Universität Regensburg, Mathematik | SS 2011
Dr. David Gepner | 8.6.2011

Introduction to Stable Homotopy Theory
6. Übungsblatt

Aufgabe 1
Given a cohomology theory R and a relative CW complex (X, X') (this means in particular that the subspace $X' \subset X$ is the inclusion of a subcomplex, and hence a cofibration), define

$$R^n(X, X') := R^n(X/X').$$

Thus we can extend the cohomology theory R from the homotopy category of pointed CW complexes to the homotopy category of relative CW complexes. Let (X, X') and (Y, Y') be relative CW complexes, and let R be a ring spectrum. Show that there exists an external product

$$R^p(X, X') \otimes R^q(Y, Y') \longrightarrow R^{p+q}(X \times Y, X' \times Y \cup X \times Y').$$

In particular, given a pair of subcomplexes Y and Z of a CW complex X, show that we obtain a product

$$R^p(X, Y) \times R^q(X, Z) \longrightarrow R^{p+q}(X, Y \cup Z).$$

Aufgabe 2
Let R be a commutative ring spectrum equipped with a complex orientation (that is, a class $t \in R^2(\mathbb{CP}^\infty)$ such that t restricts in $R^2(\mathbb{CP}^1)$ to the image of $1 \in R^0(S^0)$). Write

$$R^*(\mathbb{CP}^n_+) = \bigoplus_{p \in \mathbb{Z}} R^p(\mathbb{CP}^n_+)$$

for the graded commutative ring obtained by taking the R-cohomology of $\mathbb{CP}^n$ in each degree. Show that there exists a map of graded commutative rings

$$R^*[t]/(t^{n+1}) \longrightarrow R^*(\mathbb{CP}^n_+).$$

Use induction on n (together with the five lemma) to show that this map is an isomorphism. [Hint: to give a ring map from $R^*[t]$ it suffices to specify the image of t. Also recall that $\mathbb{CP}^n$ is covered by $n+1$ open sets, each of which is isomorphic to $\mathbb{C}^n$.]

Aufgabe 3
Calculate $R^*(\mathbb{CP}^\infty_+)$, as a graded commutative ring, using the results of Aufgabe 2. Use this to calculate the graded commutative rings

$$R^*(\mathbb{CP}^\infty_+ \wedge \cdots \wedge \mathbb{CP}^\infty_+),$$

the cohomology of the n-fold smash power of ${\mathbb{CP}^\infty}_+$ (which is the same as the n-fold cartesian product $\mathbb{CP}^\infty \times \cdots \times \mathbb{CP}^\infty$, plus a disjoint base point).

Aufgabe 4
Let $X_0 \to X_1 \to X_2 \to \cdots$ be an infinite increasing sequence of spectra with homotopy colimit $X = \mathrm{hocolim}_n X_n$ (the homotopy colimit is the cofiber of the map $\mathrm{id} - \mathrm{shift} : \bigvee_n X_n \to \bigvee_n X_n$). Show that the natural map $\mathrm{colim}_n \pi_0 X_n \to \pi_0 X$ is an isomorphism. Use this to give an explicit example of a (necessarily infinite) collection of spectra X_n such that the canonical map

$$\bigvee_n X_n \longrightarrow \prod_n X_n$$

is not an equivalence.

Due: ***29.6.2011*** *in the exercise session.*