General Relativity and Cosmology

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Mo. H34 12pm c.t. & **Wed.** PHY 9.2.01, 1pm c.t.
Thu. 1pm c.t., PHY 9.1.10

Sheet 10

1) Collapsing Binary Star System .... [12P]

In a binary star system two stars with masses M_1 and M_2 rotate on Kepler paths. This rotation causes gravitational radiation.

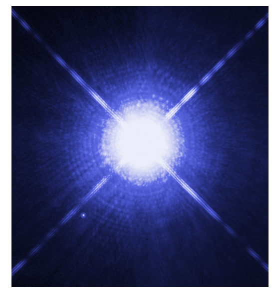

Figure 1: Hubble image of the Sirius binary system, in which Sirius B can be clearly distinguished (lower left).[1]

(a) First, we connect the radiation of a general mass distribution $\rho(\mathbf{r})$ with its moments of inertia. Recall that the quadrupole moment of $\rho(\mathbf{r})$ is defined by $(\theta_{ij}) = \int d^3r\, x_i x_j \rho(\mathbf{r})$. We can always find a coordinate system (x'_i) fixed to the body with $\theta' = \mathrm{diag}(I_1, I_2, I_3)$. Assume the system to rotate around the x'_3-axis with angular velocity ω. The quadrupole tensor transformed into an inertial system (x_i) where $x_3 = x'_3$ can be written in the form

$$\theta_{ij}(t) = \text{const.} + (q_{ij} e^{-2i\omega t} + \text{c.c.}). \tag{1}$$

Use these amplitudes q_{ij} of the oscillating quadrupole distribution to show that the radiation power P of this rotating mass distribution is given by

$$P = \frac{32 G \omega^6}{5c^5} \epsilon^2 I^2, \qquad \text{with } \epsilon = (I_1 - I_2)/(I_1 + I_2) \text{ and } I = I_1 + I_2. \tag{2}$$

Note: Using different literature one should not mix the *second moment of mass distribution* θ_{ij} as used above with the *trace-free part of the second moment of the mass distribution* $\tilde{\theta}_{ij} = \int d^3r (x_i x_j - (1/3)\delta_{ij} r^2)\rho(\mathbf{r})$ or the values $Q_{zz} = \int d^3x (3z^2 - r^2)\rho(\mathbf{r})$ which are called quadrupole moments in the textbook of Landau and Lifshitz.

(b) Consider a binary star system with $M_1 \approx M_2$ where the objects surround each other on circular orbits. Using the result above, calculate the relative energy loss $\Delta E/|E|$ per revolution as a function of distance r and orbital speed v of the stars.

(c) Which time T_c elapses until the binary star system collapses due to gravitational radiation? Calculate T_c for the binary star system PSR 1913+16: The orbital period is $T \approx 7.75h$, the masses $M \approx 1.4 M_\odot$. Assume, although it's a rough approximation, a circular orbit.

2) Robertson Expansion .... [8P]

The Robertson expansion is defined by

$$A(r) = 1 + \gamma \frac{r_s}{r} + \ldots, \tag{3}$$

$$B(r) = 1 - \frac{r_s}{r} + \frac{(\beta - \gamma)}{2} \left(\frac{r_s}{r}\right)^2 + \ldots, \tag{4}$$

$$\frac{1}{B(r)} = 1 + \frac{r_s}{r} + \frac{(2 - \beta + \gamma)}{2} \left(\frac{r_s}{r}\right)^2 + \ldots, \tag{5}$$

with $A(r)$ and $B(r)$ being the coefficients in the standard form and r_s the Schwarzschild radius. Similar to the lecture notes, show, by using the Robertson expansion, that the precession per orbit for the perihelion is given by

$$\Delta\varphi \equiv 2(\varphi_+ - \varphi_-) - 2\pi = \frac{\pi r_s}{L}(2 - \beta + 2\gamma). \tag{6}$$

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[1]NASA, ESA, H. Bond (STScI), and M. Barstow (University of Leicester).