General Relativity and Cosmology

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Mo. H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
Thu. 1pm c.t., PHY 9.1.10

Sheet 2

1) Tensors .... [5P]

(a) Show that a contraction of two indices of the same type, i.e. covariant or contravariant, does in general not yield a tensor.

(b) $X^{\mu\nu}$ fulfills the following equations in the Euclidean two-dimensional space: $X^{\mu\nu} = X^{\nu\mu}$, $X^{\mu}_{\mu} = 0$, $X^{\nu\mu}A_{\mu} = B^{\nu}$ with the parameters A^{μ}, B^{μ} fulfilling $A^{\mu}B_{\mu} = 0$. Find X.

(c) Proof the relation $\frac{\partial}{\partial A^{ij}}\det(A) = \det(A)A^{-1}_{ji}$ with $A^{ij}A^{-1}_{jk} = \delta^{i}_{k}$.

(d) Show that in general $A^{\mu}{}_{\nu} \neq A_{\nu}{}^{\mu}$ (*Hint: antisymmetric tensor*)

(e) Assume a trasformation $\Omega = \mathbb{1} + \omega + \mathcal{O}(\omega^2)$, $\omega \in \mathbb{R}$ and $x^{\mu} \underset{\Omega}{\longrightarrow} \tilde{x}^{\mu} = x^{\mu} + \alpha^{\mu} \cdot \omega + \mathcal{O}(\omega^2)$. Show that

$$\det\left(\frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}}\right) = 1 + (\partial_{\mu}\alpha^{\mu}) \cdot \omega + \mathcal{O}(\omega^2).$$

2) Accelerated Motion .... [3P]

An observer moves with constant acceleration with respect to the x-direction of some inertial frame, i.e. its 4-vector of acceleration $a^{\mu} := du^{\mu}/ds$, with $u^{\mu} = dx^{\mu}/ds$, fulfills $a^{\mu}a_{\mu} = -a^2/c^4$. The initial condition is $u(0) = (1, 0, 0, 0)$.

(a) Calculate $u^{\mu}(\tau)$, $a^{\mu}(\tau)$ as a function of proper time τ.

(b) What is the velocity $v(\tau) = dx/dt$ of the observer measured in the inertial frame?

(c) Calculate $u'^{\mu} = \Lambda(-v(\tau))^{\mu}{}_{\nu}u^{\nu}$, a'^{μ}.

3) Rocket Journey .... [4P]

A rocket of total mass M at rest in an inertial system IS starts accelerating by emitting fumes (assume a constant leakage of fume). In the rest frame of the rocket the fume particles are emitted at a constant speed u'. After a fraction ΔM of the rocket has been ejected, the rocket has reached its maximal velocity $u_{\max}$ as measured in IS. Calculate $u_{\max}$ as a function of u' and ΔM.

4) Charged Particle in the Electromagnetic Field .... [5P]

A charged particle with mass m and charge q is subjected to an arbitrary electromagnetic field. Calculate the four-acceleration of the particle

(a) in the inertial system S' where it is initially at rest, for this moment,

(b) in the inertial system S: S' moves relatively to S with speed v in the z-direction.