General Relativity and Cosmology

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Mo. H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
Thu. 1pm c.t., PHY 9.1.10

Sheet 5

1) Rotating System .... [7P]

Write down the equations of motion

$$\frac{d^2x^\kappa}{d\tau^2} = -\Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

for a rotating system as described in Ex. 2 of sheet 4. By dropping all terms of the order of $\mathcal{O}(v^2/c^2)$ and higher, identify the Coriolis and centrifugal force.

2) Covariant Derivative .... [4P]

Calculate the covariant derivative of the field tensor $A^{\alpha\beta}{}_{\nu\gamma\delta}$.

3) Lie Derivative .... [10P]

Consider a coordinate transformation $x^\mu \mapsto x'^\mu = x^\mu - \epsilon^\mu$. Thus, we have

$$A'^\mu(x) = A^\mu(x) - A^\nu(x)\partial_\nu\epsilon^\mu + \epsilon^\nu\partial_\nu A^\mu(x) + \mathcal{O}(\epsilon^2). \tag{1}$$

Now, the *Lie derivative* (Sophus Lie 1842-1899) measures the change $A'^\mu - A^\mu$. More general: It evaluates the change of a tensor field, along the flow of another vector field. It is defined for contravariant components as (see lecture notes)

$$\mathcal{L}_{\boldsymbol{\epsilon}} A^\mu := A'^\mu - A^\mu + \mathcal{O}(\epsilon^2) \tag{2}$$

$$= \epsilon^\nu\nabla_\nu A^\mu - A^\nu\nabla_\nu\epsilon^\mu + \mathcal{O}(\epsilon^2) \equiv \epsilon^\nu A^\mu{}_{;\nu} - A^\nu\epsilon^\mu{}_{;\nu} + \mathcal{O}(\epsilon^2), \tag{3}$$

$$\text{and } \mathcal{L}_{\boldsymbol{\epsilon}} A_\mu := A'_\mu + A_\mu + \mathcal{O}(\epsilon^2) \tag{4}$$

$$= \epsilon^\nu\nabla_\nu A_\mu + A^\nu\nabla_\mu\epsilon_\nu + \mathcal{O}(\epsilon^2) \equiv \epsilon^\nu A_{\mu;\nu} + A^\nu\epsilon_{\nu;\mu} + \mathcal{O}(\epsilon^2). \tag{5}$$

(a) If the Lie-derivative of a metric tensor $g_{\alpha\beta}$ along the direction of a vector-field $\boldsymbol{\xi}$ vanishes, $\mathcal{L}_{\boldsymbol{\xi}} g_{\alpha\beta} = 0$, $\boldsymbol{\xi}$ is called *Killing vector.*

 (i) Show that $g_{\alpha\beta,\mu}\xi^\mu + g_{\nu\beta}\xi^\nu_{,\alpha} + g_{\alpha\nu}\xi^\nu_{,\beta} = 0$ is equivalent to $\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0$.
 Hint: $\boldsymbol{\xi}$ defines a symmetry if an infinitesimal translation along $\boldsymbol{\xi}$ leaves ds^2 invariant: $\delta(ds^2) = 0$. Assume $\boldsymbol{\xi}$ to be a tangent vector to a curve $x^\alpha(\lambda)$, $\xi^\alpha = dx^\alpha/d\lambda$. Use (and show) $\delta(dx^\alpha) = d\xi^\alpha\delta\lambda$ and $\delta g_{\alpha\beta} = g_{\alpha\beta,\mu}\xi^\mu\delta\lambda$.

 (ii) Calculate the Killing vectors of a spherical surface with radius equal to one, $ds^2 = d\theta^2 + \sin^2(\theta)d\varphi^2$.
 Hint: You'll find three independent parameters allowing for the definition of corresponding Killing vectors.

 (iii) Calculate the Killing vectors of Minkowski spacetime. Show that the general solutions are given by $\xi_\mu = c_\mu + b_{\mu\gamma}x^\gamma$ with constants c_μ, $b_{\mu\gamma}$. Furthermore, show that this flat spacetime has ten linearly independent Killing vector fields (corresponding to the ten generators of the Poincaré algebra).

(b) Show that if $\xi_\mu(x)$ is a Killing vector field and $v^\eta(\lambda)$ is a tangent vector to a geodesic curve $m(\lambda)$ then $v^\eta\xi_\eta(x)$ is constant along m.