General Relativity and Cosmology

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Mo. H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
Thu. 1pm c.t., PHY 9.1.10

Sheet 6

1) Killing Vectors and Conservation Laws .... [5P]

Regarding the stress-energy tensor $T^{\alpha\beta}$ we learned in Special Relativity the conservation law ${T^{\alpha\beta}}_{,\beta} = 0$. A covariant expression of this equation however leads to

$$(\sqrt{-g}T^{\mu\nu})_{,\mu} = -\sqrt{-g}\Gamma^{\nu}_{\lambda\mu}T^{\mu\lambda} \tag{1}$$

which prevents a definition of conserved four-momentum as in the case of globally flat spacetime.

(a) Prove Eq.1 and

(b) show that a conserved quantity can be defined (the right hand side of Eq.1 vanishes) if the spacetime admits a Killing vector $\boldsymbol{\xi}$.
Hint: Consider instead of ${T^{\eta\nu}}_{;\nu}$ the quantity $(\xi_\eta T^{\eta\nu})_{;\nu}$.

2) Geodesics .... [3P]

Show that the quantity $g_{\alpha\beta}u^\alpha u^\beta$ is constant along a geodesic.

3) Parallel Transport on a Sphere .... [6P]

A path S on a 2-sphere with $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$ is given by the following sequence of points which are connected by paths:

$$S:\, P_1 \to P_2 \to P_3 \to P_4 \to P_1 \tag{2}$$

with $P_1 = (\theta = \pi/2, \varphi = 0)$, $P_2 = (\epsilon, 0)$, $P_3 = (\epsilon, \pi/2)$, $P_4 = (\pi/2, \pi/2)$. All but one path, $P_2 \to P_3$ where θ is kept constant, are geodesics.

(a) Write down the differential equation for the geodesics. Show that great circles are solutions.

(b) Calculate the parallel transport of the vector $\mathbf{A} = \mathbf{e}_\varphi$, $\delta\mathbf{A}$, along S starting at P_1. Assume $0 < \epsilon \ll 1$.

4) Parallel Transport and Curvature .... [6P]

We are now in the position to generalize the finding in Ex.3: Show that the parallel transport of a vector A_i along an infinitesimally small loop can be connected to the curvature (induced by the metric) via

$$\delta A_\gamma \approx \frac{1}{2} A_\mu {R^\mu}_{\gamma\lambda\nu} \oint dx^\lambda x^\nu, \tag{3}$$

with the Riemann curvature tensor ${R^\mu}_{\gamma\lambda\nu}$.
Hint: Consider a locally inertial system at some x_0 and Taylor expand the Christoffel symbols. In addition, one can assume A_μ to be constant in the vicinity of the infinitesimaly small loop.