Fakultät Mikrosystemtechnik
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# Vector calculus and numerical mathematics

## Worksheet 11

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### Problem 1: Linear functions

We consider two linear functions $f, g : \mathbb{R}^2 \to \mathbb{R}$, defined on the $xy$-plane $\mathbb{R}^2$,

$$f(x,y) \;=\; x + y, \qquad\qquad g(x,y) \;=\; 2x - y,$$

and a region $\Sigma$ in the $xy$-plane,

$$\Sigma \;=\; \Big\{ (x,y) \;\Big|\; x, y \geq 0 \quad \underline{\text{and}} \quad y \leq h(x) \equiv 3 - \tfrac{1}{2}x \Big\}.$$

(a) Draw the contour lines $f(\mathbf{r}) = n$, $\quad g(\mathbf{r}) = n$ for $n \in \{-2, 0, 2, 4, 6\}$.

(b) Evaluate the integrals

$$I_1 \;=\; \int_\Sigma \mathrm{d}^2 r\, f(\mathbf{r}), \qquad\qquad I_2 \;=\; \int_\Sigma \mathrm{d}^2 r\, g(\mathbf{r}).$$

What is the average value $\langle f(\mathbf{r}) \rangle_{\mathbf{r} \in \Sigma}$ of $f$ within $\Sigma$ ?

(c) For the constant vector $\mathbf{k} = \binom{a}{b}$, evaluate the integral

$$I_3 \;=\; \int_\Sigma \mathrm{d}^2 r\, \mathbf{k} \cdot \nabla Q(\mathbf{r}), \qquad Q(\mathbf{r}) = g(\mathbf{r})^2.$$

(d) Evaluate the integrals

$$I_4 \;=\; \int_\Sigma \mathrm{d}^2 r\, \big[\nabla f(\mathbf{r})\big] \cdot \big[\nabla g(\mathbf{r})\big], \qquad I_5 \;=\; \int_\Sigma \mathrm{d}^2 r\, \big|\nabla f(\mathbf{r})\big|^2,$$

$$I_6 \;=\; \int_\Sigma \mathrm{d}^2 r\, \big[\nabla f(\mathbf{r})\big] \cdot \big[\nabla Q(\mathbf{r})\big], \qquad Q(\mathbf{r}) = g(\mathbf{r})^2.$$

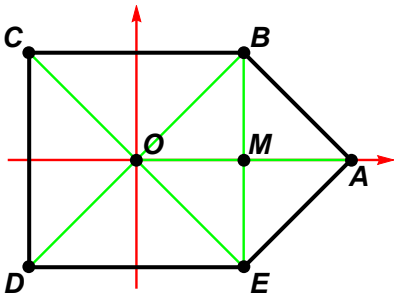
### Problem 2: Tent functions

We consider 7 points O, M, A, B, C, D, E in the $xy$-plane:

| | O | M | A | B | C | D | E |
|---|---|---|---|---|---|---|---|
| $n$ | 1 | 2 | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ |
| $\mathbf{r}_n$ | $(0\|0)$ | $(1\|0)$ | $(2\|0)$ | $(1\|1)$ | $(-1\|1)$ | $(-1\|-1)$ | $(1\|-1)$ |

The region $\Omega \subset \mathbb{R}^2$, with $\partial\Omega = \text{ABCDEA}$, is divided up into 7 triangles (green lines),

MAB, MBO, MOE, MEA, OBC, OCD, ODE.




Defining $\mathbf{r}_1 = \mathbf{r}_{\mathrm{O}}$ and $\mathbf{r}_2 = \mathbf{r}_{\mathrm{M}}$, we obtain two **tent functions** $w_1(\mathbf{r})$ and $w_2(\mathbf{r})$. Compute the following integrals:

$$
\begin{aligned}
\eta_1 &= \int \mathrm{d}^2 r \, w_1(\mathbf{r}), \\
A_{11} &= \int_\Omega \mathrm{d}^2 r \, \left|\nabla w_1(\mathbf{r})\right|^2, \\
A_{12} &= \int_\Omega \mathrm{d}^2 r \, \left[\nabla w_1(\mathbf{r})\right] \cdot \left[\nabla w_2(\mathbf{r})\right].
\end{aligned}
$$