Diquark condensation and the quark-quark interaction

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We employ a bispinor gap equation to study superfluidity at nonzero chemical potential, $\mu \neq 0$, in two- and three-color QCD, exploring the gap's sensitivity to the nature of the quark-quark interaction. The two-color theory, QC$_2$D, is an excellent exemplar; the order of truncation of the quark-quark scattering kernel K has no qualitative impact, which allows a straightforward elucidation of the effects of μ when the coupling is strong. In the three-color theory the rainbow-ladder truncation admits diquark bound states, a defect that is eliminated by an improvement of K. The corrected gap equation describes a superfluid phase that is semiquantitatively similar to that obtained using the rainbow truncation. A model study suggests that the width of the superfluid gap and the transition point in QC$_2$D provide reliable quantitative estimates of those quantities in QCD. [S0556-2813(99)07112-5]

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I. INTRODUCTION

In the application of Dyson-Schwinger equations [1] (DSEs) extensive use has been made of models based on the rainbow-ladder truncation, with contemporary variants [2] providing improved links with QCD. This truncation is also implicit in the class of model field theories with four-fermion interactions, such as the Nambu–Jona-Lasinio (NJL) model [3] and the global color model [4], which have been used successfully in describing aspects of the strong interaction. Such models admit the construction [5] of a meson-diquark auxiliary-field effective action, which is important in developing an understanding of nucleons using the relativistic Faddeev equation [6]. In addition, it is immediately apparent that the action's steepest-descent equations admit the possibility of diquark condensation, i.e., quark-quark Cooper pairing, and that was first explored using a simple version of the NJL model [7].

A nonzero chemical potential $\mu \neq 0$ promotes Cooper pairing in fermion systems, and earlier and independent of these developments in QCD phenomenology, the possibility that it is exhibited in quark matter was considered [8] using the rainbow-ladder truncation of the gap equation. Interest in this possibility has been renewed [9]. A quark-quark Cooper pair is a composite boson with both electric and color charge, and hence superfluidity in quark matter entails superconductivity and color superconductivity. However, the last feature makes it difficult to identify an order parameter that can characterize a transition to the superfluid phase: the Cooper pair is gauge dependent and an order parameter is ideally describable by a gauge-invariant operator.

Determining the (T,μ) phase diagram of QCD is an important goal. At $(T,\mu)=0$ there is a quark-antiquark condensate $\langle \bar{q}q \rangle \neq 0$, but it is undermined by increasing T and μ, and there is a domain of the (T,μ) plane for which $\langle \bar{q}q \rangle = 0$. Increasing T also opposes Cooper pairing. However, since increasing μ promotes it, there may be a (low-T, large-μ) subdomain in which quark matter exists in a superfluid phase. That domain may not be accessible at the Relativistic Heavy Ion Collider (RHIC), which will concentrate on $\mu \simeq 0$ where all studies indicate that QCD with two light flavors exhibits a chiral symmetry restoring transition $\langle \bar{q}q \rangle \to 0$, at $T \simeq 150$ MeV. However, it may be discernible in the core of dense astrophysical objects [8], which could undergo a transition to superfluid quark matter as they cool, and in baryon-density-rich heavy ion collisions at the BNL Alternating Gradient Synchrotron (AGS) and CERN Super Proton Synchrotron (SPS) [10]. An exploration of this possibility using numerical simulations of lattice QCD is inhibited by the absence of (i) a gauge-independent order parameter for the superfluid phase and (ii) a satisfactory procedure for the numerical estimation of an integral with a complex measure, such as the $\mu \neq 0$ QCD partition function. Consequently all the nonperturbative information we have comes from models.

The rainbow-ladder truncation has the feature and defect that it generates a quark-quark scattering kernel K that is purely attractive in the color antitriplet channel $\bar{3}_c$. It therefore not only yields a $\bar{3}_c$ scalar diquark condensate but also $\bar{3}_c$ diquark bound states [11], i.e., hitherto unobserved colored quark-quark bound states with masses (in GeV) [12]:

$$m^{ud}_{J^P=0^+}=0.74,\ \ m^{ud}_{1^+}=0.95,\ \ m^{ud}_{0^-}=1.5=m^{ud}_{1^-}. \qquad (1)$$

($us=ds$ diquarks are also bound, e.g., $m^{us}_{0^+}=0.88$. Color-sextet bound states do not exist because K is purely repulsive in this channel, even in rainbow-ladder truncation [11].) All models employed to date in the analysis of quark matter superfluidity have this defect [13], and we are primarily concerned with the question of whether any model or truncation with such a flawed representation of the quark-quark interaction can be a reliable tool for exploring superfluidity in quark matter at $\mu \lesssim 1$ GeV, i.e., in the nonperturbative domain. In addressing this issue, it is important to compare QC$_2$D with QCD because the same mechanism that provides for the absence of diquark bound states in the latter *must* guarantee their existence in QC$_2$D, where the diquark is the baryon of the theory. In fact, it must ensure that flavor-nonsinglet $J^{P=\mp}$ mesons are degenerate with $J^{\pm}$ diquarks [14].

In Sec. II we describe a bispinor DSE (*gap equation*) that is particularly useful for studying quark and diquark condensation and, in Sec. III, employ it in the general analysis of QC$_2$D and also to obtain quantitative results from a pedagogical model. In Sec. IV we focus on QCD, and employ the model's analogue to exemplify the gap equation and its solution in rainbow truncation, and also when a $1/N_c$-suppressed dressed-ladder vertex correction is included. We summarize and conclude in Sec. V.

II. GAP EQUATION

A direct means of determining whether a SU$_c(N_c)$ gauge theory supports 0^+ diquark condensation is to study the gap equation satisfied by[1]

$$\mathcal{D}(p,\mu) := \mathcal{S}(p,\mu)^{-1} = \begin{pmatrix} D(p,\mu) & \Delta^i(p,\mu)\,\gamma_5\lambda^i_\wedge \\ -\Delta^i(p,-\mu)\,\gamma_5\lambda^i_\wedge & CD(-p,\mu)^{\mathrm{T}}C^\dagger \end{pmatrix} \quad (2)$$

where, with $\omega_{[\mu]} = p_4 + i\mu$,

$$D(p,\mu) = i\vec{\gamma}\cdot\vec{p}\,A(\vec{p}^{\,2},\omega^2_{[\mu]}) + i\gamma_4\,\omega_{[\mu]}\,C(\vec{p}^{\,2},\omega^2_{[\mu]}) + B(\vec{p}^{\,2},\omega^2_{[\mu]}), \quad (3)$$

$\{\lambda^i_\wedge,\ i=1,\dots,n_c^\wedge,\ n_c^\wedge = N_c(N_c-1)/2\}$ are the antisymmetric generators of SU$_c(N_c)$, and $C=\gamma_2\gamma_4$ is the charge conjugation matrix:

$$C\gamma_\mu^{\mathrm{T}}C^\dagger = -\gamma_\mu; [C,\gamma_5]=0. \quad (4)$$

Using the gap equation to study superfluidity makes unnecessary a truncated bosonization, which in all but the simplest models is a procedure difficult to improve systematically.

In addition to the usual color, Dirac, and isospin indices carried by the elements of $\mathcal{D}(p,\mu)$, the explicit matrix structure in Eq. (2) exhibits the quark bispinor index and is made with reference to

$$Q(x) := \begin{pmatrix} q(x) \\ \underline{q}(x) := \tau_f^2\, C\,\bar{q}^{\mathrm{T}} \end{pmatrix}, \quad (5)$$

$$\bar{Q}(x) := (\bar{q}(x)\quad \underline{\bar{q}}(x) := q^{\mathrm{T}}\, C\, \tau_f^2), \quad (6)$$

where $\{\tau_f^i : i=1,2,3\}$ are Pauli matrices that act on the isospin index. Herein we only consider two-flavor theories SU$_f(N_f=2)$, because N_f does not affect the question at the core of our study, and focus on $T=0$, since nonzero T can only act to eliminate a condensate. A nonzero quark condensate $\langle \bar{q}q\rangle \neq 0$ is represented in the solution of the gap equation by $B(\vec{p}^{\,2},\omega^2_{[\mu]}) \not\equiv 0$ while diquark condensation is characterized by $\Delta^i(p,\mu)\not\equiv 0$, for at least one i.

The bispinor DSE can be written in the form

$$\mathcal{D}(p,\mu) = \mathcal{D}_0(p,\mu) + \begin{pmatrix} \Sigma_{11}(p,\mu) & \Sigma_{12}(p,\mu) \\ \gamma_4\,\Sigma_{12}(-p,\mu)\,\gamma_4 & C\Sigma_{11}(-p,\mu)^{\mathrm{T}}C^\dagger \end{pmatrix}, \quad (7)$$

where, in the absence of a diquark source term,

$$\mathcal{D}_0(p,\mu) = (i\gamma\cdot p + m)\,\tau_Q^0 - \mu\,\tau_Q^3, \quad (8)$$

with m the current-quark mass. Here we have introduced additional Pauli matrices $\{\tau_Q^\alpha, \alpha=0,1,2,3\}$ with $\tau^0 = \mathrm{diag}(1,1)$, which act on the bispinor indices. The structure of $\Sigma_{ij}(p,\mu)$ specifies the theory and, in practice, also the approximation or truncation of it.

III. TWO COLORS

As an important and instructive first example we consider QC$_2$D. In this special case $\Delta^i\lambda^i_\wedge = \Delta\tau_c^2$ in Eq. (2) and it is useful to employ a modified bispinor

$$Q_2(x) := \begin{pmatrix} q(x) \\ \underline{q}_2 := \tau_c^2\,\underline{q}(x) \end{pmatrix}, \quad (9)$$

with $\bar{Q}_2$ the obvious analog of Eq. (6), so that the Lagrangian's fermion-gauge-boson interaction term is simply

$$\bar{Q}_2(x)\,\frac{i}{2}\,g\,\gamma_\mu\tau_c^k\tau_Q^0\,Q_2(x)\,A_\mu^k(x) \quad (10)$$

because SU$_c(2)$ is pseudoreal, i.e., $\tau_c^2(-\vec{\tau}_c)^{\mathrm{T}}\tau_c^2 = \vec{\tau}_c$, and the fundamental and conjugate representations are equivalent.

A. Zero chemical potential

The gap equation at arbitrary order in the systematic, Ward-Takahashi identity preserving truncation scheme of Ref. [15] is readily derived. For $\mu=0$, $C=A$ in Eq. (3), all the functions in the dressed-bispinor propagator are real and the rainbow truncation yields the gap equation[2]

$$\mathcal{D}(p) = i\gamma\cdot p + m + \int \frac{d^4q}{(2\pi)^4}\, g^2 D_{\mu\nu}(p-q) \times \gamma_\mu \frac{\tau_c^k}{2}\,\mathcal{S}(q)\,\gamma_\nu \frac{\tau_c^k}{2}. \quad (11)$$

To solve Eq. (11) we consider a generalization [17] of Eq. (2),

$$\mathcal{D}(p) = i\gamma\cdot p\,A(p^2) + \mathcal{V}(-\boldsymbol{\pi})\,\mathcal{M}(p^2), \quad (12)$$

[1]In our Euclidean formulation, $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_\mu^\dagger = \gamma_\mu$, $p\cdot q = \sum_{i=1}^4 p_i q_i$, and $\mathrm{tr}_D[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = -4\,\epsilon_{\mu\nu\rho\sigma}$, $\epsilon_{1234}=1$.

[2]Renormalization is straightforward, even at $\mu\neq 0$ [16]; however, since it is not relevant to our central theme, we neglect it here.

$$\mathcal{V}(\boldsymbol{\pi}) = \exp\left\{ i\gamma_5 \sum_{l=1}^{5} T^l \pi^l(p^2) \right\} = \mathcal{V}(-\boldsymbol{\pi})^{-1}, \tag{13}$$

where $\{T^{1,2,3} = \tau_Q^3 \otimes \vec{\tau}_f, T^4 = \tau_Q^1 \otimes \tau_f^0, T^5 = \tau_Q^2 \otimes \tau_f^0\}$, $\{T^i, T^j\} = 2\delta^{ij}$, so that

$$\mathcal{S}(p) = \frac{-i\gamma\cdot p A(p^2) + \mathcal{V}(\boldsymbol{\pi})\mathcal{M}(p^2)}{p^2 A^2(p^2) + \mathcal{M}^2(p^2)} \tag{14}$$

$$:= -i\gamma\cdot p\, \sigma_V(p^2) + \mathcal{V}(\boldsymbol{\pi})\, \sigma_S(p^2). \tag{15}$$

[$\boldsymbol{\pi} = (0,0,0,0,-\frac{1}{4}\pi)$ produces an inverse bispinor propagator with the simple form in Eq. (2).]

For $\boldsymbol{\pi}(p^2) = \text{const}$, substituting Eq. (12) into Eq. (11) yields

$$\gamma\cdot p\,[A(p^2)-1] = -\int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q) \times \gamma_\mu \frac{\tau_c^k}{2} \gamma\cdot q\, \sigma_V(q^2)\, \gamma_\nu \frac{\tau_c^k}{2}, \tag{16}$$

$$\mathcal{M}(p^2) - m\,\mathcal{V}(\boldsymbol{\pi}) = \int \frac{d^4q}{(2\pi)^4} g^2 D_{\mu\nu}(p-q) \times \gamma_\mu \frac{\tau_c^k}{2} \sigma_S(q^2)\, \gamma_\nu \frac{\tau_c^k}{2}. \tag{17}$$

It is clear from these equations that the gap equation in rainbow truncation is *independent* of $\boldsymbol{\pi}$ in the chiral limit. As this result is true order by order in the truncation scheme of Ref. [15], it is a general property of the complete QC_2D gap equation. Hence, if the interaction is strong enough to generate a mass gap, then that gap describes a five-parameter continuum of degenerate condensates,

$$\langle \overline{Q}_2 \mathcal{V}(\boldsymbol{\pi}) Q_2 \rangle \neq 0, \tag{18}$$

and there are five associated Goldstone bosons: three pions, a diquark, and an anti-diquark, which is a well-known consequence of the Pauli-Gürsey symmetry of QC_2D.

For $m \neq 0$, it is clear from Eq. (17) that the gap equation requires $\mathrm{tr}_{fQ}[T^i \mathcal{V}] = 0$, i.e., in this case only $\langle \overline{Q}_2 Q_2 \rangle \neq 0$. The Goldstone bosons are now massive but remain degenerate.

The Landau gauge dressed-gauge-boson propagator is

$$g^2 D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \mathcal{F}(k^2), \tag{19}$$

and to explore $\mu \neq 0$ we employ a pedagogical model for the vacuum polarization in QC_2D:

$$\mathcal{F}_2(k^2) = \tfrac{64}{9} \pi^4\, \hat{\eta}^2\, \delta^4(k). \tag{20}$$

This form was introduced [18] for the modeling of confinement in QCD. However, it is also appropriate here because the string tension in QC_2D is nonzero, and that is represented implicitly in Eq. (20) via the mass scale $\hat{\eta}$. Further, a simple extension of the model has been used efficaciously [19,20] as a heuristic tool for the analysis of QCD at nonzero (T,μ). The mass of the model's $J=1$ composites is a useful reference scale and, for $m=0$ in rainbow-ladder truncation,

$$m_{J=1}^2 = \tfrac{1}{2}\, \hat{\eta}^2. \tag{21}$$

$m_{J=1}$ is weakly dependent on m, changing by $\lesssim 2\%$ on $m \in [0,0.01]\,\hat{\eta}$, while adding $1/N_c$-suppressed vertex corrections produces an increase of $<10\%$ [15].

B. Nonzero chemical potential

We now consider the $\mu \neq 0$ gap equation and suppose a solution of the form

$$\mathcal{D}(p,\mu) = \begin{pmatrix} D(p,\mu) & \gamma_5 \Delta(p,\mu) \\ -\gamma_5 \Delta^*(p,\mu) & \tilde{D}(p,\mu) \end{pmatrix}, \tag{22}$$

with $D(p,\mu)$ defined in Eq. (3) and $\tilde{D}(p,\mu) := C\, D(-p,\mu)\, C^\dagger$. In the *absence* of a diquark condensate; i.e., for $\Delta \equiv 0$,

$$[U_B(\alpha), \mathcal{D}(p,\mu)] = 0, \quad U_B(\alpha) := \mathrm{e}^{i\alpha \tau_Q^3 \otimes \tau_f^0}, \tag{23}$$

which is a manifestation of baryon number conservation in QC_2D.

The inverse $\mathcal{S}(p,\mu)$ is sufficiently complicated that it provides little insight directly. However, that can be obtained using Eq. (20), which yields an algebraic gap equation. Using the rainbow truncation we find, at $p^2 = |\vec{p}|^2 + p_4^2 = 0$,

$$A - 1 = \tfrac{1}{2}\, \hat{\eta}^2 K \{ A\,(B^{*2} - C^{*2}\mu^2) + A^* |\Delta|^2 \}, \tag{24}$$

$$\mu(C-1) = \frac{\mu}{2} \hat{\eta}^2 K \{ C\,(B^{*2} - C^{*2}\mu^2) - C^* |\Delta|^2 \}, \tag{25}$$

$$B - m = \hat{\eta}^2 K \{ B\,(B^{*2} - C^{*2}\mu^2) + B^* |\Delta|^2 \}, \tag{26}$$

$$\Delta = \hat{\eta}^2 K \{ \Delta\,(|B|^2 + |C|^2\mu^2) + \Delta\, |\Delta|^2 \}, \tag{27}$$

with $K^{-1} = |B^2 - C^2\mu^2|^2 + 2|\Delta|^2(|B|^2 + |C|^2\mu^2) + |\Delta|^4$. These equations illustrate the $B \leftrightarrow \Delta$ degeneracy described above for $(m,\mu) = 0$, that Δ is real for all μ, and also the action of μ, enhancing the coupling in the Δ equation but suppressing it for B, which is how an increasing μ promotes diquark condensation at the expense of the quark condensate.

For $(m,\mu) = 0$ the rainbow gap equation is Eq. (11) and with Eq. (20) the solution is

$$\mathcal{M}^2(p^2) := B^2(p^2) + \Delta^2(p^2) = \begin{cases} \hat{\eta}^2 - 4p^2, & p^2 < \dfrac{\hat{\eta}^2}{4} \\ 0, & \text{otherwise,} \end{cases} \tag{28}$$

$$A(p^2)=C(p^2)=\begin{cases} 2, & p^2<\dfrac{\hat{\eta}^2}{4} \\ \frac{1}{2}\left(1+\sqrt{1+\dfrac{2\hat{\eta}^2}{p^2}}\right), & \text{otherwise.} \end{cases} \tag{29}$$

The dynamically generated mass function $\mathcal{M}(p^2)$ is tied to the existence of quark and/or diquark condensates and breaks chiral symmetry. Further, in *combination* with the momentum-dependent vector self-energy, it ensures that the quark propagator does not have a Lehmann representation and hence can be interpreted as describing a confined quark [1]. The interplay between the scalar and vector self-energies is the key to realizing confinement in this way [21], and is precluded in studies that discard the vector self-energy.

For $\mu\neq 0$ and arbitrary p we solve the rainbow gap equation numerically, and determine whether quark or diquark condensation is stable by evaluating

$$\delta P := P(\mu,\mathcal{S}[B=0,\Delta]) - P(\mu,\mathcal{S}[B,\Delta=0]), \tag{30}$$

where the pressure is calculated using a steepest-descent approximation [22]:

$$P[\mathcal{S}] = -\mathrm{Tr}\ln[\mathcal{S}] - \tfrac{1}{2}\mathrm{Tr}[(\mathcal{D}-\mathcal{D}_0)\mathcal{S}]. \tag{31}$$

$\delta P>0$ indicates that diquark condensation is favored.

The calculation of δP is facilitated by employing the μ-dependent ‘‘bag constants’’ [17]

$$\mathcal{B}_B(\mu) := P(\mu,\mathcal{S}[B,\Delta=0]) - P(\mu,\mathcal{S}[B=0,\Delta=0]), \tag{32}$$

with $\mathcal{B}_\Delta(\mu)$ an obvious analog. They measure the stability of a quark- or diquark-condensed vacuum relative to that with chiral symmetry realized in the Wigner-Weyl mode. The $(m,\mu)=0$ degeneracy of the quark and diquark condensates is manifest in

$$\mathcal{B}_B(0)=\mathcal{B}_\Delta(0)=(0.092\,\hat{\eta})^4=(0.13\,m_{J=1})^4. \tag{33}$$

Increasing μ at $m=0$ and excluding diquark condensation one finds [19] chiral symmetry restoration at

$$\mu_{2c}^{B,\Delta=0}=0.28\,\hat{\eta} \tag{34}$$

when $\mathcal{B}_B(\mu)=0$; i.e., the pressures of the Wigner and quark-condensed phases are equal. However,

$$\text{for all}\quad \mu>0,\quad \delta P>0,\quad \mathcal{B}_\Delta(\mu)>0, \tag{35}$$

with

$$\mathcal{B}_\Delta(\mu_{2c}^{B,\Delta=0})=(0.20\,m_{J=1})^4>\mathcal{B}_\Delta(0). \tag{36}$$

Therefore, the vacuum is unstable with respect to diquark condensation for all $\mu>0$ and one has confinement and dynamical chiral symmetry breaking to arbitrarily large values. Of course, we have ignored the possibility that $\hat{\eta}$ is μ dependent. In a more realistic model, the μ dependence of $\hat{\eta}$ would be significant in the vicinity of $\mu_{2c}^{B,\Delta=0}$, with $\hat{\eta}\to 0$ as $\mu\to\infty$, which would ensure deconfinement and chiral symmetry restoration at large μ.

$\Delta\neq 0$ in Eq. (22) corresponds to $\boldsymbol{\pi}=(0,0,0,0,\frac{1}{2}\pi)$ in Eq. (18), i.e., $\langle \overline{Q_2} i\gamma_5\tau_Q^2 Q_2\rangle\neq 0$, and although the $\mu\neq 0$ theory is invariant under

$$Q_2\to U_B(\alpha)\,Q_2,\quad \overline{Q_2}\to\overline{Q_2}\,U_B(-\alpha), \tag{37}$$

which is associated with baryon number conservation, the diquark condensate breaks this symmetry:

$$\langle \overline{Q_2} i\gamma_5\tau_Q^2 Q_2\rangle \to \cos(2\alpha)\,\langle \overline{Q_2} i\gamma_5\tau_Q^2 Q_2\rangle - \sin(2\alpha)\,\langle \overline{Q_2} i\gamma_5\tau_Q^1 Q_2\rangle. \tag{38}$$

Hence, for $(m=0,\mu\neq 0)$, one Goldstone mode remains.

For $m\neq 0$ and small values of μ, the gap equation only admits a solution with $\Delta\equiv 0$; i.e., diquark condensation is blocked. However, with increasing μ a diquark condensate is generated; e.g., we find the following minimum chemical potentials for diquark condensation:

$$m=0.013\,m_{J=1}\Rightarrow\mu^{\Delta\neq 0}=0.051\,m_{J=1},$$
$$m=0.13\,m_{J=1}\Rightarrow\mu^{\Delta\neq 0}=0.092\,m_{J=1}. \tag{39}$$

Improving on rainbow-ladder truncation may yield quantitative changes of $\lesssim$20% in the illustrative results provided by our model of QC$_2$D. However, the pseudoreality of QC$_2$D and the equal dimension of the color and bispinor spaces, which underly the theory's Pauli-Gürsey symmetry, ensure that the entire discussion remains qualitatively unchanged. QCD, however, has two significant differences: the dimension of the color space is greater than that of the bispinor space and the fundamental and conjugate representations of the gauge group are not equivalent. The latter is of obvious importance because it entails that the quark-quark and quark-antiquark scattering matrices are qualitatively different.

IV. THREE COLORS

In canvassing superfluidity in QCD we choose $\Delta^i\lambda^i_\wedge=\Delta\,\lambda^2$ in Eq. (2) so that

$$\mathcal{D}(p,\mu)=\begin{pmatrix} D_\parallel(p,\mu)P_\parallel + D_\perp(p,\mu)P_\perp & \Delta(p,\mu)\gamma_5\lambda^2 \\ -\Delta(p,-\mu)\gamma_5\lambda^2 & \tilde{D}_\parallel(p,\mu)P_\parallel+\tilde{D}_\perp(p,\mu)P_\perp \end{pmatrix}, \tag{40}$$

where $P_\parallel=(\lambda^2)^2$, $P_\perp+P_\parallel=\mathrm{diag}(1,1,1)$, and $D_\parallel$, $D_\perp$, $\tilde{D}_\parallel$, $\tilde{D}_\perp$ are defined via obvious generalizations of Eqs. (2), (3), and (22). The evident, demarcated block structure makes explicit the bispinor index. Here each block is a 3×3 color matrix and the subscripts $\parallel$, $\perp$ indicate whether or not the subspace is accessible via λ_2. The bispinors associated with this representation are given in Eqs. (5) and (6), and in this case the Lagrangian's quark-gluon interaction term is $\bar{Q}(x)ig\Gamma^a_\mu Q(x)A^a_\mu(x)$,

$$\Gamma^a_\mu=\begin{pmatrix} \frac{1}{2}\gamma_\mu\lambda^a & 0 \\ 0 & -\frac{1}{2}\gamma_\mu(\lambda^a)^{\mathrm{T}} \end{pmatrix}. \tag{41}$$

It is again straightforward to derive the gap equation at arbitrary order in the truncation scheme of Ref. [15] and it is important to note that because

$$D_\parallel(p,\mu)P_\parallel+D_\perp(p,\mu)P_\perp=\lambda^0\left\{\frac{2}{3}D_\parallel(p,\mu)+\frac{1}{3}D_\perp(p,\mu)\right\} + \frac{1}{\sqrt{3}}\lambda^8\{D_\parallel(p,\mu)-D_\perp(p,\mu)\}, \tag{42}$$

the interaction $\Gamma^a_\mu\mathcal{S}(p,\mu)\Gamma^a_\nu$, necessarily couples the $\parallel$ and $\perp$ components. That interplay is discarded in models that ignore the vector self-energy of quarks, which is a qualitatively important feature of QCD [19–21,23].

A. Rainbow truncation

Diquark condensation at $\mu=0$ was studied in Ref. [24] using a minor quantitative adjustment of the confining model gluon propagator defined via Eq. (20):

$$\mathcal{F}(k^2)=4\pi^4\,\eta^2\,\delta^4(k), \tag{43}$$

with which the rainbow-truncation gap equation is

$$\mathcal{D}(p,\mu)=\mathcal{D}_0(p,\mu)+\tfrac{3}{16}\eta^2\,\Gamma^a_\rho\,\mathcal{S}(p,\mu)\,\Gamma^a_\rho. \tag{44}$$

Solving this and the ladder-truncation Bethe-Salpeter equation one obtains [19,20]

$$m^2_\omega=m^2_\rho=\tfrac{1}{2}\,\eta^2, \tag{45}$$

$$\langle\bar{q}q\rangle=(0.11\,\eta)^3, \tag{46}$$

$$\mathcal{B}_B(\mu=0)=(0.10\,\eta)^4, \tag{47}$$

and momentum-dependent vector self-energies, Eq. (29), which lead to an interaction between the $\parallel$ and $\perp$ components of $\mathcal{D}$ that blocks diquark condensation. This is in spite of the fact that $\lambda^a\lambda^2(-\lambda^a)^T=\frac{1}{2}\lambda^a\lambda^a$, which entails that the ladder-truncation quark-quark scattering kernel is purely attractive and strong enough to produce diquark bound states [11].

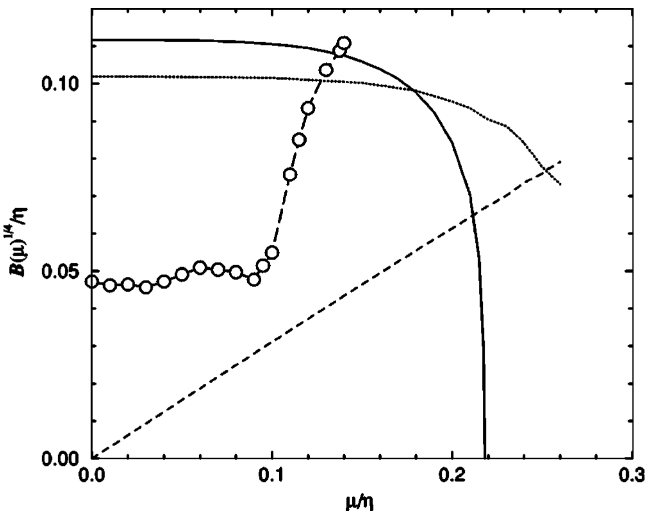

FIG. 1. μ-dependent ''bag constants'' in the QCD model defined via Eq. (43). Rainbow-truncated gap equation: dotted line, $\mathcal{B}_B(\mu)$; short-dashed line, $\mathcal{B}_\Delta(\mu)$. At the intersection, where the system flips to the superfluid phase, $\mathcal{B}_\Delta(\mu_c^{B=0,\Delta})=(0.75)^4\,\mathcal{B}_B(0)$. Vertex-corrected gap equation: solid line, $\mathcal{B}_B(\mu)$; long-dashed line with circles, $\mathcal{B}_\Delta(\mu)$. At the intersection, $\mathcal{B}_\Delta=(0.96)^4\,\mathcal{B}_B(0)$. The structure evident in $\mathcal{B}_\Delta(\mu)$ is an artifact characteristic of Eq. (43) [19].

For $\mu\neq0$ and in the *absence* of diquark condensation, the model defined via Eq. (43) exhibits [19] coincident, first order chiral symmetry restoring and deconfining transitions at

$$\mu_{c,\,\mathrm{rainbow}}^{B,\Delta=0}=0.28\,\eta, \tag{48}$$

which is where $\mathcal{B}_B=0$.

For $\mu\neq0$, however, Eq. (44) admits a solution with $\Delta(p,\mu)\not\equiv0$ and $B(p,\mu)\equiv0$. δP in Eq. (30) again determines whether the quark-condensed or superfluid phase is the stable ground state. With increasing μ, $\mathcal{B}_B(\mu)$ decreases, very slowly at first, and $\mathcal{B}_\Delta(\mu)$ increases rapidly from zero. As illustrated in Fig. 1, that evolution continues until

$$\mu_{c,\,\mathrm{rainbow}}^{B=0,\Delta}=0.25\,\eta=0.89\,\mu_{c,\,\mathrm{rainbow}}^{B,\Delta=0}, \tag{49}$$

where $\mathcal{B}_\Delta(\mu)$ becomes greater than $\mathcal{B}_B(\mu)$. This signals a first order transition to the superfluid ground state[3] and at the boundary

$$\langle\bar{Q}i\gamma_5\tau_Q^2\lambda^2Q\rangle_{\mu=\mu_{c,\,\mathrm{rainbow}}^{B=0,\Delta}}=(0.65)^3\,\langle\bar{Q}Q\rangle_{\mu=0}. \tag{50}$$

These results are typical [25] of rainbow truncation models in which the parameters in the dressed-gluon propagator are tuned to yield the correct π-ρ mass splitting. The solution of the rainbow gap equation, $\Delta(p,\mu_c^{B,\Delta})$, which is real and characterizes the diquark gap, is plotted in Fig. 2. It vanishes at $p^2=0$ as a consequence of the $\parallel$-$\perp$ coupling that blocked

[3]With η independent of μ, quark confinement, expressed as the absence of a Lehmann representation for $\mathcal{S}$, persists in the superfluid phase.

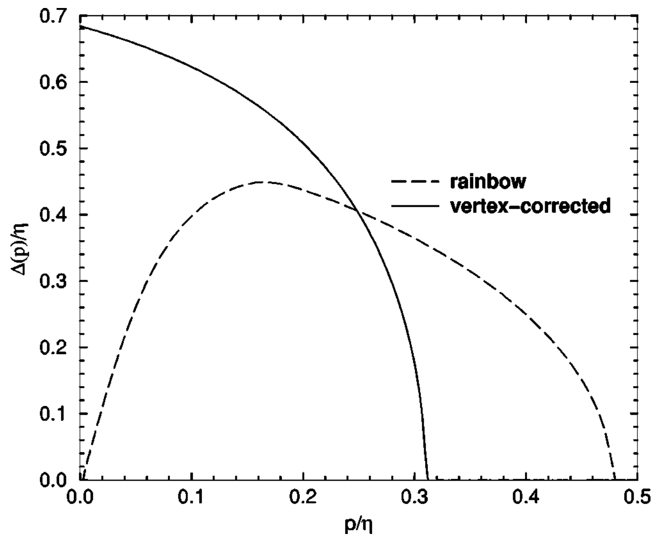

FIG. 2. Dashed line: $\Delta(z,\mu_c^{B,\Delta})$ obtained in rainbow truncation with the QCD model defined via Eq. (43), plotted for $\alpha=0$ as a function of p, where $z=p\,(0,0,\sin\alpha,i\mu+\cos\alpha)$. As μ increases, the peak position shifts to larger values of p and the peak height increases. Solid line: $\Delta(z,\mu=0)$ obtained as the solution of Eq. (51), the vertex-corrected gap equation, also with $\alpha=0$.

diquark condensation at $\mu=0$, and also at large p^2, which is a manifestation of the model's version of asymptotic freedom.

B. Vertex-corrected gap equation

The next-order term in the gap equation corresponds to adding a $1/N_c$-suppressed dressed-ladder correction to the quark-gluon vertex, and using Eq. (43) this yields

$$\mathcal{D}(p,\mu)=\mathcal{D}_0(p,\mu)+\tfrac{3}{16}\,\eta^2\,\Gamma^a_\rho\,\mathcal{S}(p,\mu)\,\Gamma^a_\rho - \tfrac{9}{256}\,\eta^4\,\Gamma^a_\rho\,\mathcal{S}(p,\mu)\,\Gamma^b_\sigma\,\mathcal{S}(p,\mu)\,\Gamma^a_\rho\,\mathcal{S}(p,\mu)\,\Gamma^b_\sigma, \tag{51}$$

which is illustrated in Fig. 3. The kernel of the Bethe-Salpeter equation receives three additional contributions at this order. Their net effect is repulsive at timelike total momentum and hence they prevent the formation of diquark bound states [15,26]. The η^4 term in Eq. (51) means that an algebraic solution cannot be obtained; however, a numerical solution is possible. For simplicity we only consider $m=0$ since $m\neq0$ opposes diquark condensation, as we saw in Sec. III. At this order there is a $\Delta\not\equiv0$ solution even for $\mu=0$, which is illustrated in Fig. 2, and

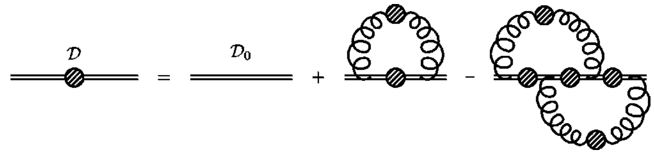

FIG. 3. Illustration of the dressed-ladder vertex-corrected gap equation, Eq. (51). Each bispinor quark-gluon vertex is bare, given by Eq. (41).

$$m_\rho^2=(1.1)^2\,m_\rho^{2\ \text{ladder}}, \tag{52}$$

$$\langle\bar{Q}Q\rangle=(1.0)^3\,\langle\bar{Q}Q\rangle^{\text{rainbow}}, \tag{53}$$

$$\mathcal{B}_B=(1.1)^4\,\mathcal{B}_B^{\text{rainbow}}, \tag{54}$$

where the rainbow-ladder results are given in Eqs. (45)–(47), and

$$\langle\bar{Q}i\gamma_5\tau_Q^2\lambda^2Q\rangle=(0.48)^3\,\langle\bar{Q}Q\rangle, \tag{55}$$

$$\mathcal{B}_\Delta=(0.42)^4\,\mathcal{B}_B. \tag{56}$$

Unsurprisingly the quark-condensed phase is favored. *Precluding* diquark condensation, the model exhibits coincident, first order chiral symmetry restoring and deconfinement transitions[4] at

$$\mu_c^{B,\Delta=0}=0.77\,\mu^{B,\Delta=0}_{c\ \text{rainbow}}. \tag{57}$$

Our numerical results[5] for the μ dependence of the ''bag constants'' are depicted in Fig. 1, which shows there is a transition to the superfluid phase at

$$\mu_c^{B=0,\Delta}=0.63\,\mu_c^{B,\Delta=0} \tag{58}$$

and at the boundary [cf. Eq. (50)]

$$\langle\bar{Q}i\gamma_5\tau_Q^2\lambda^2Q\rangle_{\mu=0.63\,\mu_c^{B,\Delta=0}}=(0.51)^3\,\langle\bar{Q}Q\rangle_{\mu=0}. \tag{59}$$

The ratio of the condensates increases by $<7\%$ on $\mu\in[0,\mu_c^{B=0,\Delta}]$. Quantitatively, the next-order correction leads to a reduction in the critical chemical potential for the transition to superfluid quark matter but does not much affect the width of the gap. Qualitatively, the transition occurs despite the significant effect that this correction has on the nature of the quark-quark interaction.

Further insight is provided by solving the inhomogeneous Bethe-Salpeter equation for the 0^+ diquark vertex in the quark-condensed phase. At $\mu=0$ and zero total momentum $P=0$, the additional contributions to the quark-quark scattering kernel generate an enhancement in the magnitude of the scalar functions in the Bethe-Salpeter amplitude. However, as P^2 evolves into the timelike region, the contributions become repulsive and block the formation of a diquark bound state. Conversely, increasing μ at any given timelike P^2 yields an enhancement in the magnitude of the scalar functions, and as $\mu\to\mu_c^{B,\Delta=0}$ that enhancement becomes large, which suggests the onset of an instability in the quark-condensed vacuum.

[4]At $\mu=0$ and $T\neq0$ these transitions are second order, and the critical temperature is reduced by $<2\%$ when calculated using Eq. (51) instead of the rainbow truncation [27].

[5]Following the evolution of the diquark gap with increasing μ is numerically difficult because of the interaction between the $\parallel$ and $\perp$ components of $\mathcal{D}$. With that interaction the gap equation yields nine coupled, quintic algebraic equations in nine variables, and of the many possible solutions one must follow the correct branch as p and μ evolve.

V. CONCLUSIONS

We have studied a confining model of QCD using a truncation of the Dyson-Schwinger equations that describes well the π-ρ mass splitting at $(T,\mu)=0$ and improves upon the rainbow-ladder truncation by ensuring that no diquark bound states appear in the spectrum. Employing a criterion of maximal pressure, we observe a first order transition to a chiral symmetry breaking superfluid ground state, which occurs at a chemical potential approximately two-thirds that required to completely eliminate the quark condensate in the absence of diquark condensation. Without fine-tuning, the superfluid gap at the transition is large, approximately one-half of that characterizing quark condensation. Thus, while completely changing the qualitative nature of the quark-quark interaction and hence the bound state spectrum in the model (eliminating unobserved colored bound states) our vertex-corrected gap equation yields a phase diagram that is semiquantitatively the same as that obtained using the rainbow truncation. This bolsters our confidence in the foundation of current speculations [13] about the phases of high-density QCD.

The procedure we used to improve the gap equation is equally applicable to two-color QCD, which we analyzed with the help of a pedagogical model for the dressed-gauge boson propagator. Diquark bound states must exist in QC_2D because they are the baryons of the theory, and the truncation procedure ensures this, with the result that flavor-nonsinglet $J^{P=\mp}$ mesons are degenerate with $J^{\pm}$ diquarks. Using a straightforward, constructive approach, we saw that at $\mu=0$ there are five Goldstone modes in QC_2D, and that one of them survives at $\mu\neq 0$. A nonzero current-quark mass opposes diquark condensation but for light fermions there is always a value of the chemical potential at which a transition to the superfluid phase takes place. Our model studies indicate that in some respects, such as the transition point and magnitude of the gap, the phase diagram of QC_2D is quantitatively similar to that of QCD. This observation can be useful because the simplest superfluid order parameter is gauge invariant in QC_2D and the fermion determinant is real and positive, which makes tractable the exploration of superfluidity in QC_2D using numerical simulations of the lattice theory [28]. The results of those studies can then be a reliable guide to features of QCD.

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