General Relativity and Cosmology

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Mo. H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
Thu. 1pm c.t., PHY 9.1.10

Sheet 4

1) Klein-Gordon Field .... [9P]

The simplest classical field theory is one with a scalar real-valued field ϕ of a point particle with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2. \tag{1}$$

(a)(2P) Show that the Euler-Lagrange equation is the Klein-Gordon equation for the field ϕ.

(b)(2P) Calculate the conjugate momentum $\Pi(x)$ and with this the Hamiltonian density $\mathcal{H}$.

(c)(2P) Based on *Noether's theorem*, calculate the energy-momentum tensor $\theta^{\mu\nu}$ and write down the conserved quantities associated with time and spatial translations. Relate $\mathcal{H}$ to a component of the Noether current.

(d)(3P) Show explicitly that $\partial_\mu \theta^{\mu\nu} = 0$ when the Euler-Lagrange equation is fulfilled for this field (as expected from *Noether's theorem*).

2) Rotating Reference Frame .... [6P]

Assume a reference frame which is constantly rotating about the z-axis with angular velocity ω.

(a)(1P) Write down the metric tensor $g_{\mu\nu}$ and ds^2 for the rotating reference frame S' in cylindrical coordinates $\{ct', r', \varphi', z'\}$.

(b)(3P) Transform into a system $\tilde{S}$ with *time-orthogonal coordinates* $\{c\tilde{t}, r', \varphi', z'\}$, i.e., $g_{00} > 0$, $g_{0i} = 0$ and $g^{00} > 0$ $g^{0i} = 0$ respectively. This allows us to define a length

$$d\tilde{l} = \sqrt{-ds^2}|_{d\tilde{t}=0}$$

of a scale at rest in the rotating system. *Hint: Define a $d\tilde{t}$ in such a way that you get rid of the term $\propto d\varphi' dt'$.* Why are here time-orthogonal coordinates necessary?

(c)(2P) Calculate the ratio of the circumference $\tilde{U}$ of a circle and its radius r' in $\tilde{S}$.

3) Locally Flat Coordinate System .... [11P]

Consider a coordinate system $\mathcal{S}'$ with coordinates x'^μ and metric $\tilde{g}_{\mu\nu}(x')$. We require

$$\left.\frac{\partial \tilde{g}_{\mu\nu}}{\partial x'^\lambda}\right|_{x'=0} = 0. \tag{2}$$

Assume further an expansion

$$x'^\mu = x^\mu + \frac{1}{2}\Gamma^\mu_{\rho\sigma} x^\rho x^\sigma + \ldots \tag{3}$$

in coordinates x of a local Minkowski system, i.e. $g_{\mu\nu}(x=0) = \eta_{\mu\nu}$ with coefficients (*Christoffel symbols*) which obey the symmetry condition $\Gamma^\mu_{\rho\sigma} = \Gamma^\mu_{\sigma\rho}$.

(a)(3P) By using the transformation between $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ and keeping only terms of zero and first order in x, show that for $x'^{\mu} = 0$ one gets $\tilde{g}_{\mu\nu}(0) = \eta_{\mu\nu}$, i.e., also in $\mathcal{S}'$ we find locally a Minkowski metric for $x'^{\mu} = 0$.

(b)(3P) Show that $\partial/\partial x^{\lambda} = \partial/\partial x'^{\lambda} + \mathcal{O}(x)$ and use, by requiring Eq.2, this relation to prove

$$\eta_{\mu\beta}\Gamma^{\mu}_{\lambda\alpha} + \eta_{\alpha\nu}\Gamma^{\nu}_{\lambda\beta} = \frac{\partial}{\partial x^{\lambda}} g_{\alpha\beta}\bigg|_{x=0} . \tag{4}$$

(c)(3P) Using the equations derived in (b), show that the *Christoffel symbols* $\Gamma^{\mu}_{\alpha\beta}$ have to fulfill the relation

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\lambda}\left(\frac{\partial}{\partial x^{\beta}} g_{\lambda\alpha} + \frac{\partial}{\partial x^{\alpha}} g_{\beta\lambda} - \frac{\partial}{\partial x^{\lambda}} g_{\alpha\beta}\right), \tag{5}$$

in order to fulfill Eq.2.
Hint: Use cyclic permutation of indices α, β, λ and make use of the symmetry property of the Christoffel symbols to combine the Eqs.4.

(d)(2P) Why isn't it possible to define coordinates x' where not only the first derivative, Eq.2, but also the second one vanishes?
Hint: In general, 20 derivatives of second order do not vanish.