University of Regensburg Summer Term 2018

Quantum Theory of Condensed Matter I

Prof. John Schliemann
Dr. Paul Wenk, M.Sc. Martin Wackerl **Mo.** 08:00-10:00 c.t., PHY 5.0.21

Sheet 9

1. Two Coupled Spins II .... [10P]

Here, we continue our calculation from Sheet 8, Ex.2. The goal is to determine $\langle S^z \rangle$.

(a)(2P) Show that the regularity of the commutator Green's functions at $E = 0$ does not allow spontaneous magnetization of the system.

(b)(2P) Calculate the spectral densities which are related to the commutator Green's functions G_{11}, G_{21} and $\Gamma \equiv \Gamma_{12} - \Gamma_{21}$, as a function of $\langle S^z \rangle$ and ρ_{12}. *Hint: Dirac identity.*

(c)(2P) Use the spectral theorem and (b) to calculate the equal-time correlation functions $\langle S_1^+ S_1^- \rangle$, $\langle S_1^+ S_2^- \rangle$ and $\langle S_1^+ S_1^z S_2^- \rangle - \langle S_1^+ S_2^z S_1^- \rangle$. The appearing related constants D_{11}, D_{21} and D_Γ will be determined in the next step.

(d)(2P) Calculate the equations of motion for the *anticommutator* Green's functions according to Sheet 8, Ex.2. Eq.(5)-(8). Using their residua of the $E = 0$ poles, determine the constants D_{11}, D_{21} and D_Γ.

(e)(2P) Having the correlation functions fully determined in (c), show that

$$\langle S^z \rangle = \frac{\hbar}{2} \frac{e^{\beta b} - e^{-\beta b}}{1 + e^{-2\beta\hbar^2 J} + e^{\beta b} + e^{-\beta b}} \tag{1}$$

with $b = g_J \mu_B B$. Discuss the case $J \to 0$ and $b = 0$.

2. Superconductivity .... [7P]

We analyze the BCS[1] model Hamiltonian which is based on mean-field theory,

$$H_{\text{BCS}}^{\text{MF}} = \sum_{\mathbf{k},\sigma} t(\mathbf{k}) a_{\mathbf{k}\sigma}^\dagger a_{\mathbf{k}\sigma} - \Delta \sum_{\mathbf{k}} (b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger), \tag{2}$$

with $t(\mathbf{k}) = (\epsilon(\mathbf{k}) - \mu)$, $t(\mathbf{k}) = t(-\mathbf{k})$, the electron creation (annihilation) operator $a_{\mathbf{k}\sigma}^\dagger$ ($a_{\mathbf{k}\sigma}$) in Bloch-representation and the Cooper pair creation operator $b_{\mathbf{k}}^\dagger = a_{\mathbf{k}\uparrow}^\dagger a_{-\mathbf{k}\downarrow}^\dagger$. Calculate the single-electron retarded Green's function

$$G_{\mathbf{k}\sigma}^{\text{ret}}(E) = \langle\langle a_{\mathbf{k}\sigma}; a_{\mathbf{k}\sigma}^\dagger \rangle\rangle_E^{\text{ret}} \tag{3}$$

and with this the excitation energy of the superconductor. Show that the spectrum is gapped.

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[1] John Bardeen, Leon Neil Cooper and John Robert Schrieffer