Brauer group and Severi-Brauer varieties

WiSe 2022/2023, University of Regensburg

Contents

The seminar is dedicated to a topic that connects Galois theory, simple (non-commutative) algebras and algebraic geometry.

A central simple algebra over k is a finite-dimensional associative algebra over k with the center equal to k and that has no non-trivial two-sided ideals. Examples of these are Hamilton quaternions over $\mathbb{R}$ and matrix algebras over any base field. One can use tensor multiplication over k to define product of such algebras and form a monoid. After imposing what's known as Morita equivalence one obtains the Brauer group of the base field k.

A Severi-Brauer variety is a smooth projective variety X defined over a field k such that after base change to an algebraic closure $\bar{k}$ it becomes isomorphic to a projective space. For example, a conic (i.e. a smooth projective curve in $\mathbb{P}^2$ of degree 2) is the main of example of a Severi-Brauer variety of dimension 1.

And finally from the point of view of Galois theory we will be interested in second Galois cohomology, i.e. the cohomology of the absolute Galois group of the base field k. Note that when studying this, one forgets the field k itself and working just with the absolute Galois group.

It will be our goal to understand that there is a one-to-one (if properly explained) correspondence between objects defined above. This opens up a possibility of using methods of one area to the other: for example, of understanding conics via quaternion algebras, or of using cohomological techniques for a better understanding of algebraic geometry of certain varieties.

Despite broad scope of the seminar, it should be accessible for students who have basic knowledge of algebra, Galois theory and some acquaintance with algebraic geometry. At least in the beginning of the seminar we will follow the book by Gille and Szamuely (see the list of the references, so it is possible) Moreover, the schedule could be slightly adapted along the way depending on the prerequisites of the students.

Literature

- Gille P., Szamuely T. "Central Simple Algebras and Galois Cohomology", Vol. 165. Cambridge University Press, 2017.
- Artin M. "Brauer-Severi varieties." Brauer groups in ring theory and algebraic geometry. Springer, Berlin, Heidelberg, 1982. 194-210.
- Kollár J. "Severi-Brauer varieties; a geometric treatment." arXiv preprint arXiv:1606.04368 (2016).
- Milne, James S. "Étale Cohomology (PMS-33), Volume 33." Étale Cohomology (PMS-33), Volume 33. Princeton university press, 2016.

Recommended previous knowledge

Galois theory, basic commutative algebra and acquaintance with algebraic geometry.

Plan of the seminar

Talk 0 (Pavel, 26.10)	Introduction and overview
Talk 1 (Lucia, 02.11)	Quaternion algebras
Talk 2 (Lorenzo, 09.11)	Central simple algebras, cyclic algebras
Talk 3 (Bowen, 16.11)	Galois descent and Brauer group I
Talk 4 (Benni, 23.11)	Cohomology of (profinite) groups and infinite Galois theory
Talk 5 (Xuefei, 30.11)	Brauer group II
Talk 6 (Pavel, 07.12)	Recap of algebraic geometry
Talk 7 (Benni, 14.12)	Severi-Brauer varieties
Talk 8 (, 11.01)	Severi-Brauer varieties II
Talk 9 (, 18.01)	Cohomological dimension and C_1-fields
Talk 10 (, 25.01)	Residue maps for Brauer groups
Talk 11 (, 01.02)	Unramified Brauer group of a smooth variety and rationality
Talk 12* (, 08.02)	TBD: Additional talk if there are enough participants

Detailed plan of the seminar

All references below (if not stated otherwise) refer to the book of Gille, Szamuely. If there is no additional indication, the reference is the same for both the first (published in 2006) and the second (published in 2017) editions, otherwise 1.1.1 1st/ 1.1.2 2nd means 1.1.1 in the first edition and 1.1.2 in the second edition.

Please present additional material only if there is time left after discussing the material in the book.

0 Introduction and overview

(Pavel)

I will make an overview of the goals and methods of this seminar.

1 Quaternion algebras

Quaternion "numbers" were invented by Hamilton in order to understand the rotations in three-dimensional space, but from algebraic point of view they are just a 4-dimensional algebra over $\mathbb{R}$. It turns out that one can define 4-dimensional algebras by similar relations over any field k, just take i, j as free generators and impose the following relations:

$$i^2 = a, j^2 = b, ij = -ji,$$

where $a, b \in k$. It turns out that this algebra can be studied through the geometry of the projective algebraic curve $ax^2 + by^2 = z^2$ in $\mathbb{P}^2$.

The goal of this talk is to introduce all these objects and prove the first fundamental results about them that will be generalized in later talks.

The main results are Prop. 1.1.7, 1.2.1, 1.2.3, 1.3.2. State and prove the Theorem of Witt (1.4.2). If you have time, also discuss Theorem of Albert (1.5.5) and state Merkurjev's theorem (1.5.8).

2 Central simple algebras, cyclic algebras

With this talk we start to study one of the main object of the seminar, central simple algebras. These are generalizations of quaternion algebras and of matrix algebras $M_n(k)$ over a field k.

Prove the Wedderburn's theorem (2.1.3), Theorem 2.2.1, Theorem 2.2.7 2nd and Corollary 2.2.6 1st/2.2.12 2nd (you should assume that the base field is perfect).

To give some interesting examples explain Proposition 2.5.2 that gives a presentation of cyclic algebras (as we do not discuss cocycles at this point, you just have to show that the given algebra is indeed central simple).

If time permits, prove 2.5.3 and 2.5.5. State 2.5.7.

3 Galois descent and Brauer group I

This talk introduces the techniques of Galois descent, namely, how the classification of "forms" over k of K-linear objects can be studied through 1-cocycles with coefficients in the automorphism group of this object (here K/k is a Galois extension of fields). This part should include all, or almost all results of 2.3.

After that you can apply this to define the Brauer group (section 2.4) and prove its basic properties.

4 Cohomology of (profinite) groups and infinite Galois theory

This talk might be divided into two!

The goal of this talk is to introduce/recall cohomology of groups (and profinite groups) with their basic properties. This material is covered with many details in Chapter 3, however, there will be not enough time to go through all of them.

Cohomology of profinite groups is just a correct way to extend considerations for discrete groups into this topological world. One should explain how the proofs are generally done but not reprove all the things.

The following topics should be covered:

- definition, methods of computations,
- long exact sequences in modules;
- restriction-corestriction and Shapiro's lemma;
- cohomology of (pro-)finite groups in $\mathbb{Q}$-modules.

You should also provide a computation of cohomology of cyclic groups via explicit resolutions, and explain Kummer and Artin-Scherier theory in terms of Galois cohomology.

To relate this abstract nonsense to the questions at hand, you have to explain how the absolute Galois group is naturally a profinite group, and also the "Infinite Galois theory", Theorem 4.1.10.

5 Brauer group II

This talk follows sections 4.4 and 4.5.

The goal of this talk is to relate the Brauer group previously defined via central simple algebras to the second cohomology of the absolute Galois group with coefficients in an abelian module. As cohomology with abelian modules exist in all degrees, this approach gives many new tools that allow to prove results about the Brauer group via cohomological techniques.

If time permits, one should also explain the "cohomological meaning" of cyclic algebras introduced in Talk 2 via Prop. 4.7.1. Deduce corollaries 4.7.4-.4.7.6 1st/4.7.4, 4.7.5, 4.7.7 2nd.

6 Recap of algebraic geometry

(Pavel)

I will remind necessary constructions and methods of algebraic geometry that are needed to work with Severi-Brauer varieties.

7 Severi-Brauer varieties

This talk follows sections 5.1 and 5.2.

Severi-Brauer varieties are smooth projective varieties over a field k such that over an algebraic closure of k they are isomorphic to a projective space.

It turns out that such varieties up to isomorphism are in a 1-to-1 correspondence with the elements of the cohomology group $H^1(k, PGL_n)$ (Theorem 5.2.1). This is the reason why will be interested in the geometry of these varieties, for example, they are isomorphic to a projective space iff they have a rational point (Theorem 5.1.3),

Additional material:

- There is another construction of Severi-Brauer varieties from a central simple algebra, see Remark 5.2.5 (1 in 2nd) and the references there.
- There is also a nice geometric construction of a central simple algebra from a Severi-Brauer variety, see Kollár, Corollary 10.

8 Severi-Brauer varieties II

This talk follows sections 5.3 and 5.4.

In the previous talk it was shown that a Severi-Brauer variety X gives a class $[X]$ in the Brauer group of the base field k. The first result of this talk is to explain in geometric terms, when two Severi-Brauer varieties yield the same classes (Theorem 5.3.3).

Another fundamental result to be covered is the following result of Amitsur that connects the birational properties of X with the class $[X]$ in $Br(k)$ (Theorem 5.4.1). The techniques of the proof will be used in later talks.

Additional material:

- Use the Hochschilde-Serre spectral sequence in etale cohomology to derive Theorem 5.4.5 1st/5.4.6 2nd, see Milne.
- The results of section 5.5.

9 Cohomological dimension and C_1-fields

This talk follows sections 6.1 and 6.2.

Cohomological dimension of a field k measures in which degree all Galois cohomology groups vanish. As we are mainly interested in the Brauer group, the interesting results for us are to try to find out for which fields the Brauer group vanishes. There is a criterion for that (6.1.8 2nd) and there are so-called C_1-fields that satisfy this property.

Among C_1-fields are finite fields (6.2.6 2nd), function fields of curves over an algebraically closed base (Tsen's theorem, 6.2.8 2nd) and totally unramified extension of the field of power series (Lang's theorem, 6.2.8 2nd). If there is a lack of time, the proof of the Lang's theorem can be just sketched.

10 Residue maps for Brauer groups

This talk follows sections 6.3 and 6.4 (with 6.3.7-6.3.8 omitted).

The main result of this talk should be the Faddeev's exact sequence (6.4.5 2nd):

$$0 \to Br(k) \to Br(k(t)) \to \oplus_P H^1(G_P, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to 0,$$

where P runs over closed points of $\mathbb{P}^1_k$, G_P is the open subgroup in $G = Gal(k)$ stabilizing a geometric point over P.

In order to reach this first we need to understand the Galois cohomology of complete discretely valued fields and to define residue maps (6.3.1-6.3.5 2nd).

Additional material:

- Faddeev exact sequence can be derived using the etale cohomology, see Milne.
- There is an interesting geometric meaning of the residue map for Brauer groups.

  The claim is that a) for a Severi-Brauer variety X over a dvf K the residue of $[X]$ is zero iff there exist a smooth projective model of X over O_K;

b) more generally, there is always a projective model with irreducible closed fiber that becomes an intersection of r copies of projective space over the algebraic closure, r is equal to the order of $res([X])$ and the Galois group acts cyclically on the components of this fiber.

The reference for this is Artin, M. "Left ideals in maximal orders." Brauer Groups in Ring Theory and Algebraic Geometry. Springer, Berlin, Heidelberg, 1982. 182-193.

11 Unramified Brauer group of a smooth variety and rationality

This talk follows sections 6.6 and 6.7.

The goal of this talk is to explain how the Faddeev exact sequence can be used to provide a counter-example to the classical problem of Lueroth's in algebraic geometry. Namely, one can ask whether a subfield K of $k(t_1, \ldots, t_n)$ of finite codegree is purely transcendental, or even a particular case of this, assuming that K is a field of invariants of the action of a finite group on $k(t_1, \ldots, t_n)$.

One can define the unramified Brauer group of a field extension K/k that has the property that it is a stable birational invariant of K. In particular, if $Br_{nr}(K)$ is not $Br(k)$, then K is not a purely transcendental field extension.

Bogomolov showed that $Br_{nr}(k(t_1, \ldots, t_n)^G)$ can be computed in terms of the Brauer groups of $k(t_1, \ldots, t_n)^G$ and $k(t_1, \ldots, t_n)^H$ where H is an abelian subgroup of G. The latter groups can be identified with cohomology groups of G and H, and so the counter-example to the Lueroth's problem arises purely out of a group-theoretic problem.

The main part of the talk is to explain Bogomolov's theorems (6.6.10 and 6.6.12), and to sketch the construction of G from 6.7.