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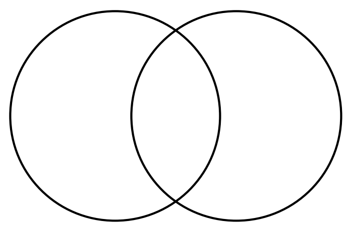
Exercises on Algebraic Curves

2. April 2019

Exercise 1. Let $p \in \mathbb{C}[x]$ and $a \in \mathbb{C}$ such that $p(a) = 0$. Show that there exists $q \in \mathbb{C}[x]$ such that $p(x) = (x - a)q(x)$.

Exercise 2. $2 \cdot 2 = 4$. Where are the two missing points?



Exercise 3. Let $X := \{(x, y, z) \in \mathbb{C}^3 \mid xy = yz = xz = 0\}$. Show that there do not exist two polynomials $f, g \in \mathbb{C}[x, y, z]$ such that $X = \{(x, y, z) \in \mathbb{C}^3 \mid f(x, y, z) = g(x, y, z) = 0\}$.
Hint: $\mathbf{P}^2(\mathbb{C})$ and Bézout's theorem.
Remark: X is an example of an algebraic curve in $\mathbb{C}^3$ which is not a *complete intersection*: its codimension does not equal the minimal number of equations needed to describe it.

Exercise 4. Let $p_1, p_2, \ldots, p_k \in \mathbb{R}[x]$ be pairwise distinct polynomials. Let a be a positive real number such that for all $x \in (-a, a) \setminus \{0\}$, $p_i(x) \neq p_j(x)$ for $i \neq j$. Up to renumbering the p_i, we may assume that $p_1(x) < p_2(x) < \ldots < p_k(x)$ for $x \in (-a, 0)$. Then there exists a permutation σ of $\{1, \ldots, k\}$ such that $p_{\sigma(1)}(x) < p_{\sigma(2)}(x) < \ldots < p_{\sigma(k)}(x)$ for $x \in (0, a)$. In other words, k polynomials define a permutation on k letters.

(a) Show that for $k \leqslant 3$, any permutation of $\{1, \ldots, k\}$ can be realised.
(b) Show that for $k \geqslant 4$, there exist permutations of $\{1, \ldots, k\}$ which cannot be realised.
Remark: this was discovered in 2009 by Maxim Kontsevich, according to Étienne Ghys.
(c) Convince yourself that for all $k \in \mathbb{N}$, every permutation of $\{1, \ldots, k\}$ can be realised (in the above sense) by k smooth functions $f_1, \ldots, f_k$.

Exercise 5.

(a) Recall that a polynomial $p \in \mathbb{R}[x]$ with $p(x) > 0$ for all $x \in \mathbb{R}$ always has a minimum. Now let $q := x^2y^2+x^2+2xy+1 \in \mathbb{R}[x,y]$. Show that $q(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$, but q has no minimum.

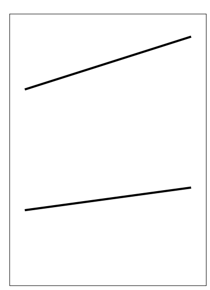
(b) Recall that a polynomial $p \in \mathbb{R}[x]$ with two distinct local minima always has a local maximum. Consider the polynomial $(x^2y-x-1)^2+(x^2-1)^2 \in \mathbb{R}[x,y]$. Show that it has two minima but no other critical points.

9. April 2019

Exercise 6. Fix $d, n \in \mathbb{N}$. The set of all homogeneous polynomials in n variables of degree d, together with the zero polynomial, forms a vector space. Compute its dimension in terms of d and n. What is the dimension of the space of lines (degree 1), of conics (degree 2) and of cubics (degree 3) in $\mathbf{P}^2(\mathbb{C})$?

Exercise 7. Let $F \in \mathbb{C}[x,y,z]$ be homogeneous of degree d. Show that $\mathrm{d}F_{(x,y,z)}(x,y,z) = d \cdot F(x,y,z)$. Note that the d on the left side denotes the differential, which is applied to the function $F : \mathbb{C}^3 \to \mathbb{C}$ at the point (x,y,z), evaluated on the vector (x,y,z).

Exercise 8. Given are two non-parallel lines on the floor of a room:



Unfortunately, their intersection point is outside the room. Without leaving the room, construct a third line going through the intersection point, only using a pen and a straight edge.

Exercise 9. Let $X := \{[x:y:z] \in \mathbf{P}^2(\mathbb{C}) \mid x^3+y^3+z^3=0\}$.

(a) Show that X is a smooth compact oriented surface.

(b) Compute its genus.

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Exercise 10. Prove that $\mathbf{P}^1(\mathbb{C})$ is diffeomorphic to S^2.

Exercise 11. Let $Z := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_1^2 + \ldots + z_n^2 = 1\} \cap B(0,2)$, where $B(0,2)$ is the closed ball in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ of radius 2 centered at $0 \in \mathbb{C}^n$.

(a) Consider first the case $n = 2$. Show that Z is homeomorphic to the cylinder $S^1 \times [-1, 1]$.
(b) In general, show that Z is homeomorphic to $T_{\leqslant 1}S^{n-1}$, the space of tangent vectors of length $\leqslant 1$ to the $(n-1)$-sphere.

Exercise 12. Observe that the action of $GL_3(\mathbb{C})$ on $\mathbb{C}^3$ by matrix-vector multiplication descends to an action of $PSL_3(\mathbb{C})$ on $\mathbf{P}^2(\mathbb{C})$.

(a) Let $T \in PSL_3(\mathbb{C})$ and $C \subset \mathbf{P}^2(\mathbb{C})$ a curve of degree d. Show that $T(C)$ is a curve of degree d.
(b) Given two smooth conics (curves of degree 2) $C_1, C_2 \subset \mathbf{P}^2(\mathbb{C})$, show that there exists $T \in PSL_3(\mathbb{C})$ such that $T(C_1) = C_2$.

Exercise 13. Recall that the space $\mathcal{L}$ of projective lines in $\mathbf{P}^2(\mathbb{C})$ is itself a copy of $\mathbf{P}^2(\mathbb{C})$, thanks to the duality

$$\delta : \mathbf{P}^2(\mathbb{C}) \to \mathcal{L}, \quad [a : b : c] \mapsto \{[x : y : z] \in \mathbf{P}^2(\mathbb{C}) \mid ax + by + cz = 0\}.$$

(a) Let $L \in \mathcal{L}$ and $p, q, r \in L$ three distinct points. Show that $\delta(p) \cap \delta(q) \cap \delta(r) = \delta^{-1}(L)$.
(b) Show that the subset of $\mathcal{L}$ given by all tangent lines to a given smooth conic C is itself dual to a smooth conic in $\mathbf{P}^2(\mathbb{C})$.
(c) What is the "image" of Pascal's theorem under the map δ?

Exercise 14. An *elliptic curve* (not to be confused with an ellipse...) is a smooth cubic (curve of degree 3) in $\mathbf{P}^2(\mathbb{C})$. Let C be an elliptic curve and $e \in C$ a point on it. Define an addition on C as follows: given two points $p, q \in C$, let L_1 be the line through p, q (or, if $p = q$, the tangent line to C at p). By Bézout's theorem, there exists a unique third point $s \in C \cap L_1$ besides p and q. Now take the line M_1 through e and s and define $p+q$ to be the third point on $C \cap M_1$ besides e and s. Show that $(C, +, e)$ is an abelian group. *Hint:* to prove associativity, that is, $(p+q)+r = p+(q+r)$ for $p, q, r \in C$, consider L_1, M_1 as above, $L_2 \cap C \ni r, p+q$; $M_2 \cap C \ni q, r$. Let t be the third point on $M_2 \cap C$. Finally, $L_3 \cap C \ni e, t, q+r$ and $M_3 \cap C \ni p, q+r$. Now consider the three cubics $C_1 := L_1 \cup L_2 \cup L_3$, $C_2 := M_1 \cup M_2 \cup M_3$ and C.

23. April 2019

Exercise 15. Let $p_1, p_2, p_3, p_4 \in \mathbf{P}^2(\mathbb{C})$. Let $V \subset \mathbf{P}^5(\mathbb{C})$ be the linear system of all conics $C \subset \mathbf{P}^2(\mathbb{C})$ such that $p_1, p_2, p_3, p_4 \in C$. Show that V is two-dimensional if the points lie on a line and one-dimensional otherwise.

Exercise 16. Let $g \in \mathbb{N}$, $g \geqslant 1$ and let $a_1, a_2, \ldots, a_{2g+1} \in \mathbb{C}$ be pairwise distinct points. Define $p(x) := (x - a_1)(x - a_2) \cdots (x - a_{2g+1})$ and consider the curve

$$C := \{(x, y) \in \mathbb{C}^2 \mid y^2 = p(x)\}.$$

Show that C is a smooth (noncompact) surface of genus g.
In particular, every nonnegative integer is realised as the genus of a smooth affine curve. Consider the projectivisation of C and compare genus and degree. Explain the discrepancy with the genus-degree formula.
Remark: If $g = 1$, C is an elliptic curve. For $g \geqslant 2$, such curves are called *hyperelliptic*.

Exercise 17. Let $S^3 = \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\}$ be the 3-dimensional sphere in $\mathbb{C}^2 \cong \mathbb{R}^4$ and $V = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3\}$. Draw the set $V \cap S^3$.
Hint: Think of S^3 as the one-point-compactification of $\mathbb{R}^3$, via the stereographic projection.

30. April 2019

Exercise 18. Let $\gamma : t \mapsto (t^4, t^6 + t^7)$. Find $f \in \mathbb{C}[x, y]$, $f \neq 0$, such that $f(\gamma(t)) = 0$ for all $t \in \mathbb{C}$.

Exercise 19. Consider the curve $C = \{(x, y) \in \mathbb{C}^2 \mid x^n - y^m = 0\}$, where $m, n \in \mathbb{N}$. How many components does the link $T(m, n) := C \cap S^3$ have (in terms of m, n)?

Exercise 20. Draw the link of $(x^2 - y^3)(y^2 - x^3) = 0$ near $(0, 0)$.

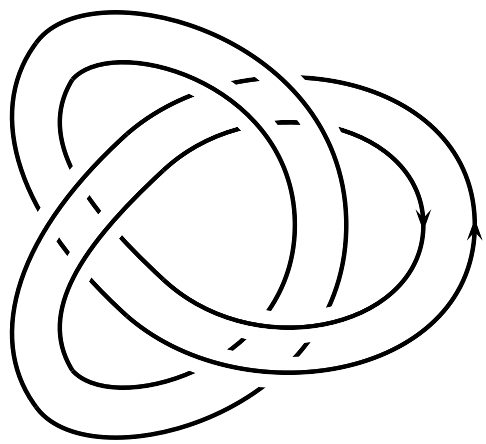
7. May 2019

Exercise 21. Try Newton's algorithm to approximate the branches of the following curves near $(0, 0)$.

(a) $-x^{10} + x^9 + 6x^8 y - 3x^6 y^2 + 2x^5 y^3 + 3x^3 y^4 - y^6 = 0$
(b) $2x^4 + x^2 y + 4xy^2 + 4y^3 = 0$

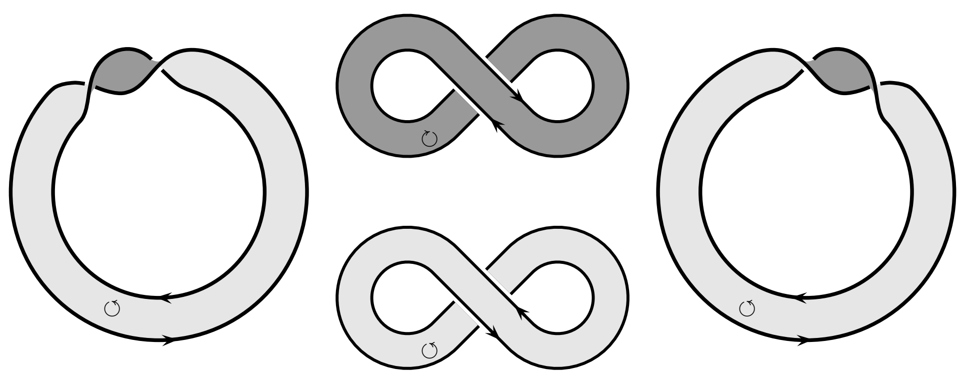
14. May 2019

Exercise 22. Apply the Seifert algorithm to the following diagram. Compute the genus of the resulting surface. Find another Seifert surface with smaller genus for the same link. Compute the linking number of the two components. Reverse the orientation on one of the two components and compute the linking number for the new link.



21. May 2019

Exercise 23. Which of the following four Seifert surfaces are isotopic?



Exercise 24. Recall the definition of the (a, b)-cable of a knot K, denoted $K_{(a,b)}$, and the definition of the torus knot $T(a, b)$.

(a) Draw a diagram of the knot $T(2, 3)_{(2,3)}$.

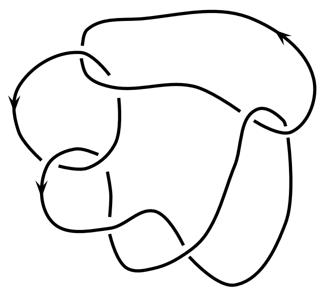
(b) Draw a diagram of the knot associated to the Newton series $y = x^{3/2}(1 + x^{1/4})$.

Exercise 25. Show that

(a) $K_{(1,n)} = K$, for all $n \in \mathbb{Z}$.
(b) The (a,b)-cable of the unknot is the torus knot $T(a,b)$.
(c) $T(2,3)_{(2,12)} = T(4,6)$. Can you generalise this phenomenon to other cables of other torus links?

(Here the equality sign means isotopy between the corresponding links.)

Exercise 26. Show that the diagram represents a torus link $T(a,b)$. Find the parameters (a,b).



Exercise 27. Show that $T(a,b)$ is isotopic to $T(b,a)$.

25. June 2019

Exercise 28. If K is a fibred knot with fibre surface F, denote the genus of F by $g(K) := g(F)$.

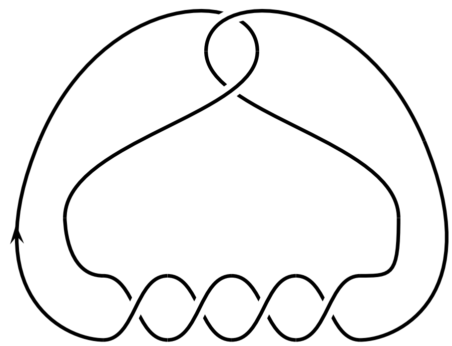
(a) Compute $g(T(a,b))$.
(b) Show that $g(K_{(a,b)}) = a \cdot g(K) + g(T(a,b))$.

Exercise 29. Let $C \subset \mathbb{C}^2$ be the zero set of a weighted-homogeneous polynomial $f \in \mathbb{C}[x,y]$ with weights $w(x) = a$, $w(y) = b$ and $w(f) = n$, that is, $f(\lambda^a x, \lambda^b y) = \lambda^n f(x,y)$ for all $(x,y) \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}$. Let $K = C \cap S^3$ be its link at $(0,0)$, where S^3 is a small sphere. Consider the Milnor fibration $p := \frac{f}{|f|} : S^3 \setminus C \to S^1$ with fibres $F_\theta := p^{-1}(\theta) \cup K$ and monodromy $\varphi : F_1 \to F_1$. Show that φ is isotopic to a periodic map, by an isotopy $h : F_1 \times [0,1] \to F_1$ (which is allowed to move the points of $\partial F_1 = K$).
Hint: Let $\zeta_t := e^{2\pi i t/n}$ and $\Phi_t : \mathbb{C}^2 \to \mathbb{C}^2$, $(x,y) \mapsto (\zeta_t^a x, \zeta_t^b y)$. Modify $\Phi_t|_{S^3}$ in a neighbourhood $N(K) \cong S^1 \times D^2$ of K to obtain a monodromy flow.

2. July 2019

Exercise 30. Compute the matrix M of the homological monodromy of T(2,3), with respect to some basis of $H_1(S,\mathbb{Z})$, where S is the fibre surface of $T(2,3)$. Verify that M has finite order.

Exercise 31. Let K be the following knot.



Show that

(a) K is not the trivial knot,
(b) K is not fibred,
(c) K is not algebraic.

Hint: Compute the Alexander polynomial of K.

Exercise 32. Let S be a fibre surface with homological monodromy $M : H_1(S,\mathbb{Z}) \to H_1(S,\mathbb{Z})$ and let $A : H_1(S,\mathbb{Z}) \times H_1(S,\mathbb{Z}) \to \mathbb{Z}$ be the Seifert form of S.

(a) Show that M is an "isometry" with respect to A, that is, for all $x, y \in H_1(S,\mathbb{Z})$, we have $A(Mx, My) = A(x, y)$.
(b) Assume that the symmetrised Seifert form $A + A^\top$ is positive definite. Show that M has finite order and conclude that the eigenvalues of M are roots of unity.
Hint: Let M act on a ball of radius r in $H_1(S,\mathbb{Z}) \cong \mathbb{Z}^n \subset \mathbb{R}^n$, with respect to the norm induced by the bilinear form $A + A^\top$.

9. July 2019

Exercise 33. Describe the monodromy of the following singularities at $0 \in \mathbb{C}^2$, using a resolution by iterated blow-up.

(a) xy
(b) $y^{12} - x^{30}$
(c) $xy^2 - x^4$
(d) $(x^2 - y^3)(x^3 - y^2)$

Specifically, determine the decomposition of the Milnor fibre induced by the resolution, describe the topological type of the pieces (number of connected components, genus and number of boundary components of each piece), how they are permuted under the monodromy, determine the periods of the monodromy on each piece and describe the amount of twisting that occurs at each curve along which the pieces are glued.

Exercise 34. Read the following article about a discovery of Étienne Ghys relating Lorentz knots, the modular flow and the Rademacher function (or watch a recording of Ghys' ICM talk from 2006).
`https://www.ams.org/publicoutreach/math-history/hap7-new-twist.pdf`