General Relativity and Cosmology

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Mo. H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
Thu. 1pm c.t., PHY 9.1.10

Sheet 3

1) Infinitesimal Lorentz Transformation .... [3P]

The infinitesimal Lorentz transformation is given by $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$. Show that $\omega_{\mu\nu} = -\omega_{\nu\mu}$ (up to $\mathcal{O}(\omega^2)$).

2) Energy Flux from a Star .... [7P]

A spherical star has total luminosity L. An observer is located at a distance r from its center and is at rest relative to the star. Assume flat space-time and that the radius R of the star can be neglected ($r \gg R$). Thus, the observer measures the energy flux $\phi = L/(4\pi r^2)$ from the star.

(a)(3P) Write down the energy-momentum tensor $T^{\mu\nu}$ in spherical coordinates (centered at the star, $\mu, \nu \in \{r, \theta, \varphi\}$). *Hint: Since $r \gg R$, assume the electromagnetic waves to be plane, thus, $E = E_\theta$, $H = H_\varphi$, with $\sqrt{\epsilon_0}E = \sqrt{\mu_0}H$.*

(b)(4P) The observer is moving now with velocity $\mathbf{v}$. Define the component of $\mathbf{v}$ along the radial direction as $v_{\parallel}$ and the one perpendicular to this as $v_{\perp}$. Show that the energy flux ϕ' the observer measures is given by

$$\phi' = \frac{L}{4\pi r^2}\left(1 - \frac{v_{\parallel}}{c}\right)^2 \left(1 - \frac{v_{\parallel}^2 + v_{\perp}^2}{c^2}\right)^{-1}. \tag{1}$$

Hint: Notice that $\mathbf{E} \cdot \mathbf{B}$ is invariant under Lorentz transformation. Use the Poynting vector.

3) Lienard-Wiechert Potential .... [8P]

The potentials of a particle with charge q moving along the path $\mathbf{r}_0(t)$ are given by

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R \mp \mathbf{R}\cdot\mathbf{u}(t')/c}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi}\frac{\mathbf{u}(t')}{R \mp \mathbf{R}\cdot\mathbf{u}(t')/c}. \tag{2}$$

The sign is according to the advanced $(+)$ and retarded $(-)$ potential with $\mathbf{R} = \mathbf{r} - \mathbf{r}_0(t')$, $\mathbf{u}(t') = d\mathbf{r}_0(t')/dt'$, and the time t' of the field-emission at particle-coordinates, $t' = t \mp |\mathbf{r} - \mathbf{r}_0(t')|/c$. The trajectory of the particle can be parametrized by the proper time, $\{X_0^\alpha(\tau)\} = \{ct_0(\tau), \mathbf{r}_0(\tau)\}$.

(a)(4P) Show that the vector field can be represented in a covariant form given by

$$A^\alpha_{\text{ret/adv}}(X) = \frac{\mu_0 q}{4\pi}\frac{U^\alpha}{\rho_{\text{ret/adv}}}, \tag{3}$$

with $U^\alpha(\tau) = \dot{X}_0^\alpha(\tau)$ and a Lorentz-invariant scalar $\rho_{\text{ret/adv}}$.

(b)(4P) Calculate the associated electromagnetic field tensor $F^{\mu\nu}$.
Hint: Keep in mind $\tau = \tau(X)$ and use the fact that for both R^α_{ret} and R^α_{av} we have $R_\lambda R^\lambda = {R^0}^2 - \mathbf{R}^2 = 0$. Calculate the differential of this scalar.