General Relativity and Cosmology

Prof. John Schliemann
Dr. Paul Wenk

Mo. H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
Thu. 1pm c.t., PHY 9.1.10

Sheet 7

---

1) Harmonic Gauge .... [6P]

The harmonic gauge is defined by $\Gamma^\lambda = g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$.

(a) A function ψ is harmonic if it satisfies $\Box\psi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\psi = 0$. Show that in case of harmonic gauge the coordinates x^η itself are harmonic functions.

(b) Recall Einstein's equations for a small perturbation $h_{\mu\nu}$. Show that if harmonic gauge condition is satisfied one finds up to first oder in $h_{\mu\nu}$

$$h^\mu{}_{\nu,\mu} = \frac{1}{2}h^\mu{}_{\mu,\nu}. \tag{1}$$

2) Gravitational Waves: Quadrupole Radiation .... [12P]

Assume a finite mass distribution located in a sphere of radius r_0 which is oscillating in time with the frequency ω. The stress-energy tensor is given by

$$T_{\mu\nu}(\mathbf{r},t) = T_{\mu\nu}(\mathbf{r})\exp(-i\omega t) + \text{c.c.} \begin{cases} \neq 0 : r \leq r_0 \\ = 0 : r > r_0 \end{cases} \tag{2}$$

In the following, use the retarded potentials for weak fields $h_{\mu\nu}(\mathbf{r},t)$,

$$h_{\mu\nu}(\mathbf{r},t) = -\frac{4G}{c^4}\int d^3r' \frac{S_{\mu\nu}(\mathbf{r}', t - |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}, \tag{3}$$

with $S_{\mu\nu} = T_{\mu\nu} - \eta_{\mu\nu}T/2$. We know further that for $r \gg r_0$ the retarded potentials can be approximated by

$$h_{\mu\nu}(\mathbf{r},t) \approx e_{\mu\nu}(\mathbf{r},\omega)e^{-ik_\lambda x^\lambda} + \text{c.c} \tag{4}$$

with the gravitational wave amplitude

$$e_{\mu\nu}(\mathbf{r},\omega) = -\frac{4G}{c^4 r}\left(T_{\mu\nu}(\mathbf{k}) - \frac{T(\mathbf{k})}{2}\eta_{\mu\nu}\right) \tag{5}$$

and $(k^\lambda) = \left(\frac{\omega}{c}, \mathbf{k}\right)$.

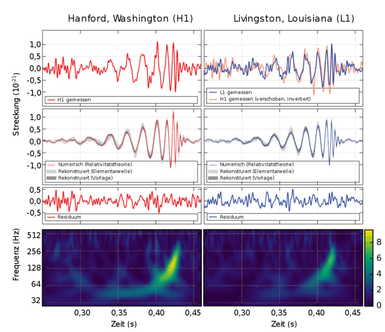

Figure 1: LIGO measurement of the gravitational waves at the Hanford (left) and Livingston (right) detectors, compared to the theoretical predicted values.[1]

(a) Assuming further $r \gg (1/k_\mu)$, show that the energy flow P through the area element $r^2 d\Omega$ is given by

$$dP = \frac{G\omega^2}{\pi c^5}\left(T^{\mu\nu}(\mathbf{k})^* T_{\mu\nu}(\mathbf{k}) - \frac{1}{2}|T(\mathbf{k})|^2\right) d\Omega \tag{6}$$

and thus $k_\nu T^{\mu\nu}(\mathbf{k}) = 0$.

(b) Show that the result in (a) can be rewritten in the following form

$$dP = \frac{G\omega^2}{\pi c^5} \Lambda_{ijlm} T^{ij}(\mathbf{k})^* T^{lm}(\mathbf{k}). \tag{7}$$

Here, the temporal components of the stress-energy tensor have been replaced by spatial components. The function Λ_{ijlm} depends only on $\mathbf{k}/k$.

Hint: Since we assume weak fields: $T^{\mu\nu}{}_{,\nu} = 0$. *To rewrite* dP *into the form Eq.(7) use* $T^{i0} = -\hat{k}_j T^{ij}$ *and* $T^{00} = \hat{k}_i \hat{k}_j T^{ij}$ *with* $\hat{k}_i = k_i/k_0$.

(c) Using the long wavelength approximation we simplify

$$T^{ij}(\mathbf{k}) \approx \int d^3r\, T^{ij}(\mathbf{r}) =: -\frac{\omega^2}{2} Q^{ij}. \tag{8}$$

Show that the total power P radiated by this mass system is give by

$$P = \frac{2G\omega^6}{5c^5} \left(\sum_{i,j=1}^{3} |Q^{ij}|^2 - \frac{1}{3} |\sum_{i=1}^{3} Q^{ii}|^2 \right). \tag{9}$$

Hint: Show that $Q^{ij} = \int d^3r\, x^i x^j \rho(\mathbf{r})$ *and use* $T^{00} \approx \rho c^2$. *Further, for the integration over the solid angle* Ω *use the fact that in spherical coordinates* Λ_{ijlm} *only depends on* θ *and* ϕ.

---

[1]B. P. Abbott et al.(LIGO Scientific Collaboration and Virgo Collaboration) - full list at the end of the article - http://physics.aps.org/featured-article-pdf/10.1103/PhysRevLett.116.061102