EXTENSION OF P-ADIC DEFINABLE LIPSCHITZ FUNCTIONS

We would like to prove

**Theorem.** 15 We work either in the Macintyre language or in the subanalytic language. Let $X \subset K^n$ and $f : X \to K$ be definable and $\lambda$-Lipschitz. There exists some $\lambda$-Lipschitz extension $\tilde{f} : K^n \to K$.

1. NOTATIONS AND PRELIMINARIES

We consider $K$ a finite extension of $\mathbb{Q}_p$ equipped with the canonical metric inherited from the $p$-adic norm $| \cdot |$. We equip $K^n$ with the product metric. So, if $x = (x_1 \ldots x_n)$ and $y = (y_1 \ldots y_n) \in K^n$, by definition $d(x, y) = \max_{i=1 \ldots n} |x_i - y_i|$.

Write $\mathcal{O}_K$ for the valuation ring, $\mathcal{M}_K$ for the maximal ideal of $K$ and $k_K$ for the residue field. Let us fix $\varpi$ some uniformizer of $K$. We denote by $\overline{\varpi}_m : K \to \mathcal{O}_K/(\mathcal{M}_K^m)$ the map sending some nonzero $x \in K$ to $x\varpi^{-\text{ord}(x)} \mod \mathcal{M}_K^m$, and sending zero to zero. This is a definable map. We denote by $RV$ the union of $K^\times/(1 + \mathcal{M}_K)$ and $\{0\}$ and by $rv : K \to RV$ the quotient map. More generally, if $m \in \mathbb{N}^*$, we set $RV_m = K^\times/(1 + \mathcal{M}_K^n) \cup \{0\}$ and $rv_m : K \to RV_m$ the quotient map.

For $m, n > 0 \in \mathbb{N}$, we set

$$Q_{m,n} = \{ x \in K^\times \mid \text{ord}(x) \in n\mathbb{Z} \text{ and } \overline{\varpi}_m(x) = 1 \}.$$

A Lipschitz map will mean a 1-Lipschitz map.

**Definition 1.** A ball of $K^n$ is a ball with respect to the metric $d$. In other words a ball is a set of the form

$$B = \{ (x_1 \ldots x_n) \in K^n \mid |x_i - c_i| \leq r \}$$

where $(c_1 \ldots c_n) \in K^n$ and $r > 0$ is a real number.

**Lemma 2.** If $\varphi : K^n \to K^n$ is an isometry, balls of $K^n$ are stable under $\varphi$.

**Proof.** OK. \qed

**Remark 3.** Let us say that a box of $K^n$ is a product of balls, that is to say some

$$\{ (x_1 \ldots x_n) \in K^n \mid |x_i - c_i| \leq r_i \}.$$  \{lemma:ball_isometry\}

We want to stress out that the above lemma does not hold for boxes. For instance, if $K = \mathbb{Q}_p$, any permutation of $\mathbb{F}_p^2$ induces an isometry of $\mathbb{Z}_p^2$, which can be extended to an isometry of $\mathbb{Q}_p^2$ extending by identity. For instance, let us consider an isometry $\varphi$ which sends, for $i = 0 \ldots p - 1$, $p\mathbb{Z}_p \times (i + p\mathbb{Z}_p)$ to $(i + p\mathbb{Z}_p)^2$. Then $p\mathbb{Z}_p \times \mathbb{Z}_p$ is a box, but $\varphi(p\mathbb{Z}_p \times \mathbb{Z}_p)$ is not a box. \{lemma:R\}

**Lemma 4.** Let $x \in K^\times$, $B \subset K$ some ball and $\lambda \in K^\times$. Let $j \geq 2$ be an integer.

(1) If $x + B \subset \lambda Q_{j,1}$, then $rv(x + z)$ is constant for $z \in B$, and takes the value $rv(x)$.

(2) Let us assume that $k_K \neq \mathbb{F}_2$. If $x + B \subset \lambda Q_{1,1}$, then $rv(x + z)$ is constant on $B$, and takes the value $rv(x)$.
Proof. Let us prove (1). This amounts to prove that the the diameter of $B$ is strictly less than $|x|$. Let us assume the contrary. Then $-x + wx \in B$. So $x + (-x + wx) = wx \in \lambda Q_{j,1}$. But this contradicts the fact that $x \in \lambda Q_{j,1}$.

(2) can be proven similarly. □

Remind [HM97, 3.1] that if $X \subset K^n$ is definable, the dimension of $X$, denoted by $\text{dim}(X)$ is the greatest integer $k$ for which there is a coordinate projection $\pi : K^n \to K^k$ such $\pi(X)$ has non-empty interior. Some general properties of this dimension are studied in [HM97].

Lemma 5 ([HM97]). Let $X \subset K^n$ be definable.

(1) Let $f : X \to K$ be definable. There exists a definable $Y \subset X$ such that $\text{dim}(Y) < \text{dim}(X)$ and $f|_{X \setminus Y}$ is continuous.

(2) There exists a decomposition in definable sets $X = A \cup B$ where $A$ is open and $\text{dim}(B) < n$.

Proof. (1) [HM97, Theorem 5.4].

(2) Indeed, take $A = \overset{5}{\overset{\circ}{X}}$, the topological interior of $X$, and $B = X \setminus A$. □

Proposition 6. Let $m \in \mathbb{N}^*$. Let $X \subset K^n$ and $f : X \to K^\times$ be some definable function. There exists a decomposition $X = A_1 \cup \ldots \cup A_M \cup B$ in definable sets such that $\text{dim}(B) < n$, the $A_i$’s are open, and $rv_m(f)$ is constant on any ball contained in some $A_i$.

We will implicitly use Lemma 5 (2) in this proof.

Proof. Since $RV_m \simeq \mathbb{Z} \times (\mathcal{O}_K / \mathcal{M}_K^m)^\times \cup \{0\}$, and that under this identification $rv_m \simeq | \cdot | \times |x|$, it is enough to prove local constancy for $| \cdot |$ and $|x|$. The last one is easy to perform since $(\mathcal{O}_K / \mathcal{M}_K^m)$ is finite, so we reduce to prove the proposition for $| \cdot |$ only. We do it by induction on $n$.

For $n = 0$ this is OK.

Let us assume this is true for $n$, and let $X \subset K^{n+1}$. Up to shrinking $X$, we can apply the p-adic cell decomposition Theorem, (see [Clu04, Theorem 2.3] for instance). So we can assume that $X$ is a cell of the form

$X = \{(x,t) \in X' \times K \mid x \in X', |\alpha(x)|t - c(x)|1| \text{ and } t - c(x) \in \lambda Q_{j,k}\}$

where $X' \subset K^n$ is definable, $\lambda \in K^\times$, $\alpha, \beta : X' \to K$ are definable, $\square_i$ are $<$ or no condition, and on $X$

$|f(x,t)| = |g(x)| \cdot |(t - c(x))^{a} \lambda^{-a}|^{\frac{1}{b}}$

for some integers $a, b \in \mathbb{N}$.

By induction hypothesis, and shrinking $X'$ if necessary, we can assume that $|g(x)|, |\alpha(x)|, |\beta(x)|$ and $rv(c(x))$ are constant on balls of $X'$, and so also on balls of $X$. Now, we add new conditions such as $t - c(x) \in \lambda Q_{j,1}$ for some $j \geq 2$. Then, thanks to Lemma 4, $rv(t - c(x))$ will be constant on balls of $X$, so in particular, $|t - c(x)|$ will be constant on balls of $X$. □
2. Technical results

Lemma 7. Let \( f : X \to K \) be some definable \( \lambda \)-Lipschitz function. There exists a unique \( \lambda \)-Lipschitz extension \( \overline{f} : \overline{X} \to K \) which is also definable.

Proof. OK.

Lemma 8. Let \( X \subseteq Y \subseteq K^n. \) Let \( r : Y \to X \) be some Lipschitz retraction. Then for all \( x \in Y \), \( |r(x) - x| = d(x, X) \) and \( X \) is necessarily closed in \( Y \).

Proof. Let us assume that \( |r(x) - x| > d(x, X) \), and let \( x' \in X \) such that \( |r(x) - x| > |x' - x| \). Then \( |r(x) - r(x')| = |r(x) - x'| + |x' - x| = |r(x) - x| + |x - x'| > |x - x'| \). This contradicts the fact that \( r \) is Lipschitz.

And \( X \) is closed in \( Y \) because if \( x_n \to x \) where \( x_n \in X \), then \( |r(x_n) - r(x)| = |x_n - r(x)| \leq |x_n - x| \to 0 \). So \( r(x) = x \in X \).

Lemma 9 (Tristan’s Gluing Lemma). Let \( X \subseteq K^n. \) Let \( X_i \subseteq X \) be a finite collection of definable sets and \( r_i : X \to X_i \) some definable Lipschitz retractions. Then there exists a definable Lipschitz retraction \( r : X \to \bigcup_i X_i \).

Proof. For simplicity, we can assume that there are two sets \( X_1, X_2 \). Define \( r \) by

\[
  r : X \to X_1 \cup X_2 \quad \begin{cases} r_1(x) & \text{if } d(x, X_1) \leq d(x, X_2) \\ r_2(x) & \text{otherwise.} \end{cases}
\]

Let \( x, y \in K^n \) and let us assume that \( |r(x) - r(y)| > |x - y| \). By definition of \( r \) this implies (up to permutation of \( x \) and \( y \)) that \( d(x, X_1) \leq d(x, X_2) \) and that \( d(y, X_1) > d(y, X_2) \), so that \( x' := r(x) \in X_1 \) and \( y' := r(y) \in X_2 \).

\[
  (1) \quad |x - y'| \geq d(x, X_2) \geq d(x, X_1) = |x - x'|.
\]

This again implies that

\[
  (2) \quad |x - y'| \geq \max(|x - x'|, |x - y'|) \geq |x' - y'|.
\]

Moreover

\[
  (3) \quad |y - x'| \geq d(y, X_1) > d(y, X_2) = |y - y'|.
\]

This again implies that

\[
  (4) \quad |y' - x'| = |(y' - y) + (y - x')| = |y - x'|.
\]

Finally

\[
  (5) \quad |x - y'| \geq |x' - y'| = |y - x'| > |y - y'|.
\]

This implies that

\[
  (6) \quad |x - y| = |(x - y') + (y' - y)| = |(x - y')| \geq |x' - y'|.
\]

Definition 10 ((Open) Cell centred in zero). If \( G \subseteq (RV_m \setminus \{0\})^n \) is non-empty and definable, we say that

\[
  C := rv_m^{-1}(G) \subseteq (K^\times)^n
\]
is an open cell centred in zero. We also say that
\[ C' := C \times \{0, \ldots, 0\} \subset K^{n+k} \]
k times
is a cell centred in zero.

\{defi:monomial\}

**Definition 11** (Monomial function). We say that a definable map \( f : X \subset (K^\times)^n \to \mathbb{Z} \) is monomial if there exists \( m \in \mathbb{N} \) such that \( f \) factorizes by some \((RV_m)^n\). By this we mean that there exists some definable map \( \tilde{f} : (RV_m)^n \to \mathbb{Z} \) such that \( f = \tilde{f} \circ \pi \) where \( \pi : K^n \to (RV_m)^n \) is the quotient map. This is equivalent to say that piecewise locally \( f \) is given by a monomial \( \prod_i |y_i|^{a_i} \) for some \( a_i \in \mathbb{Q} \).

The following lemma illustrates how this definition is useful to build centred cells.

\{lemma:newcell\}

**Lemma 12.** Let \( C = rv_m^{-1}(G) \subset K^n \) be some open centred cell. Let \( C' \subset K^{n+1} \) be a 1-cell over \( C \) defined by
\[ C' = \{(y, t) \in C \times K \mid |\alpha(y)| \leq |t| \leq |\beta(y)| \text{ and } t \in \lambda Q_{m,n}\} \]
and let us assume that the definable maps \(|\alpha|, |\beta| : C \to \mathbb{Z} \) are monomial. Then \( C' \) is an open centred cell.

**Proof.** Follows from the definitions. \( \square \)

\{lemma:cellmon\}

**Lemma 13.** Let \( C \subset (K^\times)^n \) be some open centred cell. Let \( f : C \to \mathbb{Z} \) be some definable function, and let us assume that \( f \) is constant on any ball contained in \( C \). Then \( f \) is monomial.

**Proof.** Let us write \( C = rv_m^{-1}(G) \) for some definable \( G \subset (RV_m \setminus \{0\})^n \). Then for any \( g \in G \), \( rv_m^{-1}(g) \) is a ball. So \( f \) is constant on \( rv_m^{-1}(g) \), so \( f \) factorizes by \((RV_m)^n\), which by definition means that \( f \) is monomial. \( \square \)

\{lemma:Tristan\}

**Lemma 14.** Let \( G \subset (RV_m \setminus \{0\})^n \) be non-empty and definable. Let us assume that
\[ (E) \quad (\overline{w}_m(g_1) \ldots \overline{w}_m(g_n)) \text{ is constant for all } (g_1 \ldots g_n) \in G. \]

Let
\[ C := rv_m^{-1}(G) \subset (K^\times)^n \]
be the associated open cell centred in zero. Let \( G' \subset \mathbb{Z}^n \) be the projection of \( G \), and let
\[ C' := \text{ord}^{-1}(G') \subset (K^\times)^n. \]

Then there is a definable Lipschitz retraction \( r \) from \( C' \) to \( C \).

**Proof.** The set \((O_{K^\times}/1 + M_K^n)\) is finite, of size \( N = (q-1)q^{n-1} \) to be precise. Let
\[ 1 = \xi_1, \xi_2, \ldots, \xi_N \subset O_{K^\times} \]
be a set of representatives. So if \((u_1 \ldots u_n) \in (RV_m)^n\) and \((\gamma_1 \ldots \gamma_n) = \text{ord}(u_1 \ldots u_n) \in \mathbb{Z}^n \), and if we set \( A = rv_m^{-1}(u_1 \ldots u_n) \) and \( B = \text{ord}^{-1}(\gamma_1 \ldots \gamma_n) \), one has the following decomposition
\[ B = \prod_{(\xi_1 \ldots \xi_n)} (\xi_1 \ldots \xi_n) \cdot A. \]
So by definition of $C$ and $C'$, if $x \in C'$, there exists some unique $n$-uple $(i_1 \ldots i_n)$ such that $(\xi_{i_1} \ldots \xi_{i_n})x \in C$. We define the retraction $r$ as follows.

$$r : \ C' \to C \quad x \mapsto (\xi_{i_1} \ldots \xi_{i_n})x \text{ where } (i_1 \ldots i_n) \text{ is such that } (\xi_{i_1} \ldots \xi_{i_n})x \in C.$$ 

Let us check that $r$ is Lipschitz by a tedious disjunction case. Let $x = (x_1 \ldots x_n)$ and $y = (y_1 \ldots y_n) \in C'$. Let $(\xi_{i_1} \ldots \xi_{i_n})$ (resp. $(\xi'_{i_1} \ldots \xi'_{i_n})$) be the tuple that appears in the definition of $r$. Let us fix some index $j \in \{1 \ldots n\}$.

Case 1. Let us assume that $|x_j| = |y_j|$.

Case 1.1. Let us assume that $\overline{rv}_m(x_j) = \overline{rv}_m(y_j)$. Then the condition (E) implies that $\xi_{i_j} = \xi'_{i_j}$. So

$$|r(x_j) - r(y_j)| = |\xi_{i_j}(x_j - y_j)| = |x_j - y_j|.$$ 

Case 1.2. Let us assume that $\overline{rv}_m(x_j) \neq \overline{rv}_m(y_j)$. This implies that $|x_j - y_j| \geq |\overline{v}^{m-1}| |x_j|$. Finally, by construction, $rv_m(r(x)_j) = rv_m(r(y)_j)$, so $|r(x)_j - r(y)_j| \leq |\overline{v}^m| |x_j|$. So

$$|r(x_j) - r(y_j)| \leq |\overline{v}^m| |x_j| < |x_j - y_j|.$$ 

Case 2. $|x_j| < |y_j|$. Then

$$|x_j| = |r(x)_j| < |r(y)_j| = |y_j|.$$ 

So

$$|r(x)_j - r(y)_j| = |r(y)_j| = |y_j| = |x_j - y_j|.$$ 

□

3. STATEMENT OF THE MAIN RESULT

A retraction from $K^n$ to a subset $X \subset K^n$ is a map $r : K^n \to X$ which is the identity on $X$.

Theorem 15 ($T_n$). Let $X \subset K^n$ and $f : X \to K$ be definable and $\lambda$-Lipschitz. Then there exists some $\lambda$-Lipschitz extension $\tilde{f} : K^n \to K$.

This is a direct corollary of this proposition.

Proposition 16 ($P_n$). Let $X \subset K^n$ be definable and closed. There exists a definable Lipschitz retraction $r : K^n \to X$.

Remark 17. (1) If we do not assume $X$ to be closed, this is false. For instance there is no continuous retraction from $K^\times$ to $K$.

(2) The Archimedean analogue of this proposition is false. For instance there is no continuous retraction $r : \mathbb{R} \to \{-1, 1\}$. However, when $X \subset \mathbb{R}^n$ is a closed convex set, the projection $r : \mathbb{R}^n \to X$ to the closest point of $X$ is a Lipschitz retraction.

(3) In this form, the analogue of Proposition 16 does not hold for ACVF. Indeed let $L$ is an algebraically closed valued field, and let $X = \{x \in L \mid |x| > 1\}$. Then $X$ is a closed set, but one can check that there is no Lipschitz retraction from $r : L \to X$. 


If we drop the definability condition, it is easy to prove that when $X \subset K^n$ is any closed subset, there is a Lipschitz retraction $r : K^n \to X$. This follows by Zorn’s Lemma, and the remark that if $z \notin X$, if $x \in X$ satisfies $d(z, x) = d(z, X)$, then the map $X \cup \{z\} \to X$ which sends $z$ to $x$ and is the identity on $X$ is Lipschitz.

Proof of Theorem 15. Let $\overline{f} : \overline{X} \to K$ be the definable $\lambda$-Lipschitz extension of $f$ (see Lemma 7). Let $r : K^n \to \overline{X}$ be some definable Lipschitz retraction as in Proposition 16. Then $\overline{f} = \overline{f} \circ r$ extends $f$, and is $\lambda$-Lipschitz.

Proof. We start the proof of Proposition 16. We do this by induction on $\dim(X)$. According to Proposition 6, we can find a decomposition $X = \bigcup_{i=1}^{m} X_i$ in definable sets, and for each $i$ some definable isometry $\varphi_i : K^n \to K^n$ such that if we set $C_i := \varphi_i(X_i)$, then $C_i$ is a cell centred in zero.

**Case 1.** Let us assume that $X$ is a 0-cell over $Y$. This means that $X = Y \times \{0\}$. By induction hypothesis (\(Q_{n-1}\)), there exists a definable isometry $\psi : K^{n-1} \to K^{n-1}$ such that $Y' := \psi(Y)$ is a centred cell. Then $\varphi = \psi \times \text{id} : K^n \to K^n$ is a definable isometry such that $\varphi(X) = Y' \times \{0\}$ which is a centred cell.

**Case 2.** Let us assume that $X$ is a 1-cell over $Y$. This means that there exist some definable maps $\alpha, \beta : Y \to K$, some $\lambda \in K^{\times}$ and $m, n > 0$ some integers such that

$$\chi(X) = \{(y, t) \in Y \times K \mid |\alpha(y)| \leq |t| \leq |\beta(y)| \text{ and } t \in \lambda Q_{m,n}\}.$$ 

By induction hypothesis (\(R_{n-1}\)) (see Lemma 19 below), cutting $Y$ in finitely many pieces, and up to isometry, we can assume that $Y \subset K^{n-1}$ is a centred cell and that $|\alpha|, |\beta| : Y \to \mathbb{Z}$ are monomial. So we conclude with Lemma 12.

(\(Q_{n-1}\)), there exists a definable isometry $\psi : K^{n-1} \to K^{n-1}$ such that $Y' := \psi(Y)$ is a centred cell.

Proof. According to Proposition 6, we can find a decomposition $X = A_1 \cup \ldots \cup A_m$ in definable sets such that for each $A = A_i$, there is a definable isometry $\varphi : K^n \to K^n$ such that $\varphi(A)$ is a centred cell and $\varphi_{*}(f)$ is monomial.
constant on each ball of $A_i$. Thanks to $(Q_n)$, for each $i$ we can find a decomposition
$A_i = \bigcup_{j=1}^{n_i} A_{i,j}$ and definable isometries $\varphi_{i,j}$ of $K^n$, such that $\varphi_{i,j}(A_{i,j})$ is a centred
cell. For each $(i, j)$ we face two possibilities.

**Case 1.** $A_{i,j}$ is an open centred cell. Then $(\varphi_{i,j})_*(f)$ takes constants values on
balls, because this property is invariant under isometry. In addition, the domain
of $(\varphi_{i,j})_*(f)$ is $\varphi_{i,j}(A_{i,j})$ which is an open centred cell. Thanks to Lemma 13,
$(\varphi_{i,j})_*(f)$ is monomial.

**Case 2.** $A_{i,j}$ is not an open centred cell. Then $\dim(A_{i,j}) < n$.

**Conclusion.** Let us denote by $B_k$ the $A_{i,j}$ which are open cells around zero, and
by $C$ the union of $B$ and the others $A_{i,j}$. Thanks to case 1, we are done with the
$B_k$’s. Moreover $\dim(C) < n$. Thanks to $(Q_n)$ we can find a decomposition $C = \bigcup C_l$
such that up to some isometry, $C_l$ is a centred cell such that $\dim(C_l) < n$. This
means that (up to a permutation of the coordinates), we can write $C_l = C'_l \times \{0\}$ for
some $C' \subset K^{n-1}$. So we apply $(R_n-1)$ to the induced maps $C'_l \to Z$ to conclude.

**Corollary 20 ($S_n$).** Let $X \subset K^n$ a definable set such that $\dim(X) < n$. Then
there exists a definable Lipschitz retraction $r : K^n \to \overline{X}$. In addition, any definable
Lipschitz function $f : X \to K$ can be extended in a definable and Lipschitz way.

**Proof.** Indeed, thanks to Lemma 18, we can assume (up to an isometry) that there
exists some definable $X' \subset K^{n-1}$ such that $X = X' \times \{0\}$. So we can apply our
induction hypothesis $(P_{n-1})$ to conclude.

Thanks to Lemma 18 and the above result, it is enough to prove Proposition 16
for $C$ where $C$ is an open cell centred in zero.

**Lemma 21.** Let $U \subset \mathbb{Z}^{n-1}$ be definable. Let $\alpha, \beta : \mathbb{Z}^{n-1} \to \mathbb{Q}$ some linear forms
(with possibly $\alpha = -\infty$ and $\beta = +\infty$) such that for any $u \in U$ there exists some
$v \in \mathbb{Z}$ with $\alpha(u) \leq v \leq \beta(u)$. Let us set

$$V = \{(u, v) \in \mathbb{Z}^n \mid u \in U \text{ and } \alpha(u) \leq v \leq \beta(u)\}.$$ 

Let us also assume that

$$(C) \quad \{(\gamma_1, \ldots, \gamma_n) \in V, \text{ one has } \gamma_n \geq \gamma_i \text{ for } i = 1 \ldots n.$$ 

Then there exists some definable Lipschitz retraction $r : K^{n+1} \to \overline{\text{ord}^{-1}(V)}$.

**Proof.** **Step 1.** We first construct a retraction $\text{ord}^{-1}(U \times \mathbb{Z}) \to \overline{\text{ord}^{-1}(V)}$.

Let $f_+ : \text{ord}^{-1}(U) \to \overline{\text{ord}^{-1}(V)}$ be some Skolem function which satisfies $\text{ord}(u, f_+(u)) = (\text{ord}(u), [\alpha(u)])$ for all $u \in \text{ord}^{-1}(U)$, where $[\cdot]$ stands for the ceiling function.

Similarly, let $f_- : \text{ord}^{-1}(U) \to \overline{\text{ord}^{-1}(V)}$ be a Skolem function which satisfies
$\text{ord}(u, f_-(u)) = (\text{ord}(u), [\beta(u)])$ for any $u \in \text{ord}^{-1}(U)$. Let us set

$$H = \{(u, f_+(u)) \mid u \in \text{ord}^{-1}(U)\} \cup \{(u, f_-(u)) \mid u \in \text{ord}^{-1}(U)\}.$$ 

By construction, $H \subset \text{ord}^{-1}(V)$. Then $\dim(H) = \dim(\overline{H}) = n-1$ and $\overline{H} \subset \overline{\text{ord}^{-1}(V)}$. By induction hypothesis $(S_n)$, we can find $s : K^n \to \overline{H}$ some Lipschitz retraction.

We then define our retraction like this:

$$r : \text{ord}^{-1}(U \times \mathbb{Z}) \to \overline{\text{ord}^{-1}(V)}$$

$$z \mapsto \begin{cases} z \text{ if } \text{ord}(z) \in V \\ s(z) \text{ if } \text{ord}(z) \notin V \end{cases}$$
Let us prove that \( r \) is Lipschitz. Let us consider \( z = (y, x) \) and \( z' = (y', x') \in ord^{-1}(U \times Z) \) and let us prove that \( |r(z) - r(z')| \leq |z - z'| \).

If \( z = (y, x) \) and \( z' = (y', x') \) belong simultaneously to \( ord^{-1}(V) \) or \( ord^{-1}(V)^c \), then \( |r(z) - r(z')| \leq |z - z'| \) because identity and \( s \) are Lipschitz. So we will assume that \( ord(z) \in V \) and \( ord(z') \notin V \). This implies that \( ord(z) \neq ord(z') \).

Since \( |(y, x) - (y', x')| = \max |y - y'|, |x - x'| \) it follows that \( |(y, x) - (y', x)| \) and \( |(y', x) - (y, x')| \) are less or equal than \( |(y, x) - (y', x')| \). So we can assume that \( x = x' \) or \( y = y' \).

**Case 1:** \( x = x' \). So \( z = (y, x) \in ord^{-1}(V) \) and \( z' = (y', x) \notin ord^{-1}(V) \). So \( |y| \neq |y'| \). For simplicity, let us assume that \( |y| \neq |y'| \). In particular, \( |z - z'| \geq |y_1| \). Let \( z'' = (y, f_-(y)) \in ord^{-1}(V) \) (here we might also have taken \( f_+ \)). Since \( z, z'' \in ord^{-1}(V) \), according to condition \( (C) \), \( |f_-(y)| \leq |y_1| \) and \( |x| \leq |y_1| \).

So \( |z - z''| = |x - f_-(y)| \leq |y_1| \leq |z - z'| \).

Likewise

\[ |z'' - z'| = \max(|y - y'|, |x - f_-(y)|) \leq \max(|y - y'|, |y_1|) \leq |z - z'| \]

So it suffices to show that \( |r(z) - r(z'')| \leq |z - z''| \) and that \( |r(z'') - r(z')| \leq |z'' - z'| \).

Since \( z, z'' \in ord^{-1}(V) \), \( r(z) = z \) and \( r(z'') = z'' \) and this implies the first inequality. Finally, \( z'' \in H \) and \( z' \notin ord^{-1}(V) \) so by definition of \( r \), \( r(z'') = s(z'') \) and \( r(z') = s(z') \). Since \( s \) is Lipschitz, \( |r(z'') - r(z')| = |s(z'') - s(z')| \leq |z'' - z'| \).

**Case 2:** \( y = y' \). So \( z = (y, x) \in ord^{-1}(V) \) and \( z' = (y, x') \notin ord^{-1}(V) \).

**Case 2.1.** Let us assume that

\[ |x' < \alpha(|y|) \leq |x| \leq \beta(|y|) \]

So \( |z - z'| = |x - x'| = |x| \). Let us consider \( z'' = (y, f_-(y)) \). Then

\[ |x'| < |f_-(y)| = \frac{|\alpha(|y|)|}{|x|} \leq |x| \]

So

\[ |z' - z''| = |f_-(y)| \leq |x| \leq |z - z'| \]

Likewise

\[ |z'' - z| = |f_-(y) - x| = |x| \leq |z - z'| \]

So it suffices to show that \( |r(z'') - r(z')| \leq |z'' - z'| \) and that \( |r(z) - r(z'')| \leq |z - z'| \). The first inequality is true because \( r(z') = z' \) and \( r(z'') = z'' \) since \( z', z'' \in ord^{-1}(V) \). The second inequality is true because \( z'' \in H \) and \( ord(z) \notin V \) so by definition of \( r \), \( r(z'') = s(z'') \) and \( r(z) = s(z) \) and \( s \) is Lipschitz.

**Case 2.2.** Let us assume that

\[ \alpha(|y|) \leq |x| \leq \beta(|y|) < |x'| \]

So \( |z - z'| = |x - x'| = |x'| \). Let us consider \( z'' = (y, f_+(y)) \). Then

\[ |f_+(y)| = |\beta(|y|)| < |x'| \]

So

\[ |z' - z''| = |x' - f_+(y)| = |x'| = |z - z'| \]

Likewise

\[ |z'' - z| = |f_+(y) - x| \leq |f_+(y)| = |\beta(|y|)| < |x'| = |z - z'| \]

So it suffices to show that \( |r(z'') - r(z')| \leq |z'' - z'| \) and that \( |r(z) - r(z'')| \leq |z - z'| \). The first inequality is true because \( r(z') = z' \) and \( r(z'') = z'' \) since
Lemma 22. Let \( U \subset \mathbb{Z}^{n-1} \) be definable. Let \( a, b \in \mathbb{Z} \) with \( a > 0 \). Let \( \alpha, \beta : \mathbb{Z}^{n-1} \to \mathbb{Q} \) some linear forms (with possibly \( \alpha = -\infty \) and \( \beta = +\infty \)), such that for any \( u \in U \) one has \( \alpha(u), \beta(u) \in a\mathbb{Z} + b \) and there exists some \( v \in \mathbb{Z} \) with \( \alpha(u) \leq v \leq \beta(u) \). Let us set

\[
V = \{(u, v) \in \mathbb{Z}^n \mid u \in U, v \in a\mathbb{Z} + b \text{ and } \alpha(u) \leq v \leq \beta(u)\}
\]

\[
V' = \{(u, v) \in \mathbb{Z}^n \mid u \in U \text{ and } \alpha(u) \leq v \leq \beta(u)\}
\]

Then there exists some definable Lipschitz retraction \( r : \text{ord}^{-1}(V') \to \text{ord}^{-1}(V) \).

Proof. Define \( r \) by

\[
r : \text{ord}^{-1}(V') \to \text{ord}^{-1}(V),
\]

\[
(x_1 \ldots x_n) \mapsto (x_1 \ldots x_{n-1}, \varpi^i x_n) \text{ where } i \in \{0 \ldots a - 1\}
\]

is the unique index such that

\[
(x_1, \ldots x_{n-1}, \varpi^i x_n) \in \text{ord}^{-1}(V).
\]

Proof of Proposition 16: \((\mathcal{P}_n)\). Thanks to Lemma 9 and 18, it is enough to prove that for any centred cell \( C = rv_m^{-1}(G) \subset K^n \), there exists a definable Lipschitz retraction \( r : K^n \to \overline{C} \).

In fact, cutting \( G \) in finitely many parts, we can even assume that \( G \) satisfies the condition (E) of Lemma 14. So we can apply Lemma 14, and then assume that \( C \) is of the form \( \text{ord}^{-1}(G) \) for some definable \( G \subset \mathbb{Z}^n \).

For each \( i \) we introduce

\[
G_i = \{ (\gamma_1 \ldots \gamma_n) \in \mathbb{Z}^n \mid \gamma_i \geq \gamma_j \text{ for } j = 1 \ldots n\}.
\]

Then \( G = \bigcup_{i=1}^n G_i \). If we fix \( i \), up to a permutation of the coordinates, we can assume that \( G_i \) satisfies the condition (C) in Lemma 21. Finally, one can cut a definable set of \( \mathbb{Z}^n \) in a finite number of pieces of the form \( \text{ord}^{-1}(V) \) as in Lemma 22. So if we use 22 and 21, we can apply Lemma 21 which ends the proof.

4. Pedagogical example: the case of dimension 1

We want to sketch a proof of Proposition 16 when \( n = 1 \).

Proof of Proposition 16 when \( n = 1 \). We can assume that \( X = rv_m^{-1}(G) \).

If \( G \subset RV_m \setminus 0 \) is a singleton, then it is a ball. Let us pick some point \( x \in C \). The map extended outside \( C \) by the constant map to \( x \) is a definable Lipschitz retraction.
Let us assume that there exist $a \in \mathbb{N}^*$, $b_0, b_1 \in \mathbb{Z}$ and $c \in (O_K/M_{MK}^m \mathbb{K})^\times$ such that

$$G = \{ x \in RV_m \mid b \leq \text{ord}(x) \in -a\mathbb{N} + b \text{ and } \overline{w}_m(x) = c \}.$$ 

So $G$ has a component at infinity. Let us consider $S$ some system of representatives for $K^\times/Q_{m,a}$, say $\varpi^i\xi$ for $i = 0 \ldots a-1$, and $\xi$ some representatives of $(O_K/MK^m)^\times$. Then one easily checks that if ord$(x) \leq b$, there is a unique $\varpi^i\xi \in S$ such that $p^i\xi x \in C$. Finally let us pick $d$ a point in

$$B = \{ z \in K \mid \text{ord}(z) = b \text{ and } \overline{w}_m(z) = c \}.$$ 

So $B$ is the ball of smallest radius of $C$. We then define $r$ as follows.

$$r : \quad K^\times \rightarrow C$$

$$x \mapsto \begin{cases} x & \text{if } x \in C \\ \varpi^i\xi x & \text{if ord}(x) \leq b \\ d & \text{if ord}(x) > b \end{cases}$$

One checks by a tedious case disjunction that $r$ is Lipschitz.

Let us assume that there exist $a \in \mathbb{N}^*$, $b_0, b_1 \in \mathbb{Z}$ and $c \in (O_K/M_{MK}^m \mathbb{K})^\times$ such that

$$G = \{ x \in RV_m \mid a\mathbb{N} + b \geq \text{ord}(x) \in \text{ and } \overline{w}_m(x) = c \}.$$ 

Let us consider $S$ some system of representatives for $K^\times/Q_{m,a}$, say $\varpi^i\xi$ for $i = 0 \ldots a-1$, and $\xi$ some representatives of $(O_K/MK^m)^\times$. Then one easily checks that if ord$(x) \leq b$, there is a unique $\varpi^i\xi \in S$ such that $p^i\xi x \in C$. Finally let us pick $d$ a point in

$$B = \{ z \in K \mid \text{ord}(z) = b \text{ and } \overline{w}_m(z) = c \}.$$ 

So $B$ is the ball of greatest radius of $C$. We then define $r$ as follows.

$$r : \quad K^\times \rightarrow C$$

$$x \mapsto \begin{cases} x & \text{if } x \in C \\ \varpi^i\xi x & \text{if ord}(x) \geq b \\ d & \text{if ord}(x) < b \end{cases}$$

One checks by a tedious case disjunction that $r$ is Lipschitz.

Any definable set $G \subset RV_m$ is a finite union of sets of the above form. So we can conclude using Lemma 9. $\square$

References

