Appendix B. Mackey functors and orthogonal G-spectra

This appendix provides a fairly self-contained proof of the fact that, for a finite group G, the symmetric monoidal ∞-categories afforded by orthogonal G-spectra and by spectral Mackey functors are equivalent. This result is due originally to Guillou and May [GM13] (ignoring the monoidal structure), and was revisited by Barwick and Barwick-Glasman-Shah [Bar17, BGS15] in the context of more general parametrized homotopy theory, see specifically [Nar16, Thm. A.4]. Compared to their work, our approached is streamlined by ignoring all models (as used by Guillou-May), and by not addressing any universal properties of Mackey functors (as in Barwick and Barwick-Glasman-Shah).

The motivation for giving our proof of their result is the immediate need of the present paper: We use categorical methods to construct Mackey functors, and then apply descent results proven for the homotopy theory of orthogonal G-spectra to them. Somewhat surprisingly, our work also yields a new proof of the equivariant Barratt-Priddy-Quillen theorem (which however uses the non-equivariant one).

Throughout, let G denote a finite group. We refer the reader to [MNN17b, Sec. 5] for a quick account of the symmetric monoidal ∞-category Sp_G extracted from the model category of orthogonal G-spectra. We denote by $\mathcal{O}(G)$ the orbit category of G, by $\mathcal{S}$ the ∞-category of spaces, by $\mathcal{S}_G := \mathrm{Fun}(\mathcal{O}(G)^{\mathrm{op}}, \mathcal{S})$ the presentable, cartesian closed ∞-category of G-spaces, by $\mathcal{S}_{G,\bullet} \simeq \mathcal{S}_{G,*/}$ the presentable, closed symmetric monoidal ∞-category of based G-spaces, and by $\Sigma^\infty_G \colon \mathcal{S}_{G,\bullet} \longrightarrow \mathrm{Sp}_G$ the suspension spectrum functor. We consider $\mathcal{S}_G$ with its cartesian monoidal structure.

To set the notation for Mackey functors, we denote by Fin_G the category of finite G-sets, by $\mathrm{Span}(\mathrm{Fin}_G)$ the $(2,1)$-category of spans on Fin_G (cf. [BH17, App. C]) and set

$$\mathrm{Mack}_G := \mathrm{Mack}_G(\mathrm{Sp}) = \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G)^{\mathrm{op}}, \mathrm{Sp}),$$

the category of finite product-preserving presheaves on $\mathrm{Span}(\mathrm{Fin}_G)$ with values in the ∞-category Sp of spectra. We note that $\mathrm{Mack}_G \simeq \mathcal{P}_\Sigma(\mathrm{Span}(\mathrm{Fin}_G)) \otimes \mathrm{Sp}$, cf. (INTERNAL REF). Below, we will recall the suspension functor in the Mackey context, to be denoted

$$\Sigma^\infty_{\mathcal{M}} \colon \mathcal{S}_{G,\bullet} \longrightarrow \mathrm{Mack}_G.$$

The cartesian product on Fin_G induces a symmetric monoidal structure on $\mathrm{Span}(\mathrm{Fin}_G)$, we endow Mack_G with the symmetric monoidal structure given by Day-convolution and denote it by $\otimes$. This is the unique symmetric monoidal structure which is bicontinuous and compatible with $\Sigma^\infty_{\mathcal{M}}$.

Our main result is the following.

Theorem B.1. There is a unique symmetric monoidal left-adjoint $L \colon \mathrm{Sp}_G \to \mathrm{Mack}_G$ such that $L \circ \Sigma^\infty_G \simeq \Sigma^\infty_{\mathcal{M}}$, and L is an equivalence.

The rest of this section will provide a proof of this result.

The construction of L rests on the following folklore result, and we thank Markus Hausmann for providing a key reference in its proof.

Theorem B.2. The suspension $\Sigma^\infty_G \colon \mathcal{S}_{G,\bullet} \longrightarrow \mathrm{Sp}_G$ is the initial example of a presentably symmetric monoidal functor[2] which inverts the functor $S^V \otimes -$ for all finite-dimensional, orthogonal representations V of G.

[2] In other words, a map in $\mathrm{CAlg}(\mathrm{Pr}^L)$.

Proof. This is not tautological only because we insisted that Sp_G be extracted from the model category of orthogonal G-spectra. Starting to argue, it is well known that the ∞-category underlying the model category of G-spaces is $\mathrm{Fun}(\mathcal{O}(G)^{op}, \mathcal{S})$ ("Elmendorf's theorem"). Applying [Rob15, Prop. 2.9] (cf. also [BH17, Lem. 4.1]), we see that there is an initial example $\mathrm{Fun}(\mathcal{O}(G)^{op}, \mathcal{S}) \longrightarrow \mathrm{Sp}'_G$ of a symmetric monoidal functor that inverts all representation spheres. Moreover, according to [Rob15, Thm. 2.26], the symmetric monoidal ∞-category Sp'_G is modeled by spectrum objects in the sense of Hovey, which in turn by [Man04, paragraph after Thm. 1] gives a model category equivalent to orthogonal G-spectra. In summary, we obtain the desired equivalence $\mathrm{Sp}_G \simeq \mathrm{Sp}'_G$. □

To construct $\Sigma^\infty_{\mathcal{M}}$, recall from [BH17, §9.1 before Lem. 9.4] the canonical cartesian monoidal functor $\iota\colon \mathrm{Fin}_{G,+} \to \mathrm{Span}(\mathrm{Fin}_G)$ and the symmetric monoidal equivalence $\mathcal{P}_\Sigma(\mathrm{Fin}_{G,+}) \simeq \mathcal{S}_{G,\bullet}$ ([BH17, Lem. 2.1]). This induces $\Sigma^\infty_{\mathcal{M}}$, to be defined as the composition

$$\Sigma^\infty_{\mathcal{M}} := \left(\mathcal{S}_{G,\bullet} \simeq \mathcal{P}_\Sigma(\mathrm{Fin}_{G,+}) \xrightarrow{\mathcal{P}_\Sigma(\iota)} \mathcal{P}_\Sigma(\mathrm{Span}(\mathrm{Fin}_G)) = \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_G))^{op}, \mathcal{S}) \to \mathrm{Mack}_G\right),$$

where the final map is the stabilization. By construction, Σ^∞_M is a map in $\mathrm{CAlg}(\mathrm{Pr}^L)$.

Theorem B.2 tells us that to construct the functor L in Theorem B.1, we need to see that $\Sigma^\infty_{\mathcal{M}}$ inverts all representation spheres. We will do this by constructing from scratch on Mack_G what will a posteriori turn out to be geometric fixed point functors, and by establishing some of their basic properties. Denote by $\mathcal{P}$ the family of proper subgroups of G and recall the cofiber sequence in $\mathcal{S}_{G,\bullet}$, defining $\tilde{E}\mathcal{P}$:

$$\operatorname*{colim}_{G/H \in \mathcal{O}(G)_{\mathcal{P}}} G/H_+ \simeq E\mathcal{P}_+ \longrightarrow *_+ = S^0 \longrightarrow \tilde{E}\mathcal{P},$$

(cf. [MNN15, Appendix A.1]). The least formal part of our argument is the following.

Lemma B.1. We have an equivalence $\left(\Sigma^\infty_{\mathcal{M}}(\tilde{E}\mathcal{P})\right)(G/G) \simeq S^0$ in Sp.

To see this, we will need the following result on manipulating colimits, and we are indepted to Rune Haugseng for help with its proof. For an ∞-category $\mathcal{C}$, we denote by $\mathcal{C}^\triangleright$ the result of freely adjoining a final object to $\mathcal{C}$, and by $\mathcal{C}^\simeq$ the maximal underlying subgroupoid of $\mathcal{C}$. The construction $\mathcal{C} \mapsto \mathcal{C}^\simeq$ is right adjoint to the inclusion $\mathcal{S} \simeq \mathcal{G}rp_\infty \subseteq \mathcal{C}at_\infty$ of ∞-groupoids into all ∞-categories.

Proposition B.2. Let $\mathcal{C}$ be an ∞-category and $F\colon \mathcal{C}^\triangleright \to \mathcal{S}$ the functor defined by $F(c) = \left((\mathcal{C}^\triangleright)_{/c}\right)^\simeq$. Then the canonical map of spaces $\mathrm{colim}_{\mathcal{C}} F \to \mathrm{colim}_{\mathcal{C}^\triangleright} F$ is equivalent to the inclusion $\mathcal{C}^\simeq \subseteq (\mathcal{C}^\triangleright)^\simeq$.

Proof. The closely related functor $F'\colon \mathcal{C}^\triangleright \to \mathcal{C}at_\infty$ defined by $F'(c) := (\mathcal{C}^\triangleright)_{/c}$ classifies the cocartesian codomain fibration $cd\colon \mathrm{Fun}(\Delta^1, \mathcal{C}^\triangleright) \longrightarrow \mathcal{C}^\triangleright$ given by evaluation on 1 [Lur09, Cor. 2.4.7.12]. It follows that $F = (-)^\simeq \circ F'$ classifies the left fibration $cd'\colon \mathrm{Fun}(\Delta^1, \mathcal{C}^\triangleright)^{\mathrm{left}} \longrightarrow \mathcal{C}^\triangleright$ obtain by passing from $\mathrm{Fun}(\Delta^1, \mathcal{C}^\triangleright)$ to the sub-simplicial set $\mathrm{Fun}(\Delta^1, \mathcal{C}^\triangleright)^{\mathrm{left}} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C}^\triangleright)$ consisting of all simplices all of whose edges are cd-cocartesian. Informally then, the objects of $\mathrm{Fun}(\Delta^1, \mathcal{C}^\triangleright)^{\mathrm{left}}$ are the morphisms in $\mathcal{C}^\triangleright$, and the morphisms are the commuting squares in which the map between sources is an equivalence.

We now observe that evaluation *at zero*, $ez\colon \mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{left}} \to \mathcal{C}^\simeq$, is a right fibration which has all fibers contractible (because each of them has an initial object). In particular, ez is a weak equivalence, and an inverse equivalence is provided by sending objects to identity morphisms. We have thus seen that $\mathrm{colim}_{\mathcal{C}} F \simeq \mathcal{C}^\simeq$.

Furthermore, the canonical map $\operatorname{colim}_{\mathcal{C}} F \to \operatorname{colim}_{\mathcal{C}^{\triangleright}} F$ is equivalent to the obvious map $\operatorname{Fun}(\Delta^1, \mathcal{C})^{\mathrm{left}} \to \operatorname{Fun}(\Delta^1, \mathcal{C}^{\triangleright})^{\mathrm{left}}$, the target of which is equivalent to the fiber over the cone point, namely $(\mathcal{C}^{\triangleright})^{\simeq}$. One checks that this identifies the canonical map with the inclusion $\mathcal{C}^{\simeq} \subseteq (\mathcal{C}^{\triangleright})^{\simeq}$, as claimed. □

Proof of Lemma B.1. Applying Proposition B.2 with $\mathcal{C} = \mathcal{O}(G)_{\mathcal{P}}$ the category of orbits with proper isotropy (hence $\mathcal{C}^{\triangleright} = \mathcal{O}(G)$), we obtain a cofiber sequence in spaces

$$\operatorname*{colim}_{G/H \in \mathcal{O}(G)_{\mathcal{P}}} (\mathcal{O}(G)/(G/H))^{\simeq} \simeq \mathcal{O}(G)_{\mathcal{P}}^{\simeq} \hookrightarrow \mathcal{O}(G)^{\simeq} \longrightarrow * \sqcup +$$

where the final map sends all orbits with proper isotropy to $+$, and sends G/G to $*$. We can consider this as a cofiber sequence in pointed spaces $\mathcal{S}_{\bullet}$ of the form

$$\operatorname*{colim}_{G/H \in \mathcal{O}(G)_{\mathcal{P}}} (\mathcal{O}(G)/(G/H))_{+}^{\simeq} \simeq \mathcal{O}(G)_{\mathcal{P},+}^{\simeq} \longrightarrow \mathcal{O}(G)_{+}^{\simeq} \longrightarrow S^0 = *_{+}$$

where the final map sends all orbits with proper isotropy to the base-point $+$, and sends G/G to $*$. Applying the free commutative monoid functor $\mathbb{P}\colon \mathcal{S}_{\bullet} \to \operatorname{CMon}(\mathcal{S})$ yields a cofiber sequence in $\operatorname{CMon}(\mathcal{S})$:

$$\operatorname*{colim}_{G/H \in \mathcal{O}(G)_{\mathcal{P}}} (\operatorname{Fin}_G/(G/H))^{\simeq} \longrightarrow \operatorname{Fin}_G^{\simeq} \longrightarrow \operatorname{Fin}^{\simeq} \tag{B.3}$$

in which the final map is identified with taking G-fixed points. To see this, observe that $\mathcal{O}(G)_{/(G/H)} \simeq \mathcal{O}(H)$ and that $\mathbb{P}(\mathcal{O}(H)_{+}^{\simeq}) \simeq \operatorname{Fin}_H^{\simeq}$, as can be checked most easily using the general formula $\mathbb{P}(Z) = \bigvee_{n \geq 0} \left(Z^{\wedge n} \wedge_{\Sigma_n} E\Sigma_{n+}\right)$.

We denote by $(-)^{+}$ the group completion on $\operatorname{CMon}(\mathcal{S})$, and observe that

$$\Omega^{\infty}\left(\Sigma^{\infty}_{\mathcal{M}}(G/H_{+})(G/G)\right) = \operatorname{map}_{\operatorname{Span}_G}(G/G, G/H)^{+} \simeq (\operatorname{Fin}_G/(G/H))^{\simeq,+}.$$

We thus see that the delooping of the group completion of the cofiber sequence (B.3) is a cofiber sequence in Sp of the form

$$\left(\Sigma^{\infty}_{\mathcal{M}}(E\mathcal{P})\right)(G/G) \longrightarrow \Sigma^{\infty}_{\mathcal{M}}(S^0)(G/G) \longrightarrow \Sigma^{\infty}_{\mathcal{M}}(\tilde{E}\mathcal{P})(G/G) \simeq S^0,$$

using that $\Omega^{\infty}(S^0) \simeq \operatorname{Fin}^{\simeq,+}$. □

Next, we will need to discuss restriction for Mackey functors.

Construction 7. Let $H \subseteq G$ be a subgroup.

(1) We have a symmetric monoidal and coproduct preserving functor

$$\operatorname{Res}^G_H\colon \operatorname{Span}(\operatorname{Fin}_G) \longrightarrow \operatorname{Span}(\operatorname{Fin}_H).$$

This sends a G-set U to the underlying H-set of U, and behaves accordingly on correspondences. (cf. [BH17, App. C.3]).

(2) We also have a functor

$$G \times_H (-)\colon \operatorname{Span}(\operatorname{Fin}_H) \longrightarrow \operatorname{Span}(\operatorname{Fin}_G),$$

which takes a H-set T to the G-set $G \times_H T$, and behaves analogously on correspondences. Note that the construction $T \mapsto G \times_H T$ on finite H-sets preserves fiber products.

Proposition B.4. Both, the functors $(\operatorname{Res}^G_H(-), G \times_H (-)) : \operatorname{Span}(\operatorname{Fin}_G) \rightleftarrows \operatorname{Span}(\operatorname{Fin}_H)$ and the functors $(G \times_H (-), \operatorname{Res}^G_H(-)) : \operatorname{Span}(\operatorname{Fin}_H) \rightleftarrows \operatorname{Span}(\operatorname{Fin}_G)$ are biadjoint.

Proof. If S is a finite H-set and T is a finite G-set, then we have an equivalence of categories

$$(\mathrm{Fin}_G)_{/(G\times_H S)\times T} \xrightarrow{\simeq} (\mathrm{Fin}_H)_{/S\times \mathrm{Res}^G_H(T)},$$

given by pulling back along the H-map $S\times T\to (G\times_H S)\times T$. Using that $\mathrm{map}_{\mathrm{Span}(\mathrm{Fin}_G)}(X,Y)\simeq (\mathrm{Fin}_G)^{\simeq}_{/X\times Y}$, the result follows. See also [BH17, App. C.3] for a more general treatment of $\mathrm{Span}(-)$ as an $(\infty,2)$-functor. □

We now define restriction for Mackey functors, essentially by left Kan extension. Namely, we define the symmetric monoidal, cocontinuous functor $\mathrm{Res}^G_{H,\mathcal{M}}\colon \mathrm{Mack}_G\to\mathrm{Mack}_H$ to be $\mathrm{Res}^G_{H,\mathcal{M}} := \mathcal{P}_\Sigma(\mathrm{Res}^G_H)\otimes \mathrm{id}_{\mathrm{Sp}_G}$.

As an example, note that for $F\in\mathrm{Mack}_G$ and subgroups $H'\subseteq H\subseteq G$ we have

$$\mathrm{Res}^G_{H,\mathcal{M}}(F)(H/H')\simeq F(G/H').$$

To see this, since both sides are colimit preserving functors of F, it suffices to check the case when F is the suspension of an orbit, and then the claim is immediate from the second adjunction in Proposition B.4.

Proposition B.5. Assume $F\in\mathrm{Mack}_G$ is such that for all proper subgroups $H\subseteq G$ we have $\mathrm{Res}^G_{H,\mathcal{M}}(F)\simeq *$. Then the canonical map

$$F(G/G)\longrightarrow \left(\Sigma^\infty_{\mathcal{M}}(\tilde{E}\mathcal{P})\otimes F\right)(G/G)$$

is an equivalence.

Proof. First observe that for all subgroups $H\subseteq G$, the suspension $\Sigma^\infty_{\mathcal{M}}(G/H_+)\in\mathrm{Mack}_G$ is self-dual. It then follows that for all proper subgroups $H\subseteq G$, the spectrum

$$(F\otimes\Sigma^\infty_{\mathcal{M}}(G/H_+))(G/G)\simeq F(G/H)\simeq \mathrm{Res}^G_{H,\mathcal{M}}(F)(H/H)\simeq *$$

is contractible, and hence that $(F\otimes\Sigma^\infty_{\mathcal{M}}(E\mathcal{P}_+))(G/G)\simeq *$. □

We next introduce geometric fixed points in the Mackey context. The fixed point functor $(-)^G$: $\mathrm{Fin}_G\longrightarrow\mathrm{Fin}$ commutes with pullbacks and hence induces a functor on span categories. This functor preserves finite coproducts and the cartesian product, hence the functor

$$\Phi^G_{\mathcal{M}} := \mathcal{P}_\Sigma(\mathrm{Span}((-)^G))\otimes\mathrm{id}_{\mathrm{Sp}}:\mathrm{Mack}_G\longrightarrow \mathcal{P}_\Sigma(\mathrm{Span}(\mathrm{Fin}))\otimes\mathrm{Sp}\simeq\mathrm{Sp}$$

commutes with all colimits and is symmetric monoidal. By construction, it takes the expected values on orbits, namely $\Phi^G_{\mathcal{M}}(\Sigma^\infty_{\mathcal{M}}(G/H_+))$ is contractible for a proper subgroup $H\subseteq G$, and equivalent to S^0 for $H=G$. In fact, it is clear that, more generally, for each $X\in\mathcal{S}_{G,\bullet}$ we have

$$\Phi^G_{\mathcal{M}}(\Sigma^\infty_{\mathcal{M}}(X))\simeq\Sigma^\infty(X(G/G)).$$

For a subgroup $H\subseteq G$, we denote $\Phi^{G,H}_{\mathcal{M}} := \Phi^H_{\mathcal{M}}\circ\mathrm{Res}^G_{H,\mathcal{M}}$.

We will need to know that our geometric fixed points are given by the familiar contruction:

Proposition B.6. We have an equivalence of functors $\Phi^G_{\mathcal{M}}(-)\simeq\left(\Sigma^\infty_{\mathcal{M}}(\tilde{E}\mathcal{P})\otimes(-)\right)(G/G)$.

Proof. Both functors preserve colimits and take the same values on all orbits, because for a subgroup $H\subseteq G$ we can compute that

$$\left(\Sigma^\infty_{\mathcal{M}}(\tilde{E}\mathcal{P})\otimes\Sigma^\infty_{\mathcal{M}}(G/H_+)\right)(G/G)\simeq\left(\Sigma^\infty_{\mathcal{M}}\left(\tilde{E}\mathcal{P}\wedge(G/H_+)\right)\right)(G/G)$$

is contractible if H is proper, and is S^0 if $H=G$ by Lemma B.1. □

This allows to easily establish the basic properties of geometric fixed points in the Mackey context:

Proposition B.7. The family $\{\Phi^{G,H}_{\mathcal{M}}\}_{H\subseteq G}$ of symmetric monoidal left adjoints is jointly conservative.

Proof. It only remains to see the joint conservativity, so assume $\Phi^{G,H}_{\mathcal{M}}(F) \simeq *$ for all $H \subseteq G$, and we need to see that $F \simeq *$.

This is clear for trivial G, and we argue by induction on the group order in general. We can thus assume that $\mathrm{Res}^G_{H,\mathcal{M}}(F) \simeq *$ for all proper subgroups $H \subseteq G$. In particular then, for all proper subgroups $H \subseteq G$ we know that

$$F(G/H) = \mathrm{Res}^G_{H,\mathcal{M}}(F)(H/H) = *$$

is contracible, and need to see that $F(G/G)$ is as well. But combining Proposition B.5 and Proposition B.6, wee see that $F(G/G) \simeq \Phi^G_{\mathcal{M}}(F) = \Phi^{G,G}_{\mathcal{M}}(F)$, and this is contractible by assumption. $\square$

This finally lets us check that suspension for Mackey functors inverts all representation spheres. We are indepted to Marc Hoyois for very helpful comments on the following proof.

Proposition B.8. For every representation V of G, $\Sigma^\infty_{\mathcal{M}}(S^V) \in \mathrm{Mack}_G$ is invertible.

Proof. We first note that $\Sigma^\infty_{\mathcal{M}}(S^V) \in \mathrm{Mack}_G$ is at least dualizable. Since Mack_G is stable, the dualizable objects are stable under finite colimits, and S^V is a finite colimits of orbits. It thus suffies to remark that the orbits are dualizable (in fact, self-dual) already in $\mathrm{Span}(\mathrm{Fin}_G)$. Once we know $\Sigma^\infty_{\mathcal{M}}(S^V) \in \mathrm{Mack}_G$ is dualizable, it will be invertible if and only if it becomes so after applying anyone family of jointly conservative symmetric monoidal functors. By Proposition B.7 it will thus suffice to see that for every subgroup $H \subseteq G$, the spectrum $\Phi^{G,H}_{\mathcal{M}}(\Sigma^\infty_{\mathcal{M}}(S^V))$ is invertible, but this follows from a direct computation:

$$\Phi^{G,H}_{\mathcal{M}}(\Sigma^\infty_{\mathcal{M}}(S^V)) = \Phi^H_{\mathcal{M}}(\mathrm{Res}^G_{H,\mathcal{M}}(\Sigma^\infty_{\mathcal{M}}(S^V))) \simeq \Sigma^\infty((S^V)^H) \simeq S^{\dim(V^H)}.$$

$\square$

We can now complete the proof of our main result.

Proof. (proof of Theorem B.1) Theorem B.2 and Proposition B.8 show that there is a unique symmetric monoidal left adjoint $L : \mathrm{Sp}_G \longrightarrow \mathrm{Mack}_G$ such that $L \circ \Sigma^\infty_G \simeq \Sigma^\infty_{\mathcal{M}}$. It remains to see that L is an equivalence. Denote by R the right adjoint of L. Since both Sp_G and Mack_G are generated under colimits by dualizable objects (namely the suspensions of orbits), it follows from [BDS16, Thm. 1.3] that R admits itself a right-adjoint, hence preserves colimits, and that the adjunction (L, R) satisfies a projection formula. Furhtermore, R is conservative because the image of its left adjoint L contains a set of generators. We can thus apply [MNN17a, Prop. 5.29] to conclude that the adjunction (L, R) induces an adjoint equivalence

$$\mathrm{Mod}_{\mathrm{Sp}_G}(R(1_{\mathrm{Mack}_G})) \simeq \mathrm{Mack}_G,$$

and it remains to see that the counit of the adjunction

$$1_{\mathrm{Sp}_G} \longrightarrow R(L(1_{\mathrm{Sp}_G})) \simeq R(1_{\mathrm{Mack}_G})$$

is an equivalence. By induction on the group order, it suffices to see that Φ^G (geometric fixed points for orthogonal spectra) turns this map into an equivalence. Since $\Phi^G(1_{\mathrm{Sp}_G}) = S^0$ and we are looking at a map of commutative algebras, it suffices in fact to see that there is an equivalence of spectra $\Phi^G(R(1_{\mathrm{Mack}_G})) \simeq S^0$. This follows from the following computation:

$$\Phi^G(R(1_{\mathrm{Mack}_G})) \simeq \left(\Sigma^\infty_G(\tilde{E}\mathcal{P}) \otimes R(1_{\mathrm{Mack}_G})\right)^G \simeq \left(R\left[L(\Sigma^\infty_G(\tilde{E}\mathcal{P})) \otimes 1_{\mathrm{Mack}_G})\right]\right)^G \simeq$$
$$\simeq L(\Sigma^\infty_G(\tilde{E}\mathcal{P}))(G/G) \simeq (\Sigma^\infty_{\mathcal{M}}(\tilde{E}\mathcal{P}))(G/G) \simeq S^0.$$

This computation used in turn: The definition of Φ^G, the projection formula for (L, R), the fact that $(R(-))^G \simeq (-)(G/G)$ (by adjointness of L and R), the fact that $L \circ \Sigma^\infty_G \simeq \Sigma^\infty_{\mathcal{M}}$, and finally Lemma B.1. □

As promised earlier, our account yields the following proof of the equivariant Barrat-Priddy-Quillen therem (originally due to Guillou/May), which by-passes any loop-space theory (but uses the non-equivariant version).

Corollary B.9. For a finite group, there is an equivalence in $\mathrm{CMon}(\mathcal{S})$

$$\operatorname*{colim}_V \Omega^V S^V \simeq (\mathrm{Fin}_G)^{\simeq,+},$$

where the colimit is taken along any cofinal system of representations of G.

For the proof, one simply computes the endomorphism space of the unit of both Sp_G and Mack_G from the definition, and compares the result.

References

[AS69] M.F. Atiyah and G.B. Segal. Equivariant K-theory and completion. *J. Differential Geometry*, 3:1–18, 1969.

[Bar17] Clark Barwick. Spectral Mackey functors and equivariant algebraic K-theory (I). *Adv. Math.*, 304:646–727, 2017.

[BDG+] Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah. Parametrized higher category theory and higher algebra.

[BDS16] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. Grothendieck-Neeman duality and the Wirthmüller isomorphism. *Compos. Math.*, 152(8):1740–1776, 2016.

[BGS15] Clark Barwick, Saul Glasman, and Jay Shah. Spectral Mackey functors and equivariant algebraic K-theory II. *arXiv preprint arXiv:1505.03098*, 2015.

[BGT13] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.

[BH17] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. *arXiv preprint arXiv:1711.03061*, 2017.

[CMNN16] Dustin Clausen, Akhil Mathew, Niko Naumann, and Justin Noel. Descent in algebraic K-theory and a conjecture of Ausoni–Rognes. *arXiv preprint arXiv:1606.03328*, 2016.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. Universality of multiplicative infinite loop space machines. *Algebr. Geom. Topol.*, 15(6):3107–3153, 2015.

[GM13] Bertrand Guillou and J.P. May. Models of G-spectra as presheaves of spectra. *arXiv preprint arXiv:1110.3571*, 2013.

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594 (electronic), 2000.

[Ill78] Sören Illman. Smooth equivariant triangulations of G-manifolds for G a finite group. *Math. Ann.*, 233(3):199–220, 1978.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Mal15] Cary Malkiewich. Coassembly and the K-theory of finite groups. *arXiv preprint arXiv:1503.06504*, 2015.

[Man04] Michael A. Mandell. Equivariant symmetric spectra. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 399–452. Amer. Math. Soc., Providence, RI, 2004.

[Mat15] Akhil Mathew. A thick subcategory theorem for modules over certain ring spectra. *Geom. Topol.*, 19(4):2359–2392, 2015.

[Mat16] Akhil Mathew. The Galois group of a stable homotopy theory. *Adv. Math.*, 291:403–541, 2016.

[Mat17] Akhil Mathew. Examples of descent up to nilpotence. *arXiv preprint arXiv:1701.01528*, 2017.

[Mit90] Stephen A. Mitchell. The Morava K-theory of algebraic K-theory spectra. *K-Theory*, 3(6):607–626, 1990.

[MNN15] Akhil Mathew, Niko Naumann, and Justin Noel. Derived induction and restriction theory. 2015.

[MNN17a] Akhil Mathew, Niko Naumann, and Justin Noel. Nilpotence and descent in equivariant stable homotopy theory. *Adv. Math.*, 305:994–1084, 2017.

[MNN17b] Akhil Mathew, Justin Noel, and Niko Naumann. Nilpotence and descent in equivariant stable homotopy theory. *Adv. Math.*, 305:994–1084, 2017.

[Nar16] Denis Nardin. Stability with respect to an orbital infinity category. *arXiv preprint arXiv:1608.07704*, 2016.

[Rob15] Marco Robalo. K-theory and the bridge from motives to noncommutative motives. *Adv. Math.*, 269:399–550, 2015.

[Rog] John Rognes. A Galois extension that is not faithful.

[Rog08] John Rognes. Galois extensions of structured ring spectra. Stably dualizable groups. *Mem. Amer. Math. Soc.*, 192(898):viii+137, 2008.

[Swa60a] Richard G. Swan. Induced representations and projective modules. *Ann. of Math. (2)*, 71:552–578, 1960.

[Swa60b] Richard G. Swan. Induced representations and projective modules. *Ann. of Math. (2)*, 71:552–578, 1960.

[Swa70] Richard G. Swan. *K-theory of finite groups and orders.* Lecture Notes in Mathematics, Vol. 149. Springer-Verlag, Berlin-New York, 1970.

[Tho83] R. W. Thomason. The homotopy limit problem. In *Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982)*, volume 19 of *Contemp. Math.*, pages 407–419. Amer. Math. Soc., Providence, R.I., 1983.

[Tho85] R. W. Thomason. Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup. (4)*, 18(3):437–552, 1985.

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