# General Relativity and Cosmology

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**Mo.** H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
**Thu.** 1pm c.t., PHY 9.1.10

## Sheet 11

### Chandrasekhar Limit ........................................ [12P]

The goal of this exercise is to calculate the maximum mass $M_C$ of a stable white dwarf star. To accomplish this, we recall the star model in the Newtonian limit. We apply the polytropic equation of states.

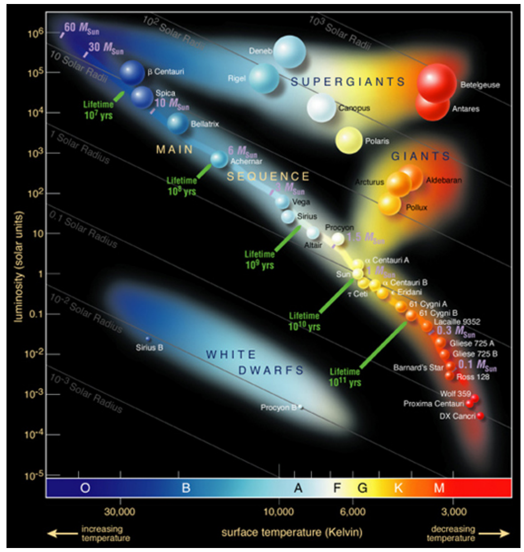


Figure 1: Hertzsprung-Russel Diagram.[1]

(a)(6P) Assume the polytrope index $n = 1/(\gamma - 1)$ to be $n < 5$. Let $x_1$ be the first zero of the Lane-Emden-function $\Theta$. Express the star radius $R$ in terms of the star density $\rho_0 \equiv \rho(r = 0)$, $\gamma$ and $K$ with preasure $P = K\rho^\gamma$. Show that the mass of the star can be written as

$$M = 4\pi\rho_0^{(3\gamma-4)/2} \left(\frac{K\gamma}{4\pi G(\gamma - 1)}\right)^{\frac{3}{2}} x_1^2 |\Theta'(x_1)|. \quad (1)$$

Find the numerical values for $x_1$ and $\Theta'(x_1)$ for $\gamma = 5/3$ and $\gamma = 4/3$. *Hint: There are many numerical tools like Mathematica[2], Matlab,... Use, e.g., a power series expansion of $\Theta$ around zero to solve the problem.*

(b)(3P) To understand the stability of a white dwarf we have to understand the equilibrium between the hydrostatic pressure and the pressure of the degenerate Fermi gas. Show that the electron degeneracy pressure $P$ is given by

$$P = \frac{m_e^4 c^5}{\pi^2 \hbar^3} \left(\frac{x_F^3}{3}\sqrt{1 + x_F^2} - f(x_F)\right), \quad (2)$$

with the electron rest mass $m_e$, $x_F := p_F/(m_e c)$, where $p_F$ is the Fermi momentum, and the function $f$ which is defined by $f(x) = \int_0^x d\,x' x'^2 \sqrt{1 + x'^2}$. Recall that, since we assume a degenerate electron gas, we have $T = 0$ and all electron states with $|\mathbf{p}| \leq p_F$ are filled. Use the thermodynamic relation $dE = Td\,S - Pd\,V$ where $E$ is the energy of the electron gas in our case.

(c)(3P) By expanding $f(x_F)$ for $x_F \ll 1$ and for $x_F \gg 1$ and using the previous results, show that the polytropic equation of state is given by

$$P = \begin{cases} K_1 \rho^{5/3}, & \rho \ll \rho_c, \\ K_2 \rho^{4/3}, & \rho \gg \rho_c, \end{cases} \quad (3)$$

where the two extrema of $x_F$ have been related to the density via $x_F \sim \rho^{1/3}$. The characteristic density is given by $\rho_c = (\sigma m_n/(3\pi^2\hbar^3))(m_e c)^3$ with $\sigma$ being the average number of nucleons (mass $m_n$) per electron.

(d)(3P) Give a simple hint why $\gamma \geq 4/3$ is a stability condition by examining the dependence of the sum of gravitational energy $E_{\text{grav}}$ and inner energy $E_{\text{mat}} \approx PV$ on the radius $R$ of a star.

(e)(3P) Using the results from (a), show that

$$M = \begin{cases} \frac{2.79}{\sigma^2} \left(\frac{\rho_0}{\rho_c}\right)^{\frac{1}{2}} M_\odot & \rho_0 \ll \rho_c, \\ \frac{5.87}{\sigma^2} M_\odot := M_C & \rho_0 \gg \rho_c, \end{cases} \tag{4}$$

$$R = \begin{cases} \frac{2.0}{\sigma} \left(\frac{\rho_c}{\rho_0}\right)^{\frac{1}{6}} 10^4\text{km} & \rho_0 \ll \rho_c \\ \frac{5.33}{\sigma} \left(\frac{\rho_c}{\rho_0}\right)^{\frac{1}{3}} 10^4\text{km} & \rho_0 \gg \rho_c. \end{cases} \tag{5}$$

Calculate critical $M_C$, $\rho_c$ and radius for a white dwarf star assuming that it consists of helium and carbon. Is it legitimate to use the non-relativistic limiting case of the Oppenheimer-Volkoff equation?
*Hint: Which radius is relevant in this case?*

(f)(2P) We measure the absolute luminosity $L$ of a white dwarf star to be $L = 10^{-4} L_\odot$. Its spectrum appears to be white. Give an approximation of its radius $R$.

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[1] By ESO [CC BY 4.0 (http://creativecommons.org/licenses/by/4.0)], via Wikimedia Commons
[2] Wolfram Demo: Lane Emden Equation In Stellar Structure