Quantum Theory of Condensed Matter I

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Sheet 10

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1. Single-Particle Matsubara Green's Function ....[5P]

Given the (fermionic or bosonic) Hamiltonian $H_0 = \sum_k (\epsilon(\mathbf{k}) - \mu) a_k^\dagger a_k$,

(a)(3P) prove the modified Heisenberg representation:

$$a_k(\tau) = a_k \exp\left(-\frac{1}{\hbar}(\epsilon(\mathbf{k}) - \mu)\tau\right), \quad a_k^\dagger(\tau) = a_k^\dagger \exp\left(\frac{1}{\hbar}(\epsilon(\mathbf{k}) - \mu)\tau\right) \quad \text{with} \quad \tau := it. \tag{1}$$

(b)(2P) Show that the free single-particle Matsubara Green's function can be written as

$$G_k^0(\tau) = -\exp\left(-\frac{1}{\hbar}(\epsilon(\mathbf{k}) - \mu)\tau\right)(\theta(\tau)(1 + \epsilon\langle n(k)\rangle^{(0)}) + \theta(-\tau)\epsilon\langle n(k)\rangle^{(0)}), \tag{2}$$

$$\text{with} \quad \langle n(k)\rangle^{(0)} = \frac{1}{e^{\beta(\epsilon(\mathbf{k})-\mu)} - \epsilon}, \quad \epsilon = \pm. \tag{3}$$

2. Calculating the Matsubara Sum .... [10P]

(a)(4P) Prove *Poisson's summation formula*,

$$\sum_{n=-\infty}^{\infty} e^{2\pi i \mu n} = \sum_{m=-\infty}^{\infty} \delta(\mu - m) \tag{4}$$

and show that

$$\sum_{m=-\infty}^{\infty} f(m) = \int_{-\infty}^{\infty} \mathrm{d}\mu \sum_{n=-\infty}^{\infty} e^{2\pi i \mu n} f(\mu) \tag{5}$$

holds for a smooth function $f(\mu)$. *Hint: What is the periodicity of the term on the RHS of Eq. (4)?*

(b)(6P) For the free particle we know from the lecture

$$G(\mathbf{r}, \tau) = \frac{1}{\beta\hbar} \sum_{\omega_m} \int \mathrm{d}^3 p\, e^{-i\omega_m \tau + i\mathbf{p}\cdot\mathbf{r}/\hbar} G(\mathbf{p}, \omega_m) \quad \text{with} \quad G(\mathbf{p}, \omega_m) = \frac{\hbar}{i\hbar\omega_m - \epsilon(\mathbf{p})} \tag{6}$$

with $\epsilon(\mathbf{p}) = \mathbf{p}^2/(2M)$ and $\tau \in (0, \beta\hbar)$. Using Eq. (5), show that

$$G(\mathbf{p}, \tau) = \sum_{n=-\infty}^{\infty} \left\{ \begin{array}{c} 1 \\ (-1)^n \end{array} \right\} \theta(\tau + n\hbar\beta) e^{-\epsilon(\mathbf{p})(\tau + n\hbar\beta)/\hbar} = e^{\epsilon(\mathbf{p})\tau/\hbar}(1 \pm f_{\epsilon(\mathbf{p})}) \tag{7}$$

with the case differentiation according to bosons/fermions choice and the Fermi/Bose distribution function $f_{\epsilon(\mathbf{p})}$.

3. 2-point Matsubara Green's Function for the Harmonic Oscillator .... [5P]

We consider the Hamiltonian for the 1D harmonic oscillator, $H = \hbar\omega(a^\dagger a + 1/2)$. Show that

$$G(\tau) = \frac{1}{Z}\mathrm{Tr}[e^{-\beta H}x(\tau)x(0)] = \frac{\hbar}{2m\omega}\frac{\cosh\left(\left(\frac{\beta\hbar}{2} - \tau\right)\omega\right)}{\sinh\left(\frac{\beta\hbar\omega}{2}\right)} \tag{8}$$

with $\tau \in (0, \beta\hbar)$.