_2 \cap \mathbb{S}_\epsilon^{2n-1}) = \Theta(\epsilon^{q_{out}}),$$

$$d_i(\gamma_1 \cap \mathbb{S}_\epsilon^{2n-1}, \gamma_2 \cap \mathbb{S}_\epsilon^{2n-1}) = \Theta(\epsilon^{q_{inn}}),$$

where d_i means inner distance in $(X, 0)$ as before.

Definition 3.8. We call $q_{out}(\gamma_1, \gamma_2)$ (resp. $q_{inn}(\gamma_1, \gamma_2)$) the *outer contact exponent* (resp. *the inner contact exponent*) between γ_1 and γ_2.

We are now ready to state the Test Curve Criterion [15, Theorem 3.8]:

Theorem 3.9 (Test curve criterion for LNE of a complex surface)**.** *A normal surface germ $(X, 0)$ is LNE if and only if the following conditions are satisfied for all generic projections $\ell \colon (X, 0) \to (\mathbb{C}^2, 0)$ and all test curves $(\gamma, 0) \subset (\mathbb{C}^2, 0)$:*

- (1*) *for every principal component $\widehat{\gamma}$ of $\ell^{-1}(\gamma)$, $\mathrm{mult}(\widehat{\gamma}) = \mathrm{mult}(\gamma)$ where mult means multiplicity at 0;*
- (2*) *for every pair (γ_1, γ_2) of distinct principal components of $\ell^{-1}(\gamma)$, $q_{inn}(\gamma_1, \gamma_2) = q_{out}(\gamma_1, \gamma_2)$.*

4. The trees T and G_0 of a superisolated singularity with ordinary projectivized tangent cone

In this section, we prove the following proposition which will enable one to describe the graphs T and G_0 for a superisolated singularity whose projectivized tangent cone has only ordinary singularities.

Proposition 4.1. *Let $(X,0) \subset (\mathbb{C}^3, 0)$ be a superisolated singularity and let $e \colon X^* \to X$ be the blow-up of the maximal ideal of X, so the projectivized tangent cone C_0X is the exceptional curve $e^{-1}(0)$. Let $p_i, i = 1, \ldots, r$ be the singular points of C_0X.*

Let $\mathcal{D} \in \Omega$, i.e., $\ell_{\mathcal{D}} \colon (X,0) \to (\mathbb{C}^2, 0)$ is a generic projection, and let us decompose the polar curve $\Pi_{\mathcal{D}}$ as the union

$$\Pi_{\mathcal{D}} = \Pi_{\mathcal{D},0} \cup \Pi_{\mathcal{D},1} \cup \ldots \cup \Pi_{\mathcal{D},r},$$

where the strict transform $\Pi_{\mathcal{D},0}^*$ of $\Pi_{\mathcal{D},0}$ by e intersects C_0X at smooth points and where for each $i = 1, \ldots, r$, $\Pi_{\mathcal{D},i}^*$ consists of the components of $\Pi_{\mathcal{D}}^*$ passing through p_i. So in particular, for every $i \neq j$, $\Pi_{\mathcal{D},i}$ and $\Pi_{\mathcal{D},j}$ are transversal.
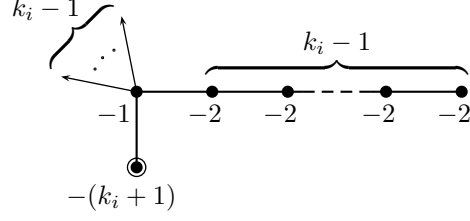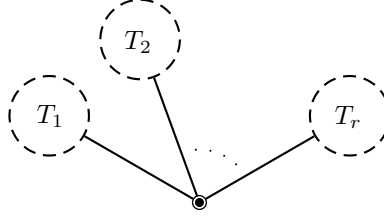
(1) *For each $i = 1, \ldots, r$, $\Pi_{\mathcal{D},i} \neq \emptyset$ and the basepoints of the family of generic polar curves $\left(\Pi_{\mathcal{D}'}\right)_{\mathcal{D}' \in \Omega}$ are exactly the points p_i, $i = 1, \ldots, r$.*

(2) *Assume that the projectivized tangent cone of $(X,0)$ has only ordinary singularities, i.e., for each $i = 1, \ldots, r$, the germ (C_0X, p_i) consists of k_i smooth transversal curves. Then*

    (a) *The germ $(\Pi_{\mathcal{D},i}, 0)$ has $k_i - 1$ components, its strict transform by e consists of smooth transversal curves passing through p_i and the basepoint p_i is resolved by just one blow-up.*

    (b) *Let $\ell \colon (X,0) \to (\mathbb{C}^2, 0)$ be a generic projection and let $\mathcal{D} \in \Omega$. For each $i = 1, \ldots, r$, the $k_i - 1$ components of the curve $\ell(\Pi_{\mathcal{D},i})$ are all $(k_i + 1, k_i)$-cusps and they have pairwise contact $\frac{k_i+1}{k_i}$. Moreover, for every generic pair $\mathcal{D}, \mathcal{D}' \in \Omega$, any two components of $\ell(\Pi_{\mathcal{D},i}) \cup \ell(\Pi_{\mathcal{D}',i})$ have contact $\frac{k_i+1}{k_i}$.*

As a consequence of Proposition 4.1, we obtain the following description of the resolution tree T and of the resolution graph G_0 introduced in Section 3 for all superisolated singularities whose projectivized tangent cones have only ordinary singularities.
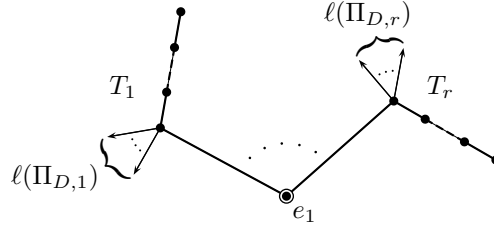
Denote by T_i the resolution tree of the $(k_i, k_i + 1)$-cusp decorated with $k_i - 1$ arrows corresponding to the components of $\ell(\Pi_{\mathcal{D},i})$ (see Figure 1). The weight decorating each vertex is the self-intersection of the corresponding exceptional curve. The root vertex, i.e., the vertex corresponding to the first blow-up of $0_{\mathbb{C}^2}$ is the circled one. The negative number weighting each vertex is the self-intersection of the corresponding exceptional curve.

By Proposition 4.1, the tree $\mathcal{T}$ of the minimal sequence ρ'_{ℓ} of blow-ups which resolves the family of projected generic polar curves $\ell(\Pi_{\mathcal{D}})$ is a bouquet of r trees T_i, $i = 1, \ldots, r$ attached by their root vertex (Figure 2). Moreover, any Δ-node of $\mathcal{T}$ which is not the root vertex is the vertex of a T_i with valency $k_i + 1$ and each such vertex is a Δ-node.

By definition, T is obtained from $\mathcal{T}$ by blowing-up each edge joining two Δ-nodes, creating separating nodes. In our situation, there are two cases.

FIGURE 1. The tree T_i.

FIGURE 2. The tree $\mathcal{T}$ is the bouquet of the r trees $T_1, \ldots, T_r$, obtained by identifying their root vertices.

Case 1. Either $C_0 X$ is a line arrangement, i.e., each of its components has degree 1, then the root vertex is not a Δ-node and there are no adjacent Δ-nodes in $\mathcal{T}$. We then have $\rho_\ell = \rho'_\ell$, $\Pi_{\mathcal{D},0} = \emptyset$ and $T = \mathcal{T}$. We obtain the tree T of Figure 3.

FIGURE 3. The tree T in Case 1. The multiplicity at the root vertex is $e_1 = -1 - \sum_{i=1}^{r} k_i$.

Case 2. Otherwise, i.e., when at least one of the components of $C_0 X$ has degree ≥ 2, then the root vertex of $\mathcal{T}$ is a Δ-node and every adjacent edge has the Δ-node of a T_i at the other extremity. The tree T is then obtained from $\mathcal{T}$ by blowing-up each edge adjacent to the root vertex, creating r new vertices. Let us denote T'_i the tree resulting from T_i by this blowing-up. The tree T is then a bouquet of trees T'_i attached by their root vertex. We obtain the tree T of Figure 4.

The exceptional divisor of $\pi_0^{-1}(0)$ is obtained by blowing-up each singular point of $C_0 X$ and the created $\mathbb{P}^1$ are the $\mathcal{P}$-curves of $\pi_0^{-1}(0)$, i.e., the curves corresponding to the components of the Nash transform of $(X, 0)$ (or equivalently, those intersecting the strict transform of the polar curve Π of a generic projection). This describes G_0.

Remark 4.2. In the case of a superisolated singularity, we have $G'_0 = G_0$ since every vertex of G_0 is either an $\mathcal{L}$-node or a $\mathcal{P}$-node.
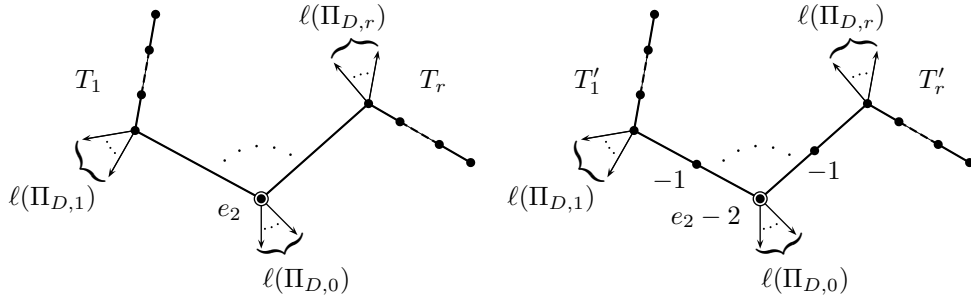
FIGURE 4. The trees $\mathcal{T}$ (left) and T (right) in Case 2. Here, $e_2 = e_1 - r$.

Example 4.3. Consider the superisolated singularity $(X, 0)$ with equation

$$xy(x + y)(x - y) + z^5 = 0.$$

Its projectivized tangent cone $C_0 X = \{xy(x + y)(x - y) = 0\} \subset \mathbb{P}^2$ consists of four lines which intersect transversely at the point $[0 : 0 : 1]$. In particular, the unique singularity of $C_0 X$ is ordinary. The circled vertices in G_0 represent components of $C_0 X$. The arrows in T represent the components of the discriminant curve Δ and arrows in G_0 represent the components of the polar curve Π. We do not write all intersections numbers (Euler classes) since we do not use them in the paper, but they can be easily computed.
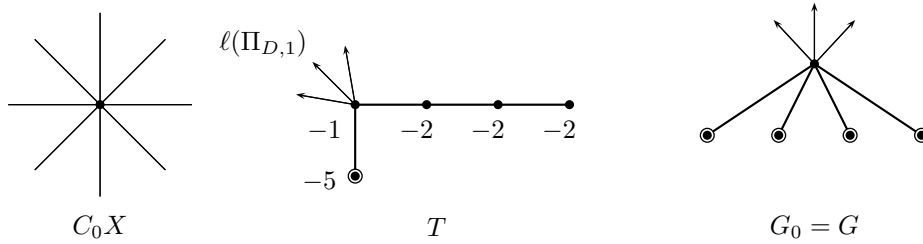

FIGURE 5. Example 4.3: Lines intersecting at a single point.

Example 4.4. Let $(X, 0)$ be given by the equation

$$xy(x^2 + y^2 + z^2) + z^5 = 0.$$

Here, the root vertex of T and all adjacent vertices are Δ-nodes, so we have to blow-up the intersection points between the corresponding Δ-curves, creating four vertices which are weighted by the Euler class -1 on the tree below.

Example 4.5. Consider the singularity $(X, 0) \subset (\mathbb{C}^3, 0)$, for which $C_0 X$ consists of two transverse ovals:

$$(x^2 + 2y^2)(2x^2 + y^2) + z^5 = 0.$$

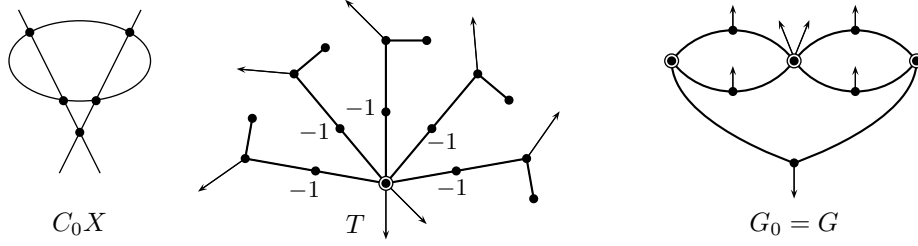The graphs T and G_0 are represented on Figure 7.

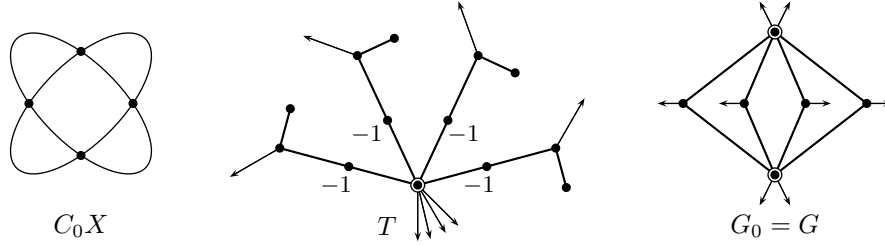FIGURE 6. Example 4.4: A pair of lines intersecting an oval.

FIGURE 7. Example 4.5: Two ovals with transversal intersections.

Example 4.6. Consider the singularity $(X, 0) \subset (\mathbb{C}^3, 0)$, for which $C_0 X$ consists of four pairwise transversal lines:

$$xy(x + y + z)(x - y - z) + z^5 = 0.$$

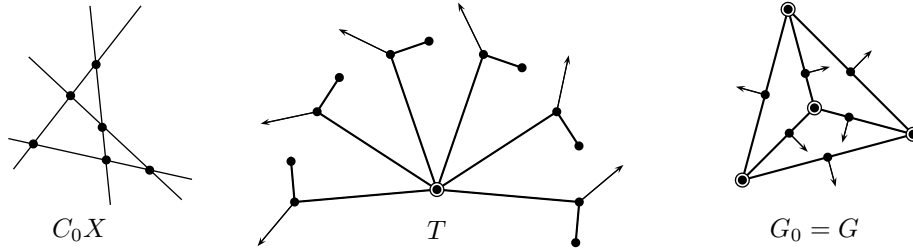The graphs T and G_0 are represented on Figure 8.

FIGURE 8. Example 4.6: Generic line arrangement.

Proof of Proposition 4.1. Assume that $(X, 0)$ has equation $f_d + f_{d+1} = 0$. We set $f = f_d$ and $g = -f_{d+1}$. Let $\widetilde{e} \colon W \to \mathbb{C}^3$ be the blow-up of the origin of $\mathbb{C}^3$, so X^* is the strict transform of X by $\widetilde{e}$ and $e \colon X^* \to X$ is the restriction of $\widetilde{e}$ to X^*.

We will work in one of the three standard affine coordinate charts of W. Let us fix notations. By definition, W is the closure in $\mathbb{C}^3 \times \mathbb{P}^2$ of the set

$$\{((x, y, z), L) \in (\mathbb{C}^3 \smallsetminus \{0\}) \times \mathbb{P}^2 \colon [x : y : z] = L\},$$

and $\widetilde{e}$ is the restriction to W of the projection $\mathbb{C}^3 \times \mathbb{P}^2 \to \mathbb{C}^3$ on the first factor. Consider the affine coordinate chart of $\mathbb{P}^2$ defined by $U_1 = \{[1 : v : w] \in \mathbb{P}^2 \colon (v, w) \in \mathbb{C}^2\}$

and set $W_1 = W \cap (\mathbb{C}^3 \times U_1)$. We can choose affine coordinates (x, v, w) in W_1 so that in this chart, $\widetilde{e}(x, v, w) = (x, y, z) = (x, xv, xw)$ and $\widetilde{e}^{-1}(0) \cong \mathbb{P}^2$ has equation $x = 0$.

Let p be a point of $C_0 X = e^{-1}(0)$ such that the strict transform $\Pi_{\mathcal{D}}^*$ passes through p. Without loss of generality, we can choose our coordinates in $\mathbb{C}^3$ so that $p = (0, 0, 0)$ in the affine coordinates of W_1 (i.e., p corresponds to the x-axis in the tangent cone of X). Then, in this chart, X^* has equation

$$f(1, v, w) - xg(1, v, w) = 0,$$

and $e^{-1}(0) = C_0 X$ has equations

$$f(1, v, w) = 0 \quad \text{and} \quad x = 0.$$

Let us identify $\mathbf{G}(1, \mathbb{C}^3)$ with $\mathbb{P}^2$. We can also assume that $\mathcal{D} = [0 : 0 : 1]$, i.e., $\ell_{\mathcal{D}} = (x, y)$. Then $\Pi_{\mathcal{D}} = X \cap \{f_z - g_z = 0\}$, so the strict transform $\Pi_{\mathcal{D}}^*$ of $\Pi_{\mathcal{D}}$ by e has equations:

$$f_z(1, v, w) - xg_z(1, v, w) = 0 \quad \text{and} \quad f(1, v, w) - xg(1, v, w) = 0.$$

Since $\Pi_{\mathcal{D}}^*$ passes through p, we have $f_z(1, 0, 0) = 0$.

Set $\widetilde{f}(v, w) = f(1, v, w)$, so $e^{-1}(0)$ has equation $\widetilde{f}(v, w) = 0$. Notice that $f_y(1, v, w) = \widetilde{f}_v(v, w)$ and $f_z(1, v, w) = \widetilde{f}_w(v, w)$.

Assume first that p is a smooth point of $e^{-1}(0)$, i.e., one of $\widetilde{f}_v(0, 0)$ or $\widetilde{f}_w(0, 0)$ is nonzero. Since $\widetilde{f}_w(0, 0) = f_z(1, 0, 0) = 0$, we have $\widetilde{f}_v(0, 0) = f_y(1, 0, 0) \neq 0$. Let us prove that p is not a basepoint of the family of generic polars. Since $\mathcal{D} = [0 : 0 : 1] \in \Omega$, for all pairs of sufficiently small complex numbers (α, β), we have $\mathcal{D}_{(\alpha, \beta)} = [\alpha : \beta : 1] \in \Omega$. The polar curve $\Pi_{\mathcal{D}_{(\alpha, \beta)}}$ has equation $\alpha(f_x - g_x) + \beta(f_y - g_y) + (f_z - g_z) = 0$, so its strict transform by $\widetilde{e}$ has equations

$$\alpha f_x(1, v, w) + \beta f_y(1, v, w) + f_z(1, v, w)$$

$$-x\big[\alpha g_x(1, v, w) + \beta g_y(1, v, w) + g_z(1, v, w)\big] = 0$$

and

$$f(1, v, w) - xg(1, v, w) = 0.$$

Hence a point $(0, v, w) \in e^{-1}(0)$ lies on the strict transform $\Pi_{\mathcal{D}_{(\alpha, \beta)}}^*$ if and only if

$$\alpha f_x(1, v, w) + \beta f_y(1, v, w) + f_z(1, v, w) = 0.$$

At $p = (x, v, w) = (0, 0, 0)$, we have $\alpha f_x(1, 0, 0) + \beta f_y(1, 0, 0) + f_z(1, 0, 0) \neq 0$ for generic values of the pair (α, β) since $f_y(1, 0, 0) \neq 0$. We conclude that for generic values of (α, β), the strict transform $\Pi_{\mathcal{D}_{(\alpha, \beta)}}^*$ does not pass through p. This proves that p is not a basepoint of the family of generic polars $(\Pi_{\mathcal{D}'})_{\mathcal{D}' \in \Omega}$.

Assume now that p is a singular point of $e^{-1}(0)$, so $p = p_i$ for some $i = 1, \ldots, r$. We then have $f_x(1, 0, 0) = f_y(1, 0, 0) = f_z(1, 0, 0) = 0$ and the previous argument shows that for all (α, β), $\Pi_{\mathcal{D}_{(\alpha, \beta)}}^*$ passes through p. Therefore p is a basepoint of the family of generic polars $(\Pi_{\mathcal{D}'})_{\mathcal{D}' \in \Omega}$. This completes the proof of (1).

Let us now prove (2). For a moment, we just assume that $(X, 0)$ is a superisolated singularity and we prove a lemma. We use again the above notations. Let p be a singular point of $e^{-1}(0)$, say $p = p_1$. We restrict ourselves to pairs $(\alpha, \beta) = (0, t)$ for $t \in \mathbb{C}$ and we set $\mathcal{D}_t = \mathcal{D}_{(0, t)}$. Since $\mathcal{D}_0 = \mathcal{D} \in \Omega$, then for t in a small neighbourhood B of 0 in $\mathbb{C}$, we still have $\mathcal{D}_t \in \Omega$, i.e. the projection $\ell_t \colon (X, 0) \to (\mathbb{C}^2, 0)$ defined by $\ell_t(x, y, z) = (x, y - tz)$ is generic for $(X, 0)$.

Recall that $\Pi_{\mathcal{D}_t,1}$ is the part of the polar curve Π_t whose strict transform by $\widetilde{e}$ passes through $p = p_1$, i.e., the germ $(\Pi_{\mathcal{D}_t,1}, 0)$ is the $\widetilde{e}$-image of the germ $(\Pi^*_{\mathcal{D}_t}, p)$. We will describe the family $(\Pi_{\mathcal{D}_t,1})_{t \in B}$.

We use again the previous coordinates chart and notations. Let $e_0 \colon Y \to \mathbb{C}^2$ be the blow-up of the origin of $\mathbb{C}^2_{(x,y)}$. We consider e_0 in the chart $(x,v) \mapsto (x,y) = (x, xv)$, we set $q = (1,0) \in Y$ in this chart and we denote by $\tilde{\ell}_t \colon (X^*, p) \to (Y, q)$ the projection $(x, v, w) \mapsto (x, v - tw)$. So we have the commutative diagram:

$$
\begin{array}{ccc}
(X^*, p) & \xrightarrow{\ e\ } & (X, 0) \\
\Big\downarrow{\tilde{\ell}_t} & & \Big\downarrow{\ell_t} \\
(Y, q) & \xrightarrow{\ e_0\ } & (\mathbb{C}^2, 0).
\end{array}
$$

The series $g(1, v, w) \in \mathbb{C}\{v, w\}$ is a unit at p since $\{g = 0\} \cap Sing(f = 0) = \emptyset$ in $\mathbb{P}^2$ and by change of the local coordinates, we can assume $g(1, v, w) = 1$. Then the equations for $(\Pi^*_{\mathcal{D}_t}, p)$ can be written:

$$
t\widetilde{f}_v(v, w) + \widetilde{f}_w(v, w) = 0 \quad \text{and} \quad \widetilde{f}(v, w) - x = 0.
$$

The first equation is nothing but the polar curve Γ_t of the morphism $(v, w) \mapsto (\widetilde{f}(v, w), v - tw)$. Consider the isomorphism $\mathrm{proj}_t \colon (X^*, p) \to (\mathbb{C}^2, 0)$ which is the restriction of the linear projection $(x, v, w) \mapsto (v - tw, w)$. We have proved:

Lemma 4.7. $(\Pi^*_{\mathcal{D}_t}, p)$ *is the inverse image by* proj_t *of the polar curve* Γ_t *of the morphism* $\phi_t \colon (\mathbb{C}^2_{(v,w)}, 0) \to (\mathbb{C}^2_{(x,v)}, 0)$ *defined by* $(v, w) \mapsto (\widetilde{f}(v, w), v - tw)$, *i.e., the relative polar curve of the map germ* $\widetilde{f}$ *for the generic projection* $(v, w) \mapsto v - tw$.

Set $q = (1, 0)$ in $\mathbb{C}^2_{(x,v)}$. The situation is summarized in the commutative diagram:

$$
\begin{array}{ccccc}
(\mathbb{C}^2, \Gamma_t, 0) & \xleftarrow{\ \mathrm{proj}_t\ } & (X^*, \Pi^*_{\mathcal{D}_t}, p) & \xrightarrow{\ e\ } & (X, \Pi_{\mathcal{D}_t}, 0) \\
& \searrow{\phi_t} & \Big\downarrow{\tilde{\ell}_t} & & \Big\downarrow{\ell_t} \\
& & (Y, \phi_t(\Gamma_t), q) & \xrightarrow{\ e_0\ } & (\mathbb{C}^2, \Delta_{\mathcal{D}_t}, 0)
\end{array}
$$

Assume now that the projectivized tangent cone of the superisolated singularity $(X, 0)$ has only ordinary singularities. Then the germ $(e^{-1}(0), p)$ consists of k smooth branches having k distinct tangent lines at p. Maybe after change of coordinates, we can assume that none of the tangent lines of $(e^{-1}(0), p)$ is tangent to $v = 0$. Hence $\widetilde{f}(v, w) = \prod_{i=1}^{k}(w + a_i v + h.o.)$, where $h.o.$ means higher order terms and where for all i, $a_i \in \mathbb{C}^*$ and $a_i \neq a_j$ for all $i \neq j$.

Assertion (2a) of Proposition 4.1 is now a direct consequence of Lemma 4.7 and of the following Claim.

Claim 1. The polar curve Γ_t consists of $k - 1$ smooth curves which are pairwise transversal. Moreover, the base point 0 of the family $(\Gamma_t)_{t \in B}$ is resolved by just blowing-up once the origin.

Proof of Claim 1. The curve Γ_t has equation $t\widetilde{f}_v(v, w) + \widetilde{f}_w(v, w) = 0$, so $(\Gamma_t)_{t \in \mathbb{C}}$ is a linear pencil of curves which are the fibers of the meromorphic function $h = \frac{\widetilde{f}_v}{\widetilde{f}_w}$.

We have

$$\widetilde{f}_v(v,w) = \sum_{i=1}^{k} a_i \prod_{j \neq i}(w + a_j v) \;\; + \;\; \widetilde{\alpha}(v,w)$$

and

$$\widetilde{f}_w(v,w) = \sum_{i=1}^{k} \prod_{j \neq i}(w + a_j v) \;\; + \;\; \widetilde{\beta}(v,w),$$

where $\widetilde{\alpha}$ and $\widetilde{\beta}$ are sums of monomials of degree $\geq k$.

Let $e' \colon Y' \to \mathbb{C}^2_{(v,w)}$ be the blow-up of the origin of $\mathbb{C}^2_{(v,w)}$ and let $C = e'^{-1}(0)$ be its exceptional curve. In order to prove that $(\Gamma_t)_{t \in \mathbb{C}}$ is resolved by e', we have to prove that $h \circ e'$ is well defined along C, i.e., that the strict transforms of $\widetilde{f}_v$ and $\widetilde{f}_w$ intersect C at distinct points (see [10, Sections 2,3]). Consider the two polynomials $\widehat{\alpha}(v,w_1) = \frac{1}{v^k}\widetilde{\alpha}(v,vw_1)$ and $\widehat{\beta}(v,w_1) = \frac{1}{v^k}\widetilde{\beta}(v,vw_1)$. In the chart $e' \colon (v,w_1) \mapsto (v,vw_1)$, the strict transforms of $\widetilde{f}_v = 0$ and $\widetilde{f}_w = 0$ have equations respectively $P(w_1) + v\widehat{\alpha}(v,w_1) = 0$ and $Q(w_1) + v\widehat{\beta}(v,w_1) = 0$, where

$$P(w_1) = \sum_{i=1}^{k} a_i \prod_{j \neq i}(w_1 + a_j) \;\; \text{and} \;\; Q(w_1) = \sum_{i=1}^{k}\prod_{j \neq i}(w_1 + a_j),$$

so their intersections with C are the points $(v,w_1) = (0,\lambda)$ where the λ's are the roots of P (resp. Q).

The end of the proof of Claim 1 uses the following:

Claim 2. The resultant of the polynomials P and Q has the form

$$\mathrm{Res}(P(X),Q(X)) = \eta \prod_{i \neq j}(a_i - a_j)^2,$$

where $\eta \in \mathbb{C} \smallsetminus \{0\}$.

Since the a_i's are pairwise distinct, P and Q do not have a common root. This proves that the strict transforms of $\widetilde{f}_v$ and $\widetilde{f}_w$ intersect C at distinct points. Since e' is a resolution of the meromorphic function h, by [10, Affirmation 1, p. 361], e' is a resolution of each generic Γ_t. Therefore Γ_t consists of $k-1$ smooth curves which are pairwise transversal. This proves Claim 1. $\qquad\square$

Proof of Claim 2. $R := \mathrm{Res}(P(X),Q(X))$ is a polynomial in the variables $a_1, \ldots, a_k$ which is symmetric since P and Q are themselves symmetric polynomials in these variables. For $a_i - a_j = 0$, the monomial $w_1 + a_i$ divides both $P(w_1)$ and $Q(w_1)$, so $-a_i$ is a double common root of $P(w_1)$ and $Q(w_1)$, hence $R = 0$. This implies that $a_i - a_j$ divides the polynomial R. Since R is symmetric, actually, $(a_i - a_j)^2$ divides R. Therefore, the polynomial $S = \prod_{i \neq j}(a_i - a_j)^2$ has to divide R. Since S has degree $k(k-1)$, we just have to prove that R has degree at most $k(k-1)$. We have

$$P = p_1 w_1^{k-1} + p_2 w_1^{k-2} + \ldots + p_{k-1}w_1 + p_k$$

and

$$Q = w_1^{k-1} + q_1 w_1^{k-2} + \ldots + q_{k-2}w_1 + q_{k-1},$$

where for each $i \in \{1 \ldots, k-1\}$, p_i and q_i are homogeneous symmetric polynomials of degree i and where $p_k = ka_1a_2 \ldots a_k$ is of degree k. The resultant R is the determinant of the following $(k-1) \times (k-1)$ matrix:

$$
\begin{pmatrix}
p_1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
p_2 & p_1 & \ddots & \vdots & q_1 & 1 & \ddots & 0 \\
\vdots & p_2 & \ddots & 0 & \vdots & q_1 & \ddots & 0 \\
p_{k-1} & \vdots & \ddots & p_1 & q_{k-2} & \vdots & \ddots & 1 \\
p_k & p_{k-1} & & p_2 & q_{k-1} & q_{k-2} & & q_1 \\
0 & p_k & \ddots & \vdots & 0 & q_{k-1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & p_{k-1} & \vdots & \ddots & \ddots & q_{k-2} \\
0 & \cdots & 0 & p_k & 0 & \cdots & 0 & q_{k-1}
\end{pmatrix}
$$

Writing this matrix as (a_{ij}), its determinant R equals

$$
R = \sum_{\sigma \in \mathcal{S}_{2k-2}} \text{sign}(\sigma) \prod_{i=1}^{2k-2} a_{i\sigma(i)},
$$

where $\mathcal{S}_{2k-2}$ is the group of permutations on a set of $2k-2$ elements. When σ is the identity permutation, the corresponding summand is the homogenous polynomial $(p_1)^{k-1}(q_{k-1})^{k-1}$, whose degree is $k(k-1)$. Now, it is easy to see from the matrix above that performing a transposition of two indices i and j either leads to a zero term or to a homogeneous monomial of the same degree $k(k-1)$. Therefore, R is a sum of homogeneous monomials of degree $k(k-1)$. This completes the proof of Claim 2. □

Let us now prove (2b). Since the family of plane curves $(\ell_{\mathcal{D}}(\Pi_{\mathcal{D}'}))_{\mathcal{D} \times \mathcal{D}'}$ is equi-singular ([21, Chap. I, 6.4.2]), it suffices to prove it for $\ell = \ell_0 = (x, y)$, $\mathcal{D} = \mathcal{D}_t$ for any t close to 0, and for a pair $(\mathcal{D}, \mathcal{D}') = (\mathcal{D}_t, \mathcal{D}_0)$ with $t \neq 0$.

Let $\lambda_{j,t}$, $j = 1, \ldots, k-1$, be the $k-1$ roots of the polynomial $tP+Q$. We denote by $\delta_{j,t}$ the $\widetilde{e}$-image of the component $\delta_{j,t}^*$ of $(\Pi_{\mathcal{D}_t}^*, p)$ which has a parametrization of the form $w = \lambda_{j,t}v + h.o.$ Substituting into the equation $\widetilde{f}(v, w) - x = 0$, we obtain that $\widetilde{\ell}(\delta_{j,t}^*)$ is a smooth curve parametrized by $x = \mu_{j,t}v^k + h.o.$, where $\mu_{j,t} = \prod_{i=1}^{k}(\lambda_{j,t} + a_i)$. Therefore $\ell(\delta_{j,t})$ is a $(k+1, k)$-cusp. Moreover, for small t, we have $\mu_{j,t} \neq \mu_{l,t}$ for every $j \neq l$. This implies that the two cusps $\ell(\delta_{j,t})$ and $\ell(\delta_{l,t})$ have contact $\frac{k+1}{k}$. So $\ell(\Pi_{\mathcal{D}_t,1})$ is a union of $k-1$ cusps of type $(k+1, k)$ having pairwise contact $\frac{k+1}{k}$.

Moreover, the two sets $\{\mu_{j,0} : j = 1, \ldots, k-1\}$ and $\{\mu_{j,t} : j = 1, \ldots, k-1\}$ are disjoint for $t \neq 0$, which means that the components of $\ell(\Pi_{\mathcal{D}_t,1})$ and $\ell(\Pi_{\mathcal{D}_0,1})$ have pairwise contact $\frac{k+1}{k}$. This completes the proof of (2b). □

5. Proof of Theorem 1.4

We first give two lemmas which will enable one to check Conditions (1^*) and (2^*) of Theorem 3.9 in the proof of the "if" direction of Theorem 1.4.

Let $\ell \colon (X, 0) \to (\mathbb{C}^2, 0)$ be a generic projection, let γ be a test curve and let $\widehat{\gamma}$ be a principal component of $\ell^{-1}(\gamma)$. The equality $\text{mult}(\gamma) = \text{mult}(\widehat{\gamma})$ of Condition (1^*)

can be checked easily in the resolution as follows. Let C be the exceptional curve of $\rho_\ell^{-1}(0)$ which intersects the strict transform γ^*. Let $\pi\colon \widetilde{X} \to X$ be a resolution of X which is also a good resolution of $\widehat{\gamma}$, i.e. $\pi^{-1}(0) \cup \widehat{\gamma}^*$ is a normal crossing divisor, and let E be the exceptional component of $\pi^{-1}(0)$ which intersects $\widehat{\gamma}^*$. Let m_C be the multiplicity of a generic linear form $h\colon \mathbb{C}^2 \to \mathbb{C}$ along C and let m_E be that of a generic linear form $k\colon (X,0) \to (\mathbb{C},0)$ along $(X,0)$.

Lemma 5.1. *([15, Lemma 11.6]) Assume that Conditions (1^*) and (2^*) are satisfied for any test curve at the root vertex (so for any generic line through the origin). Let γ be a test curve at a component C of $\rho_\ell^{-1}(0)$ which is not the root vertex. Then any pair of principal components $\widehat{\gamma}_1, \widehat{\gamma}_2$ of $\ell^{-1}(\gamma)$ whose strict transforms intersect the projectivized tangent cone $C_0 X = e^{-1}(0)$ at distinct points satisfies Condition (2^*).*

To state the second lemma, we need to introduce some notations. Let $(X,0) \subset (\mathbb{C}^n,0)$ be a complex surface and let $\mathscr{N}\colon NX \to X$ be the Nash modification of X. The *Gauss map* $\widetilde{\lambda}\colon NX \to \mathbf{G}(2,\mathbb{C}^n)$ is the restriction to NX of the projection of $X \times \mathbf{G}(2,\mathbb{C}^n)$ on the second factor.

If C is an irreducible component of the exceptional divisor of a composition of blow-ups $\rho\colon Y \to \mathbb{C}^2$, we defined the *inner rate* q_C of C as the contact in $\mathbb{C}^2$ between the ρ-image of two curvettes of C meeting C at distinct points.

Lemma 5.2. *Let $\rho\colon Y \to \mathbb{C}^2$ be a sequence of blow-ups of points which resolves the base points of the family of projected polar curves $(\ell(\Pi_\mathcal{D}))_{\mathcal{D} \in \Omega}$. Let E' be the union of components of $\rho^{-1}(0)$ which are not Δ-curves (Definition 3.5). Let $(\gamma,0) \subset (\mathbb{C}^2,0)$ be a complex curve such that $\gamma^* \cap E' \neq \emptyset$ and such that γ^* intersects $\rho^{-1}(0)$ at a smooth point. Let C be the component of $\rho^{-1}(0)$ such that $C \cap \gamma^* \neq \emptyset$ and let q_C be its inner rate. Let γ_1 and γ_2 be two components of $\ell^{-1}(\gamma)$ and consider the two points $p_1 = \gamma_1^* \cap \mathscr{N}^{-1}(0)$ and $p_2 = \gamma_2^* \cap \mathscr{N}^{-1}(0)$ where * means strict transform by the Nash modification $\mathscr{N}$. Assume that $q_{inn}(\gamma_1,\gamma_2) = q_C$ and that $\mathrm{mult}(\gamma_1) = \mathrm{mult}(\gamma_2) = \mathrm{mult}(\gamma)$. Then the pair of curves (γ_1,γ_2) satisfies Condition (2^*) if and only if $\widetilde{\lambda}(p_1) \neq \widetilde{\lambda}(p_2)$.*

Proof. Let $m = \mathrm{mult}(\gamma_1) = \mathrm{mult}(\gamma_2) = \mathrm{mult}(\gamma)$. Choose coordinates $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$ such that $z_1\colon \mathbb{C}^n \to \mathbb{C}$ is a generic linear form and $\ell = (z_1, z_2)$. Let (x,y) be the standard coordinates of $\mathbb{C}^2$. The intersection $\gamma \cap \{x = t \in \mathbb{R}^+\}$ defines m real arcs $\delta_1, \ldots, \delta_m \subset \mathbb{C}^2$, and for $i = 1, 2$, the intersection $\gamma_i \cap \{z_1 = t \in \mathbb{R}^+\}$ defines m real arcs $\delta_{i,1}, \ldots, \delta_{i,m} \subset \mathbb{C}^n$ such that $\ell(\delta_{1,j}) = \ell(\delta_{2,j}) = \delta_j$, for all $j = 1, \ldots, n$.

The outer and inner rates $q_o = q_o(\delta,\delta')$ and $q_i = q_i(\delta,\delta')$ between germs of real arcs $(\delta,0)$ and $(\delta',0)$ in $(\mathbb{C}^n,0)$ are defined by the following property, exactly as in the case of complex arcs (Definition 3.8): $d_o(\delta(t),\delta'(t)) = \Theta(t^{q_o})$ and $d_i(\delta(t),\delta'(t)) = \Theta(t^{q_i})$, respectively. By [15, Proposition 12.1], we have

$$q_{out}(\gamma_1,\gamma_2) = \max_{k,l} q_o(\delta_{1,k}, \delta_{2,l}) \quad \text{and} \quad q_{inn}(\gamma_1,\gamma_2) = \max_{k,l} q_i(\delta_{1,k}, \delta_{2,l}).$$

Since $q_{inn}(\gamma_1,\gamma_2) = q_C$, by [15, Lemma 6.18], $q_i(\delta_{1,j}, \delta_{1,j}) = q_C$ for all $j = 1, \ldots, n$. Hence, by [15, Lemma 9.1], $\widetilde{\lambda}(p_1) \neq \widetilde{\lambda}(p_2)$ implies that $q_i(\delta_{1,j}, \delta_{2,j}) = q_o(\delta_{1,j}, \delta_{2,j})$, for all $j = 1, \ldots, n$.

Since the equality $q_i(\delta_{1,j}, \delta_{2,j}) = q_o(\delta_{1,j}, \delta_{2,j})$ holds for the vertically aligned arcs $\delta_{1,j}, \delta_{2,j}$ for all $j = 1, \ldots, n$, the equalities $q_i(\delta_{1,j}, \delta_{2,k}) = q_o(\delta_{1,j}, \delta_{2,k})$ hold for every pair $j, k \in \{1, \ldots, n\}$ ([15, Section 7]). Taking the maximum on both sides, we obtain $q_{out}(\gamma_1,\gamma_2) = q_{inn}(\gamma_1,\gamma_2)$.

Conversely, assume $q_{out}(\gamma_1, \gamma_2) = q_{inn}(\gamma_1, \gamma_2)$ and let k, l such that $q_{out}(\gamma_1, \gamma_2) = q_o(\delta_{1,k}, \delta_{2,l})$. We then have $q_o(\delta_{1,k}, \delta_{2,l}) = q_{out}(\gamma_1, \gamma_2) = q_C = q_i(\delta_{1,k}, \delta_{2,l})$. By [15, Lemma 9.1], this implies $\widetilde{\lambda}(p_1) \neq \widetilde{\lambda}(p_2)$. $\square$

Proof of Theorem 1.4. The "only if" direction of the theorem is a direct consequence of the main result of [6] which states that a LNE analytic set has a LNE tangent cone. In the case of a hypersurface singularity in $(\mathbb{C}^3, 0)$, having a LNE tangent cone is equivalent to having a projectivized tangent cone which is LNE. But a complex curve singularity is LNE if and only if it consists of pairwise transversal smooth components. Therefore, if a supersisolated singularity is LNE, its projectivized tangent cone has only ordinary singularities.

Let us prove the "if" direction of the theorem.

Assume $(X, 0)$ is a superisolated singularity with equation $f_d + f_{d+1} = 0$ whose projectivized tangent cone $\{f_d = 0\} \subset \mathbb{P}^2$ has only ordinary singularities. We have to prove that for all generic projections ℓ, every test curve satisfies Conditions (1^*) and (2^*) of Theorem 3.9.

Let $\ell\colon \mathbb{C}^3 \to \mathbb{C}^2$ be a generic linear projection for $(X, 0)$, let Π be its polar curve and let $\Delta = \ell(\Pi)$ be its discriminant curve.

First, consider a test curve at the root vertex of T, i.e., γ is a generic line in $(\mathbb{C}^2, 0)$. Since the projectivized tangent cone $C_0(X) = \{f_d = 0\} \subset \mathbb{P}^2$ is reduced, the inverse image $\ell^{-1}(\gamma)$ consists of d smooth curves which meet pairwise transversally. Therefore, γ satisfies Conditions (1^*) and (2^*).

Let us now consider a test curve γ at a component C of $\rho_\ell^{-1}(0)$ which is not the root vertex. We first prove Condition (1^*).

We will use the notations of Proposition 4.1 and the description of the trees T and G_0 given just after its statement. Recall that $\pi_0\colon X_0 \to X$ denotes the minimal resolution of $(X, 0)$ which factorizes through the Nash modification and through the blow-up of the maximal ideal. By Proposition 4.1, π_0 is obtained by first considering the blow-up $e\colon X^* \to X$ of the maximal ideal of $(X, 0)$ and then blowing-up each singular point p_i, $i = 1, \ldots, r$ of the projectivized tangent cone $C_0 X$ on the smooth surface X^*.

Assume first that γ is a test curve at the Δ-node v_i of the subtree T_i. Call C_i the component of $\rho_\ell^{-1}(0)$ corresponding to v_i. Then the multiplicity m_{C_i} of a generic linear form $h\colon (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ along C_i equals k_i. On the other hand, the principal components of $\ell^{-1}(\gamma)$ are k_i curvettes of the component E_i of $\pi_0^{-1}(0)$ obtained by blowing-up the point p_i on the smooth surface X^*. Since $E_i \subset X_0$ is the exceptional curve of the blowing-up of a point on a smooth surface, its self-intersection E_i^2 in the surface X_0 equals -1. Let us compute the multiplicity m_{E_i} of a generic linear form H on $(X, 0)$ along E_i. The multiplicity of H equals 1 along each component of $C_0 X$. If (H) denotes the total transform of H by π_0, we have $(H).E = 0$ for each component E of $\pi_0^{-1}(O)$ ([4, Lemma 2.6]). Since $(H) = \sum_{j=1}^r m_{E_j} E_j + C_0 X$, we obtain for $E = E_i$: $m_{E_i} E_i^2 + E_i.C_0 X = 0$. This gives $m_{E_i} = k_i$, since $E_i.C_0 X = k_i$. Therefore $m_{C_i} = m_{E_i}$. Hence we have proved that each of the k_i principal components of $\ell^{-1}(\gamma)$ satisfies Condition (1^*).

Assume now that the root vertex of T is a Δ-node and let γ be a test curve at the separation node obtained by blowing-up the intersection point q_i of the exceptional curve $e_0^{-1}(0)$ with C_i. Let C_i' be the curve created by blowing-up q_i. Then the multiplicity of h along C_i' equals $k_i + 1$. let $\widehat{\gamma}$ be one of the principal connected components of $\ell^{-1}(\gamma)$. Then $q' = \widehat{\gamma}^* \cap \pi_0^{-1}(0)$ is the intersection point between E_i

and C_0X, and we obtain a resolution of $\widehat{\gamma}$ by blowing-up the point q'. Let E'_i be the exceptional curve of this blowing-up. Then the multiplicity $m_{E'_i}$ of H along E'_i equals $m_{E_i} + 1 = k_i + 1$. We then have $m_{C'_i} = m_{E'_i}$, i.e., $\widehat{\gamma}$ satisfies Condition (1^*).

Let us now prove Condition (2^*). By Lemma 5.1, we will then have to check Condition (2^*) just for pairs of principal components of $\ell^{-1}(\gamma)$ whose strict transforms by e intersect the same singular point of $C_0X = e^{-1}(0)$.

Assume first that γ is a test curve at the Δ-node v_i of T_i, i.e., the strict transform γ^* of γ by e_0 is of the form $x = \mu v^k$ in the local coordinates defined above, with $\mu \in \mathbb{C}$ generic. Any point of C_0X distinct from p_i is on at most one component of the strict transform γ^* of γ by e_0, so we just have to consider components of γ^* passing through p_i. Let $\widehat{\gamma}_i$ be the union of components of $\ell^{-1}(\gamma)$ whose strict transforms by e pass through p_i. If δ is one of them, it has a Puiseux expansion of the form $w = \lambda v + h.o.$ in the local coordinates (x, v, w) centered at p_i. Substituting this into the equation $\widetilde{f}(v, w) - x = 0$ of X^*, we obtain $\prod_{i=1}^{k}(\lambda + a_i)v + h.o. - \mu v^k = 0$. Therefore λ satisfies the equation $\prod_{i=1}^{k}(\lambda + a_i) - \mu = 0$. For a generic value of μ, this equation admits k_i distinct roots. This means that the curve $\widehat{\gamma}_i$ has exactly k_i components and its strict transform by π_0 consists of k_i curvettes of the $\mathcal{P}$-curve E_i meeting E_i at k_i distinct points.

Let $\delta_{i,1}$ and $\delta_{i,2}$ be two components of $\widehat{\gamma}_i$ and let $w = \lambda_1 v + h.o.$ and $w = \lambda_2 v + h.o.$ be the Puiseux expansions of their strict transforms by e. Then the curve germs $\delta_{i,1}$ and $\delta_{i,2}$ of $(\mathbb{C}^3, 0)$ are parametrized by

$$v \longmapsto (\mu v^k, \mu v^{k+1}, \lambda_j \mu v^{k+1}), \quad j = 1, 2.$$

Since $\lambda_1 \neq \lambda_2$, this implies $q_o(\delta_{i,1}, \delta_{i,2}) = \frac{k+1}{k}$.

On the other hand, by [15, 15.3], the inner rate $q_i(\delta_{i,1}, \delta_{i,2})$ equals the inner rate between two curvettes of C_i meeting C_i at distinct points. Since C_i is the node of the resolution tree T_i of the $(k+1, k)$-cusp, we obtain $q_i(\delta_{i,1}, \delta_{i,2}) = \frac{k+1}{k}$.

Finally, $q_o(\delta_{i,1}, \delta_{i,2}) = q_i(\delta_{i,1}, \delta_{i,2})$, so γ satisfies Condition (2^*).

Assume now that γ is a curvette of a curve C'_i corresponding to a separating node and let $\widehat{\gamma}_i$ be as above. The curve $\widehat{\gamma}_i$ consists of k_i curves whose strict transforms by π_0 pass through the k_i points $p_{i,1}, \ldots, p_{i,k_i}$ of $C_0X \cap E_i$. By Lemma 5.2, in order to prove that all pairs of components of $\widehat{\gamma}_i$ satisfy Condition (2^*), we have to prove that for all $r \neq s$, $\widetilde{\lambda}(p_{i,r}) \neq \widetilde{\lambda}(p_{i,s})$. Let C_j be the component of C_0X whose strict transform by the blow-up of p_i satisfies $C_j \cap E_i = p_{i,j}$. The tangent line $L_{i,j}$ to C_j at p_i corresponds to a 2-plane in $\mathbf{G}(2, \mathbb{C}^n)$, and we have $\widetilde{\lambda}(p_{i,j}) = L_{i,j}$. Since the singularity (C_0X, p_i) is ordinary, we have $L_{i,r} \neq L_{i,s}$ for all $r \neq s$, i.e., $\widetilde{\lambda}(p_{i,r}) \neq \widetilde{\lambda}(p_{i,s})$. This completes the proof. $\qquad\square$

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