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Real closures of algebraic varieties.

Talk at the Quadratic Forms Conference at Kingston August 1976.

In this talk I want to discuss an application of the theory displayed in my Kingston lectures "Symmetric bilinear forms over algebraic varieties", cited here by [*], to the Galois theory of schemes. I shall use the terminology developed in the lectures throughout.

Let us first review the basic notions and theorems of Galois theory over schemes. The theorems and their proofs can be found in [SGA] in a slightly different language. We work with connected schemes which are, say, quasicompact and separated. We do not impose noetherian conditions. A finite covering of such a scheme X is a finite etale morphism $f:Y \to X$ with Y again a connected scheme. (I shortly say "covering" instead of "connected covering".) This means the following. f is affine, i.e. Y coincides up to canonical isomorphism with the spectrum Spec $\mathfrak{A}$ (cf. [EGA II, § 1]) of the quasicoherent $\mathcal{O}_X$ -algebra $\mathfrak{A} := f_*\mathcal{O}_Y$. Moreover $\mathfrak{A}$ is a locally free $\mathcal{O}_X$ -module, i.e. a "vector bundle", and for every point *) $x \in X$ the fibre $\mathfrak{A}(x) = \mathfrak{A}_x/\mathfrak{m}_x$ is a separable algebra over the field $\mathcal{O}_x/\mathfrak{m}_x$, and $\mathfrak{A}(X)$ contains no idempotents except 0 and 1. The rank of the vector bundle $\mathfrak{A}$ will be called the <u>degree</u> $[Y:X]$ of the finite covering.

Any projective system $\{Y_\alpha \to X, f_{\alpha\beta}\}$ of finite coverings has a projective limit $Y \to X$ in the category of schemes over X, cf. [EGA IV, § 8]. Indeed, we have an inductive system $\{\mathfrak{A}_\alpha, \lambda_{\alpha\beta}\}$ of quasicoherent $\mathcal{O}_X$-algebras $\mathfrak{A}_\alpha$ such that $Y_\alpha = \operatorname{Spec}(\mathfrak{A}_\alpha)$ and $\lambda_{\alpha\beta}:\mathfrak{A}_\beta \to \mathfrak{A}_\alpha$ corresponds with $f_{\alpha\beta}:Y_\alpha \to Y_\beta$. We take for Y just the spectrum of the quasicoherent $\mathcal{O}_X$-algebra

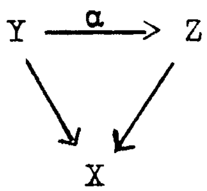
$$\mathfrak{A} := \varinjlim_{\alpha} \mathfrak{A}_\alpha .$$

All $\lambda_{\alpha\beta}$ are automatically injective. Thus we can think of the $\mathfrak{A}_\alpha$ as subsheaves of $\mathfrak{A}$ and of $\mathfrak{A}$ as the union of the $\mathfrak{A}_\alpha$.

Projective limits of finite coverings will be called <u>coverings</u> of X. The following fact about coverings is very useful.

<u>Lemma 1</u>. Assume $Y \to X$ and $Z \to X$ are coverings of X and $\alpha:Y \to Z$ is a morphism between these coverings, i.e. a morphism of schemes such that the diagram

*) It suffices to know this for the closed points.



commutes. Then $\alpha: Y \to Z$ is again a covering.

We call a scheme X' <u>simply connected</u> if X' is connected and does not admit any coverings except isomorphisms. A <u>universal covering</u> of X is a covering $\tilde{X} \to X$ with $\tilde{X}$ simply connected. It can be shown that a universal covering $\tilde{X} \to X$ always exists.

Any two universal coverings of X are isomorphic. This follows by use of the preceding Lemma 1 immediately from the following theorem, which moreover justifies the notion "universal covering".

<u>Theorem 1</u>. Let $\varphi: Y \to X$ be a covering and $\alpha: X' \to X$ be a morphism with X' simply connected. Then there exists at least one morphism $\beta: X' \to Y$ such that $\varphi \circ \beta = \alpha$. If φ is a finite covering then precisely $[Y:X]$ such morphisms β exist.

We now choose a fixed universal covering $\pi: \tilde{X} \to X$ and denote by G the group $\mathrm{Aut}(\tilde{X}|X)$ of all automorphisms of $\tilde{X}$ over X.

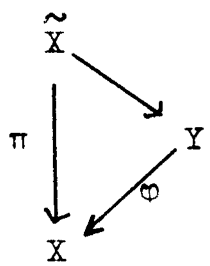
For every covering $\varphi: Y \to X$ the set $\mathrm{Mor}_X(\tilde{X}, Y)$ of morphisms from $\tilde{X}$ to Y over X is not empty according to the preceding theorem. Since any two universal coverings of X are isomorphic, we obtain easily the following statement.

Corollary. For any covering Y of X the group G operates transitively on the set $\mathrm{Mor}_X(\tilde{X},Y)$ of all morphisms from $\tilde{X}$ to Y over X.

If these morphisms are finite coverings then they all must have the same degree which we denote by $[\tilde{X}:Y]$. Otherwise we put $[\tilde{X}:Y] = \infty$.

The sets $\mathrm{Mor}_X(\tilde{X},Y)$ are in a natural way profinite spaces, i.e. projective limits of discrete topological spaces with only finitely many elements. Indeed, if $Y \to X$ is the projective limit of finite coverings $Y_\alpha \to X$, then $\mathrm{Mor}_X(\tilde{X},Y)$ can be regarded as the projective limit of the finite sets $\mathrm{Mor}_X(\tilde{X},Y_\alpha)$. In particular $(Y = \tilde{X})$ the group G is a profinite group in a natural way. We call this profinite group the absolute Galois group of X.

G operated continuously on all the spaces $\mathrm{Mor}_X(\tilde{X},Y)$. Thus for every commutative triangle



with φ a covering the subgroup $H := \mathrm{Aut}(\tilde{X}|Y)$ of G is closed in G.

Theorem 2. In this way we obtain a one-to-one correspondence between the isomorphism classes of commutative triangles as above and the closed sobgroups of G. This yields a one-to-one correspondence between the isomorphism classes of coverings of X and the conjugacy classes of closed subgroups of G.

The main point of our theory of real closures is that under mild restrictions on X *every signature* σ *of* X *yields an element of order* 2 *of the* Galois *group* G *uniquely determined by* σ *up to conjugacy*. Recall from [*] that the signatures of X are by definition the ring homomorphisms from the Witt ring W(X) to $\mathbb{Z}$, and that in the case that X is the spectrum Spec(F) of a field F these signatures correspond in a unique way with the orderings of F [*, II § 7].

We consider *pairs* (X,σ) consisting of a scheme X and a signature σ of X. A *morphism* $f:(X',\sigma') \to (X,\sigma)$ between pairs is a morphism $f:X' \to X$ of schemes such that the triangle

$$\begin{array}{ccc} W(X) & \xrightarrow{f^*} & W(X') \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\sigma'} & \\ & \mathbb{Z} & \end{array}$$

commutes. Notice that, if X and X' are spectra of fields F and F', this means, that the corresponding homomorphism from F to F' is compatible in the usual sense with the orderings corresponding to σ and σ'.

Given a morphism $f:X' \to X$ of schemes and signatures σ' on X' and σ on X we say that σ' "extends" σ with respect to f, or that σ is the "restriction" of σ' with respect to f, if f is a morphism from the pair (X',σ') to (X,σ).

Assume now that X is connected.

Definition 1. A covering of the pair (X,σ) is a morphism $\varphi:(Y,\tau) \to (X,\sigma)$ such that the morphism of schemes $\varphi:Y \to X$ is a covering in the sense explained above. A pair (S,ρ) is called real closed if S is connected and every covering of (S,ρ) is an isomorphism. A real closure of (X,σ) is a covering $\varphi:(S,\rho) \to (X,\sigma)$ with (S,ρ) real closed.

By use of Theorem 2 and Zorn's lemma we see easily that real closures exist for every pair (X,σ) with X connected.

Let now (X,σ) be a pair with X connected and divisorial (cf. [*,III § 1], e.g. X quasiprojective). Then the following two theorems hold true:

Theorem 3. If $\alpha:(S,\rho) \to (X,\sigma)$ is a morphism with (S,ρ) real closed and S divisorial and $\varphi:(Y,\tau) \to (X,\sigma)$ is a covering there exists at least one morphism β from (S,ρ) to (Y,τ) with $\varphi\circ\beta = \alpha$.

In particular, if α and φ both are real closures of (X,σ) then according to Lemma 1 the morphism β must be an isomorphism. (Notice that coverings of divisorial schemes are again divisorial.) Thus (X,σ) has up to isomorphism only one real closure.

Theorem 4 ("Fundamental theorem of algebra").
Let $\varphi:(S,\rho) \to (X,\sigma)$ be a real closure of (X,σ). Then $[\tilde{X}:S] \leq 2$. If there exists a prime number p which is a unit in $\mathcal{O}(X)$ *) then $[\tilde{X}:S] = 2$. If 2 is a unit in $\mathcal{O}(X)$ then moreover $\tilde{X}$ is isomorphic over S to $S[\sqrt{-1}]$, i.e. to the spectrum of the $\mathcal{O}_X$ -algebra $\mathcal{O}_X \oplus \mathcal{O}_X i$ with $i^2 = -1$.

Here is an example with $\tilde{X} = S$: Let X be the spectrum of the ring $\mathbb{Z}$. Then $W(X) = \mathbb{Z}$, cf. [MH, p. 90]. Thus X has a unique signature σ. On the other hand X is simply connected according to Minkowski's theorem that every number field except $\mathbb{Q}$ has a discriminant different from ± 1. Thus (X,σ) is real closed and $\tilde{X} = X$.

Theorems 3 and 4 have been proved in [K,I] for X affine. A proof for X divisorial can be performed on precisely the same lines using chapter III of [*]. {In [K] real closures of commutative rings with involution are studied. The real closures defined above are there called "strict real closures".} We shall reproduce below the proof of Theorem 3 since this proof has interesting repercussions to the theory of Witt rings of divisorial schemes.

*) e.g. X admits a morphism to the spectrum of a field.

Assume now that there exists a prime number p which is a unit in $\mathcal{O}(X)$. According to Theorems 3 and 4 we have a map Φ from the set Sign X of signatures on X to the set of conjugacy classes of elements of order 2 in G by attaching with a signature σ of X the conjugacy class $[\alpha]$ of an element α of G of order 2 such that the subgroup $\{1,\alpha\}$ of G determines a real closure of G in the sense of Theorem 2. The following direction for investigations about real closures seems to be reasonable. Given a scheme X as above of some fixed type try to obtain enough information about the coverings of X and their signatures to decide the following two questions:

Question 1. Is Φ injective ?

Question 2. Is Φ surjective ?

It has been shown in [K, II § 7 and § 10] that both questions have an affirmative answer if X is the spectrum Spec A of a connected semilocal ring A. It further has been shown in [K, II § 11] that Question 1 has an affirmative answer if X is an affine smooth connected curve over the field R of real numbers. Here the computation of the Witt rings of smooth real curves, reproduced in [*, Chap. V § 4], gives the necessary information. This computation has been generalized in [K_1, II § 10] to smooth curves over real closed fields. (The generalization seems to be non trivial.) The arguments in [K, II § 11] then yield an affirmative answer to Question 1 for connected smooth curves over arbitrary fields. One of my students works on singular curves, and very likely Question 1 has an affirmative answer also for these curves.

In all the cases described above it turned out to be true for a real closure (S,ρ) of (X,σ) that ρ is the only signature of S, which implies the affirmative answer to Question 1. If 2 is a unit in $\mathfrak{O}(X)$ – automatically true in the curve case – then moreover $W(S) = \mathbb{Z}$, and thus the signature $\sigma: W(X) \to \mathbb{Z}$ can be identified with the natural map from $W(X)$ to $W(S)$.

To prove Theorem 3 we need a supplement to the theory of Witt rings developed in [*]. Let $\varphi: Y \to X$ be a finite etale morphism with X an arbitrary scheme, and let (E,B) be a (symmetric) bilinear bundle over Y. The direct image $\varphi_*(\mathfrak{O}_Y)$ is a finite etale $\mathfrak{O}_X$-algebra $\mathfrak{A}$ and $\varphi_*(E)$ is a locally free $\mathfrak{A}$-module of finite type. Since $\mathfrak{A}$ is a locally free $\mathfrak{O}_X$-module of finite type, this implies that $\varphi_* E$ is a locally free $\mathfrak{O}_X$-module of finite type, i.e. a vector bundle over X. Let $Tr_\varphi: \mathfrak{A} \to \mathfrak{O}_X$ denote the regular trace associated with φ ([SGA, exposé I, § 4] or [EGA, IV § 18.2], which is a linear form on the vector bundle $\mathfrak{A}$ over X. The bilinear form

$$B: E \times_Y E \to \mathfrak{O}_Y$$

yields an $\mathfrak{A}$-bilinear form

$$\tilde{B}: \varphi_* E \times_X \varphi_* E \to \mathfrak{A}$$

as follows: If Z is an open subset of X then $\varphi_*(E) = E(\varphi^{-1} Z)$

and $\mathfrak{A}(Z) = \mathfrak{O}(\varphi^{-1}Z)$. Put

$$\tilde{B}_Z = B_{\varphi^{-1}Z} : E(\varphi^{-1}Z) \times E(\varphi^{-1}Z) \to \mathfrak{O}(\varphi^{-1}Z).$$

We now introduce the symmetric $\mathfrak{O}_X$-bilinear form

$$\varphi_*(B) := \mathrm{Tr}_\varphi \circ \tilde{B} : \varphi_* E \times_X \varphi_* E \to \mathfrak{O}_X,$$

and we call the bilinear bundle $(\varphi_*(E), \varphi_*(B))$ over X the <u>direct image</u> $\varphi_*(E,B)$ of the bilinear bundle (E,B) under φ.

If B is non degenerate then $\varphi_*(B)$ is again non degenerate. This follows easily from the fact that the symmetric $\mathfrak{O}_X$-bilinear form

$$\beta : \mathfrak{A} \times_X \mathfrak{A} \to \mathfrak{O}_X,$$

defined by

$$\beta(u,v) = \mathrm{Tr}_\varphi(uv)$$

for sections u and v of $\mathfrak{A}$ over some open set Z, is non degenerate since $\mathfrak{A}$ is finite etale [loc.cit.]. Thus we obtain a map

$$\varphi_* : \mathrm{Bil}(Y) \to \mathrm{Bil}(X)$$

from the set Bil(Y) of isomorphism classes of bilinear spaces over Y to Bil(X). This map φ_* clearly is compatible with orthogonal sums and thus induces an additive map

$$\varphi_* : K\,\mathrm{Bil}(Y) \to K\,\mathrm{Bil}(X).$$

{We use for this map and similar ones again the notation φ_*.} Assume now E is a bilinear space over Y and V is a subbundle of E. Then $\varphi_* V$ is a subbundle of $\varphi_* E$. Indeed, $\varphi_* E/\varphi_* V$ is canonically isomorphic to $\varphi_*(E/V)$ since φ is an affine morphism. It is easily checked that

$$(\varphi_* V)^{\perp} = \varphi_*(V^{\perp}).$$

In particular if V is a lagrangian of E, i.e. $V^{\perp} = V$, then $\varphi_* V$ is a lagrangian of $\varphi_* E$. From this observation we learn that the map φ_* from K Bil(Y) to K Bil(X) induces additive maps $\varphi_*: L(Y) \to L(X)$ and $\varphi_*: W(Y) \to W(X)$, which we call the <u>transfer maps corresponding to</u> φ

We shall make use of the transfer map $\varphi_*: W(Y) \to W(X)$. This map sends the Witt class $\{E\}$ of a space E over Y to the Witt class $\{\varphi_* E\}$. We also have, as always, the ring homomorphism $\varphi^*: W(X) \to W(Y)$ which makes W(Y) an algebra over W(X). The map $\varphi_*: W(Y) \to W(X)$ is now linear with respect to W(X) by the following lemma.

<u>Lemma 2</u>. If E is a bilinear bundle over X and F a bilinear bundle over Y then there exists a natural isomorphism from the bilinear bundle $\varphi_*(\varphi^*(E) \otimes F)$ onto the bilinear bundle $E \otimes \varphi_* F$.

<u>Proof</u>. Let Z be an affine open subset of X. Then $\varphi^{-1}Z$ is an affine open subset of Y, and

$$\mathfrak{A}(Z) = \mathcal{O}(\varphi^{-1}Z).$$

We have

$$\varphi_*(\varphi^*(E) \otimes F)(Z) = \varphi^*(E)(\varphi^{-1}Z) \otimes_{\mathfrak{A}(Z)} F(\varphi^{-1}Z) =$$

$$= (E(Z) \otimes_{\mathfrak{O}(Z)} \mathfrak{A}(Z)) \otimes_{\mathfrak{A}(Z)} F(\varphi^{-1}Z).$$

On the other hand

$$(E \otimes \varphi_* F)(Z) = E(Z) \otimes_{\mathfrak{O}(Z)} F(\varphi^{-1}Z).$$

Now it is easily checked that the natural isomorphism from the first $\mathfrak{O}(Z)$-module onto the second one is isometric with respect to the bilinear forms given on both modules.

The following base change lemma is also easily verfied.

Lemma 3. Consider a cartesian square

$$\begin{array}{ccc} Y & \xleftarrow{\alpha'} & Y \times_X X' \\ \varphi \downarrow & & \downarrow \varphi' \\ X & \xleftarrow[\alpha]{} & X' \end{array}$$

of schemes with φ finite etale, hence also φ' finite etale. Then for E a bilinear bundle over Y there exists a natural isomorphism

$$\varphi'_* \alpha'^* E \xrightarrow{\sim} \alpha^* \varphi_* E.$$

Assume now that $\varphi: Y \to X$ is a finite etale morphism and that X is divisorial.

Lemma 4. Let σ be a signature on X and assume that there exists an element z of W(Y) with $\sigma(\varphi_* z) \neq 0$. Then there exists a signature τ on Y which extends σ.

Proof. Let P denote the kernel of the homomorphism $\sigma: W(X) \to \mathbb{Z}$. According to [*, Chap.III] P is a minimal prime ideal of W(X). For every w in W(X) with $\varphi^*(w) = 0$ we have

$$w \cdot \varphi_*(z) = \varphi_*(\varphi^*(w) \cdot z) = 0$$

according to Lemma 2. Applying σ we learn that $\sigma(w) = 0$. Thus P contains the kernel of the ring homomorphism $\varphi^*: W(X) \to W(Y)$. By general commutative algebra this implies that there exists a minimal prime ideal Q of W(Y) lying over P with respect to φ^*. We have an injection

$$\mathbb{Z} = W(X)/P \to W(Y)/Q.$$

Thus by the prime ideal theory in [*, Chap. III] there exists a unique signature τ of Y with kernel Q. It is immediately verified that τ extends σ with respect to φ.

Let σ be a fixed signature on X and let $\mathrm{Sign}(\varphi,\sigma)$ denote the set of all signatures of Y which extend σ with respect to φ. Our proof of Theorem 3 will drop out from a study of "transfer formulas" for σ.

Definition 2. A transfer formula for σ and φ is a map $n: \mathrm{Sign}(\varphi,\sigma) \to \mathbb{Z}$ such that $n(\tau) = 0$ for finitely many τ in $\mathrm{Sign}(\varphi,\sigma)$ and

$$\sigma \circ \varphi_*(z) = \sum_{\tau | \sigma} n(\tau)\tau(z)$$

for all z in W(Y). Here the sum is taken over all τ in Sign(φ,σ) with the convention that the sum is zero of Sign(φ,σ) is empty.

Remark. We shall see below that actually Sign(φ,σ) is always a finite set.

Lemma 5. For given φ and σ there exists at most one transfer formula.

Proof Assume n and n' are different transfer formulas for φ and σ. We choose some τ_o in Sign(φ,σ) with $n(\tau_o) \neq n'(\tau_o)$. Let M denote the finite set consisting of all τ in Sign(φ,σ) such that $n(\tau) \neq 0$ or $n'(\tau) \neq 0$. For every τ in M let $P(\tau)$ denote the kernel of $\tau: W(Y) \to \mathbb{Z}$. Since all these $P(\tau)$ are minimal prime ideals of W(Y) [loc.cit.] the intersection of all $P(\tau)$ with τ in M, $\tau \neq \tau_o$, is not contained in $P(\tau_o)$. Thus we can find some z in W(Y) with $\tau_o(z) \neq 0$, but $\tau(z) = 0$ for all other τ in M. Now evaluating $\sigma\circ\varphi_*(z)$ using both transfer formulas n and n' we arrive at the contradiction

$$n(\tau_o)\tau_o(z) = n'(\tau_o)\tau_o(z).$$

Theorem 5.(i) For given φ and σ there always exists a transfer formula n. (ii) In this formula $n(\tau) > 0$ for every τ in S(φ,σ). In particular Sign(φ,σ) is finite. (iii) If $\alpha:(S,\rho) \to (X,\sigma)$ is a morphism with (S,ρ) real closed and S divisorial then for any τ in Sign(φ,σ) the number $n(\tau)$ is the cardinality of the set of all morphisms from (S,ρ) to (Y,τ) over (X,σ).

Remark. For every pair (X,σ) with X divisorial there exists a morphism α as above. Indeed, by [*, Chap. V, § 1] we can find such a morphism with S the spectrum of a real closed field. If X is connected then of course we can choose α as a real closure of (X,σ).

To prove Theorem 5 we consider the situation described in part (iii) of this theorem. According to Lemma 3 we have a commutative diagram

$$\begin{array}{ccc} W(Y) & \xrightarrow{\alpha'^*} & W(Y \times_X S) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi'_*} \\ W(X) & \xrightarrow{\alpha^*} & W(S) \end{array}$$

Thus for z in $W(Y)$

$$\sigma \circ \varphi_*(z) = \rho \circ \alpha^* \circ \varphi_*(z) = \rho \circ \varphi'_* \circ \alpha'^*(z).$$

Now $Y \times_X S$ is the disjoint union of finitely many connected schemes $Z_1, \ldots, Z_t$. Let $g_i : Z_i \to Y \times_X S$ denote the inclusion morphism from Z_i to $Y \times_X S$ and let

$$\alpha_i := \alpha' \circ g_i, \quad \varphi_i = \varphi' \circ g_i$$

be the components of α' and φ' corresponding to Z_i $(1 \leq i \leq t)$. The $\varphi_i : Z_i \to S$ are finite coverings. We have

$$\rho \circ \varphi'_* \circ \alpha'^*(z) = \sum_{i=1}^{t} \rho \circ \varphi_{i*} \circ g_i^* \circ \alpha'^*(z) =$$

$$= \sum_{i=1}^{t} \rho \circ \varphi_{i*} \circ \alpha_i^*(z).$$

Let i be a fixed index in $[1,t]$. If $[Z_i:S] > 1$ then ρ cannot extend to Z_i, since (S,ρ) is real closed. Thus by Lemma 4 the corresponding summand $\rho\circ\varphi_{i*}\circ\alpha_i^*(z)$ is zero. If $[Z_i:S] = 1$ then $\varphi_{i*} = (\varphi_i^*)^{-1}$, as is easily verified, and we obtain

$$\varphi_{i*}\circ\alpha_i^*(z) = \beta_i^*(z)$$

with $\beta_i := \alpha_i\circ\varphi_i^{-1}$. Now these β_i are precisely all morphisms from S to Y over X. Thus

$$\sigma\circ\varphi_*(z) = \sum_\beta \rho\circ\beta^*(z)$$

with β running through the finitely many morphisms from S to Y over X. For every such β the signature $\rho\circ\beta^*$ of Y clearly extends σ. We now <u>define</u> for τ in $\mathrm{Sign}(\varphi,\sigma)$ the natural number $n(\tau)$ as the number of all β with $\rho\circ\beta^* = \tau$. Clearly $n(\tau) = 0$ except for finitely many τ, and as we have just seen

$$\sigma\circ\varphi_*(z) = \sum_{\tau|\sigma} n(\tau)\tau(z)$$

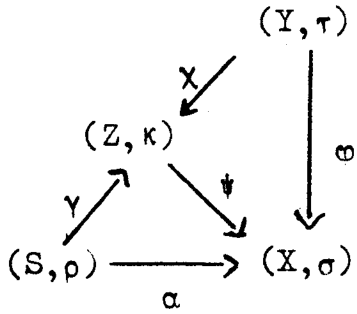
for every z in W(Y). Keeping Lemma 5 in mind the assertions (i) and (iii) of Theorem 5 are proved.

Let now τ_0 denote a fixed signature in $\mathrm{Sign}(\varphi,\sigma)$ and choose some morphism $\beta_0:(S,\rho) \to (Y,\tau_0)$ with (S,ρ) real closed (cf. the remark above). Applying assertion (iii) of Theorem 5 to the morphism $\alpha := \varphi\circ\beta_0$ we see $n(\tau_0) > 0$. Thus also assertion (ii) is proved.

We now deduce Theorem 3 from Theorem 5. Let (X,σ) be a pair with X a connected divisorial scheme, and let $\alpha:(S,\rho) \to (X,\sigma)$

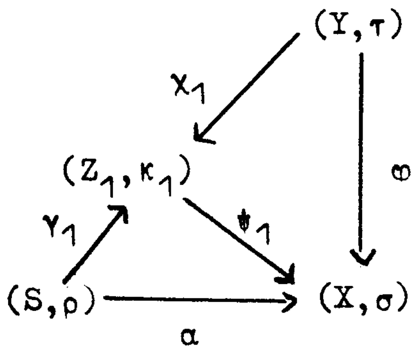
be a morphism with S divisorial and (S,ρ) real closed. Let further $\varphi:(Y,\tau) \to (X,\sigma)$ be a covering. We consider the set $\mathfrak{M}$ of isomorphism classes of commutative diagrams

D :



with ψ and hence χ a covering. If D and

D_1 :



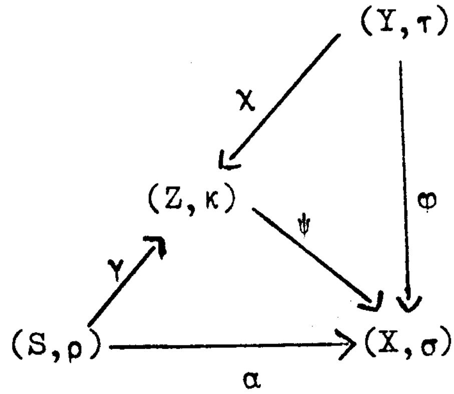
are two such diagrams, then we say that D_1 dominates D if there exists a morphism $\lambda:(Z_1,\kappa_1) \to (Z,\kappa)$ such that $\lambda\circ\chi_1 = \chi$ and $\lambda\circ\gamma_1 = \gamma$. Since χ_1 is an epimorphism this morphism λ then is uniquely determined by D and D_1. Moreover we have

$$\varphi = \psi\circ\chi = \psi\circ\lambda\circ\chi_1 = \psi_1\circ\chi_1,$$

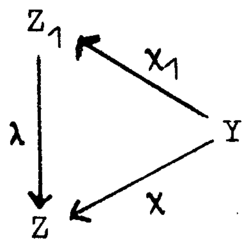
hence $\psi\circ\lambda = \psi_1$.

By this relation "D_1 dominates D" our set $\mathfrak{M}$ becomes partially ordered. It is easily seen that this ordering is inductive. Thus by Zorn's lemma there exists a maximal diagram

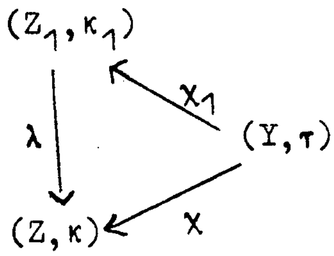
D :



Suppose χ is not an isomorphism. Then we can find a commutative triangle

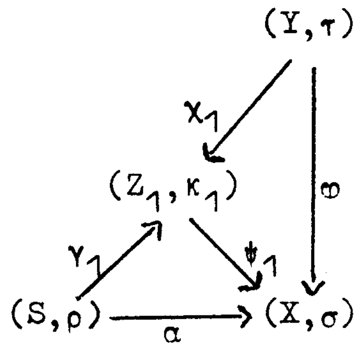


with λ a finite covering of degree > 1. Let κ_1 denote the signature $\tau \circ \chi_1^*$ on Z_1. Then $\kappa_1 \circ \lambda^* = \tau \circ \chi^* = \kappa$. Thus we have a commutative triangle

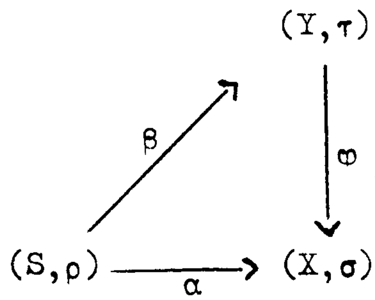


By Theorem 5, applied to the covering $\lambda:(Z_1,\kappa_1) \to (Z,\kappa)$ and the morphism $\gamma:(S,\rho) \to (Z,\kappa)$, there exists a morphism $\gamma_1:(S,\rho) \to (Z_1,\kappa_1)$ with $\lambda \circ \gamma_1 = \gamma$. We now have the commutative diagram

D_1 :



with $\psi_1 := \psi \circ \lambda$. This diagram D_1 dominates D and is not isomorphic to D, in contradiction to the maximality of D. Thus χ must be an isomorphism. With $\beta := \chi^{-1} \circ \gamma$ we have the commutative triangle



as wanted. Theorem 3 is proved.

For another proof of Theorem 3 in the special case that also α is a covering see the paper [D] of A.Dress. In this paper an axiomatic proof in the frame work of "Green functors" is given.

The main tool in our proof of Theorem 3 are the transfer maps φ_* for finite etale morphisms φ. It would be desirable to have a good definition of the transfer map φ_* for more general proper morphisms φ. Up to now no such definition exists. There seems to be no hope for such a definition for the functor W, but reasonable maps φ_* might exist for the functor L.

Theorem 5 is valuable beside its use to prove Theorem 3. Let $\varphi: Y \to X$ be a finite etale morphism, let σ be a signature on X, and assume that X is divisorial. Inserting $z = 1$ into the transfer formula we obtain

$$\sigma(\varphi_*(1)) = \sum_{\tau | \sigma} n(\tau).$$

$\sigma(\varphi_*(1))$ is more concretely the value of σ on the bilinear space $(\mathfrak{A}, \beta)$ with β the bilinear form induced by Tr_φ as described above. By an application of the corollary in [*, Chap. V, § 1] it is easily verified that

$$\sigma(\varphi_*(1)) \leq [Y:X].$$

Thus σ has at most [Y:X] extensions to Y.

If we have enough information about the "multiplicities" $n(\tau)$ Theorem 5 yields a quantitative extension theory for signatures with respect to finite etale morphisms. It has been shown in [K, II § 8] that if X is the spectrum of a semilocal ring then all multiplicities $n(\tau) = 1$. This implies for $\alpha: (S, \rho) \to (X, \sigma)$ a real closure that S has no automorphisms over X except the identity - a well known fact in the field case - since ρ is the only signature of S.

If X is a curve over $\mathbb{R}$ then $n(\tau)$ may attain arbitrary large values, cf. [K, II § 11].

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