Quantum Theory of Condensed Matter I

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Sheet 11

Lifetime of Quasiparticles .... [20P]

Consider a system describing tunneling from a lead to a localized fermionic state,

$$H = H_0 + H_t, \tag{1}$$

$$\text{where} \quad H_0 = \sum_k \epsilon(k) c_k^\dagger c_k + \Delta d^\dagger d \tag{2}$$

with the operators d, $d^\dagger$ describing the localized fermionic level. The tunneling term,

$$H_t = \gamma(d^\dagger \psi(0) + \psi^\dagger(0) d), \tag{3}$$

describes the coupling between the fermionic, non-interacting lead given by fields $\psi(x) = \sum_k e^{ikx} c_k$, $\psi^\dagger(x)$ and the fermionic level.

(a)(5P) First, we consider only the lead and assume the temperature to be $T = 0$ and set the chemical potential to $\mu = 0$. At the special position $x = 0$ we assume the local density ρ_0 to be energy independent. At this position the non-interacting Green's function $G_0(x,t) = -i\langle \mathcal{T} \psi(0,t) \psi^\dagger(x,0)\rangle$ can be rewritten in the known form

$$G_0(x = 0, t) = -i \sum_k e^{-it\epsilon(k)} (\theta(t)(1 - n_k) - \theta(-t) n_k) \tag{4}$$

with the particle occupation number n_k and the time-ordering operator $\mathcal{T}$. From this, show that

$$G_0(x = 0, \omega) = -\frac{i}{2} \rho_0 \text{sign}(\omega). \tag{5}$$

(b)(5P) We define in the next step the Green's function of the localized fermionic level

$$G_d(t) = -i\langle \mathcal{T} d(t) d^\dagger(0)\rangle, \tag{6}$$

and the mixed Green's function,

$$G_{\psi,d}(x,t) = -i\langle \mathcal{T} \psi(x,t) d^\dagger(0)\rangle. \tag{7}$$

Considering now the whole Hamiltonian, show that the following equations of motion hold:

$$(\partial_t + i\Delta) G_d(t) = -i\delta(t) - i\gamma G_{\psi,d}(0,t), \tag{8}$$

$$((\partial_t + iK) G_{\psi,d})(x,t) = -i\gamma\delta(x) G_d(t), \tag{9}$$

with the operator

$$(K G_{\psi,d})(x,t) = \int dx'\, E(x - x') G_{\psi,d}(x', t), \tag{10}$$

$$\text{and} \quad E(x - x') = \sum_k \epsilon(k) e^{-ik(x-x')}. \tag{11}$$

(c)(5P) Fourier transform the equations of motion derived in (b) and combine them to derive the Dyson equation for $G_d(\omega)$,

$$G_d(\omega) = G_d^{(0)}(\omega) + \gamma^2 G_d^{(0)}(\omega) G_0(0,\omega) G_d(\omega), \tag{12}$$

with $G_d^{(0)}(t)$ and $G_0(x,t)$ being the propagators of the non-interacting system, i.e. $\gamma = 0$. For $G_0(x,t)$ the relation $(\partial_t + iK)G_0(x,t) = -i\delta(t)\delta(x)$ holds. Using the result of (a), solve the Dyson equation for $G_d(\omega)$.
Remember: The Fourier transformation of a convolution is given by $\mathcal{F}(f * g) = (2\pi)^{n/2}\mathcal{F}(f)\mathcal{F}(g)$, *with* $f, g \in L^1(\mathbb{R}^n)$.

(d)(5P) Calculate the retarded form of $G_d(\omega)$ and show that the corresponding spectral function is given by

$$S(\omega) = \frac{1}{\pi} \frac{\Gamma}{(\omega - \Delta)^2 + \Gamma^2}, \quad \Gamma := 2\pi^2\gamma^2\rho_0. \tag{13}$$

Give an interpretation in terms of particle lifetimes.