University of Regensburg SS 17

General Relativity and Cosmology

Prof. John Schliemann
Dr. Paul Wenk

Mo. H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
Thu. 1pm c.t., PHY 9.1.10

Sheet 9

1) Thomas Precession .... [12P]

Assume a particle with classical spin $\mathbf{s} = s^i \mathbf{e}_i$ subjected to a force F^α in an inertial system IS. We find an IS$'$ where the particle is at rest, $\{u'^\alpha\} = \{c, 0\}$, for a certain moment and no torque is acting on the spin, i.e.

$$\left\{\frac{ds'^\alpha}{d\tau}\right\} = \left\{\frac{ds'^0}{d\tau}, 0\right\} \tag{1}$$

with $\{s'^\alpha\} = \{0, s'^i\}$.

Figure 1: Thomas,Llewellyn, 1926 at Kopenhagen.

(a) Show that the spin precesses (Thomas precession) if it is accelerated by proving

$$\frac{ds^\alpha}{d\tau} = -\frac{1}{c^2}\frac{u_\beta}{d\tau} s^\beta u^\alpha. \tag{2}$$

(b) The particle is accelerated in a form which keeps it on a circular orbit with an angular frequency ω and radius r_0. Show that the solution of the equation of motion found in (a) is given by

$$s^1 = \tilde{S}_0 \cos(\omega_{\mathrm{Th}} t), \qquad s^2 = -\tilde{S}_0 \sin(\omega_{\mathrm{Th}} t), \tag{3}$$

with $\omega_{\mathrm{Th}} = \omega(\gamma - 1)$, and the initial conditions $s^0(t=0) = 0$ and $\mathbf{s}(t=0) = \tilde{S}_0 \mathbf{e}_x$.
Hint: Show that $\frac{d^3 s^0}{dt^3} = -\omega^2\gamma^2 \frac{ds^0}{dt}$. Expand the amplitude by dropping second order corrections in v/c.

(c) Which geometrical conclusions can we draw when analyzing the spin after $T = 2\pi/\omega$?

2) Spin Orbit Coupling .... [8P]

In the following we make use of a classical description of an electron surrounding a nucleus. To see the coupling between the orbital motion of the electron and its spin, we transform into the current rest frame P' of the electron.

(a) The expansion of the Dirac equation at low velocities, $v \ll c$, (Pauli theory) yields, besides other terms, the Stern-Gerlach term $H_{\mathrm{SB}} = e\hbar/(2m_e)\boldsymbol{\sigma} \cdot \mathbf{B}$. Assuming $\varphi(\mathbf{r}) = \varphi(r)$, show that H_{SB} can be expressed as

$$H_{\mathrm{SB}} = -\frac{e}{m_e^2 c^2}\left(\frac{1}{r}\frac{d\varphi}{dr}\right)(\mathbf{L}\cdot\mathbf{S}). \tag{4}$$

which describes the coupling between the electron's spin $\mathbf{S}$ and its orbital motion, with the orbital angular momentum $\mathbf{L}$.
Hint: Explore the situation in the current rest frame of the electron.

(b) From quantum mechanics lecture we know that in Eq.4 we should expect a factor 1/2. Show that this can be cured if we include the Thomas precession derived in Ex.1, and remember that the time derivative of an arbitrary vector in a rotated frame is given by

$$\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{non-rot}} - \boldsymbol{\omega} \times \mathbf{A}. \tag{5}$$

3) Physically Meaningful Metric .... [8P]

Near a spherical star of mass m the metric to first order in $1/r$ is given by

$$ds^2 = \left(1 + \sigma_1 \frac{2m}{r}\right) c^2 dt^2 - \left(1 + \sigma_2 \frac{2m}{r}\right) dl^2 \qquad \text{with } \sigma_i = \pm 1. \tag{6}$$

(a) Which sign in the coefficient of dt^2 is meaningful? *Hint: Consider the Newtonian limit.*

(b) In the given coordinate system t, l, define a coordinate velocity of light $\tilde{c}$ and sketch it as a function of r for both signs σ_2. Over what range of r is this graph meaningful? To choose the correct sign σ_2, consider an electromagnetic plane wave passing a massive body.