Universität Regensburg, Mathematik — SS 2011
Dr. David Gepner — 25.5.2011

Introduction to Stable Homotopy Theory
3. Übungsblatt

Aufgabe 1

The *homotopy coequalizer* of a pair of maps $f, g : X \to Y$ of (not necessarily pointed) spaces is the pushout $Y \coprod_{X \times \partial I} X \times I$ of $f \coprod g : X \coprod X = X \times \partial I \to Y$ along the inclusion $X \to X \times I$ of X into the cylinder on X. Suppose that X and Y are CW complexes, let $f, g : X \to Y$ be a pair of maps, and let Z be the homotopy coequalizer of f and g. Show that $\pi_0(Z)$ is the coequalizer (in the category of sets) of the pair of maps $\pi_0(f), \pi_0(g) : \pi_0(X) \to \pi_0(Y)$. Similarly, the *homotopy colimit* X of a sequence of maps $f_n : X_n \to X_{n+1}$, $n \in \mathbb{N}$, is the homotopy coequalizer of the pair of maps

$$\mathrm{id}, s : \coprod_n X_n \to \coprod_n X_n,$$

where s is the "shift" map, i.e. the map which on the m^{th} component X_m of $\coprod_n X_n$ is the composite of $f_m : X_m \to X_{m+1}$ with the canonical inclusion $X_{m+1} \to \coprod_n X_n$. Deduce that the induced map $\mathrm{colim}_n \pi_0(X_n) \to \pi_0(X)$ is a bijection.

Aufgabe 2

Dually, the *homotopy equalizer* of a pair of maps $f, g : Y \to X$ of (not necessarily pointed) spaces is the pullback $Y \times_{X^{\partial I}} X^I$ of $f \times g : Y \to X \times X = X^{\partial I}$ along the projection $X^I \to X^{\partial I}$ which evaluates a (unbased) path $I \to X$ at its endpoints. Suppose that X and Y are CW complexes, let $f, g : Y \to X$ be a pair of maps, and let Z be the homotopy equalizer of f and g. Give an example to show that the canonical map from $\pi_0(Z)$ to the equalizer of $\pi_0(f), \pi_0(g) : \pi_0(Y) \to \pi_0(X)$ (in the category of sets) need not be an isomorphism.

Aufgabe 3

Suppose given a commutative diagram of pointed sets

$$\begin{array}{ccccccccc}
W & \longrightarrow & V & \longrightarrow & U & \longrightarrow & T & \longrightarrow & S \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle d} \\
W' & \longrightarrow & V' & \longrightarrow & U' & \longrightarrow & T' & \longrightarrow & S'
\end{array}$$

such that the left-hand square is a diagram of groups and group homomorphisms, the groups V and V' act on the pointed sets U and U', respectively, the maps $V \to U$ and $V' \to U'$ are the action maps (that is, $v \mapsto v \cdot *$ and $v' \mapsto v' \cdot *'$, where $* \in U$ and $*' \in U'$ denote the basepoints), $f : U \to U'$ is equivariant with respect to this action, and the images of U in T and U' in T' are the quotients of U and U' by the actions of V and V'. Show that f is epi whenever h, g, and e are epi and d is iso, and that f is iso whenever h, g, e and d are iso. What are the exact minimal conditions one needs for each of these?

Aufgabe 4

Suppose that M and N are smooth manifolds, and that $f : M \to N$ is a continuous map. Show that if $N = \mathbb{R}^n$, for some n, then f is homotopic to a smooth map $g : M \to N$ (you may use the fact that the space $C^\infty(M)$ of smooth functions on M is dense in the space $C^0(M)$ of continuous functions on M). Show that if N is an smoothly embedded submanifold of $\mathbb{R}^n$ then f is homotopic to a smooth map $g : M \to N$. Use the Whitney embedding theorem to conclude that, for any N, f is homotopic to a smooth map $g : M \to N$.

Due: ***1.6.2011*** *in the exercise session.*