$K \to \pi\pi$ and a light scalar meson

J. C. R. Bloch,[1] M. A. Ivanov,[2] T. Mizutani,[3] C. D. Roberts,[1] and S. M. Schmidt[1]
[1]*Physics Division, Argonne National Laboratory, Argonne, Illinois 60439-4843*
[2]*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia*
[3]*Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0435*

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We explore the $\Delta I = \frac{1}{2}$ rule and ϵ'/ϵ in $K\to\pi\pi$ transitions using a Dyson-Schwinger equation model. Exploiting the feature that quantum chromodynamics penguin operators direct K_S^0 transitions through 0^{++} intermediate states, we observe an enhancement of $K\to\pi\pi_{I=0}$ transitions from the contribution of a putative light σ-meson. This mechanism also affects ϵ'/ϵ.

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I. INTRODUCTION

The $\Delta I = \frac{1}{2}$ rule is an empirical observation: the widths for nonleptonic decays of kaons and hyperons that change isospin by one-half unit are significantly larger than those for other K and Λ transitions; e.g., [1]

$$\Gamma_{K_S^0\to(\pi\pi)}/\Gamma_{K^+\to\pi^+\pi^0} = 660. \tag{1}$$

In terms of the amplitudes $M_{K_S^0\to\pi^+\pi^-}$ and $M_{K_S^0\to\pi^0\pi^0}$ that describe $K_S^0\to\pi\pi$ transitions, the pure isospin-zero and isospin-two $\pi\pi$ final states are

$$A_0 = \frac{1}{\sqrt{6}}(2M_{K_S^0\to\pi^+\pi^-} + M_{K_S^0\to\pi^0\pi^0}), \tag{2}$$

$$A_2 = \frac{1}{\sqrt{3}}(M_{K_S^0\to\pi^+\pi^-} - M_{K_S^0\to\pi^0\pi^0}), \tag{3}$$

and the ratio in Eq. (1) corresponds to

$$1/w := \mathrm{Re}(A_0)/\mathrm{Re}(A_2) \approx 22. \tag{4}$$

The analogous amplitude ratio for S-wave $\Lambda\to\pi N$ transitions is $|A_{1/2}/A_{3/2}|\approx 80$.

The processes involved are nonleptonic weak decays so one necessarily encounters quantum chromodynamics (QCD) effects in their analysis and the operator product expansion (OPE) can therefore be employed to good effect. Using the OPE the amplitude, A, for a given transition is expressed as the expectation value of an effective Hamiltonian:

$$A = \langle \mathcal{H}_{\rm eff}\rangle = \sum_i a_i(\mu)\langle Q_i(\mu)\rangle, \tag{5}$$

where μ is a renormalization point. The coefficients $a_i(\mu)$, are calculable in perturbation theory and describe short-distance effects. However, the expectation values of the local effective operators $\langle Q_i(\mu)\rangle$, contain the effects of bound state structure; i.e., long-distance QCD effects, and must be calculated nonperturbatively.

The transitions of interest herein are mediated by nonleptonic strangeness changing ($\Delta S=1$) effective operators. The simplest

$$Q_1 = \bar{s}_i O_\mu^- u_j \bar{u}_j O_\mu^- d_i, \tag{6}$$

$$Q_2 = \bar{s}_i O_\mu^- u_i \bar{u}_j O_\mu^- d_j, \tag{7}$$

with $O_\mu^\pm = \gamma_\mu(1\pm\gamma_5)$ and color indices: $i,j=1,\ldots,N_c$, have the flavor structure of the standard weak four-fermion current-current interaction, and there are eight other terms representing the QCD and electroweak (ew) penguin operators. (Q_1 results from QCD corrections to the weak current-current vertex. The penguin operators are also generated by QCD and ew corrections but their flavor structure is different.) At least some of these operators must have large expectation values if the $\Delta I = \frac{1}{2}$ rule is to be understood.

Another quantity that may be much influenced by the $\Delta S=1$ effective interaction is the ratio ϵ'/ϵ. The indirect CP violating parameter:

$$\epsilon := A(K_L\to\pi\pi_{I=0})/A(K_S\to\pi\pi_{I=0}) \tag{8}$$

measures the admixture of CP-even state in K_L:

$$\text{for}\quad \epsilon=0,\quad CP|K_{L/S}\rangle = \mp|K_{L/S}\rangle, \tag{9}$$

i.e., they are CP eigenstates. ϵ appears to be primarily determined by short-distance contributions from the weak nonleptonic $\Delta S=2$ effective interaction [2].

In contrast, ϵ' measures the phase of the heavy-quark CKM matrix elements in the standard model and

$$\frac{\epsilon'}{\epsilon} = \frac{1}{\sqrt{2}|\epsilon|}\mathrm{Im}\left(\frac{A_2}{A_0}\right), \tag{10}$$

with $|\epsilon| = 0.002\,280$, experimentally.[1] A nonzero value of ϵ'/ϵ entails direct transitions between CP-even and CP-odd

[1]In Eq. (10) we follow contemporary practice and make explicit the $\pi\pi$-scattering phase shifts: $\delta_{0,2}$, in factoring out the phase $\Phi_{\epsilon'} = (\pi/2)+\delta_2-\delta_0$. Then, using the experimental observations $\delta_0\approx 37°$, $\delta_2\approx -7°$, $\Phi_\epsilon\approx\pi/4$, one has $\Phi_{\epsilon'}-\Phi_\epsilon\approx 0$ and the imaginary part in Eq. (10) relates only to an explicit CP violating phase.

eigenstates. ϵ' is sensitive to the same penguin operators that contribute to the $\Delta I=\frac{1}{2}$ rule, and hence is likely to receive significant long-distance contributions. The current generation of experiments [3] appears to be consistent and an average value of the ratio is [4]

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right)=(2.1\pm0.46)\times10^{-3}. \tag{11}$$

A standard form of the $\Delta S=1$ effective interaction at a renormalization scale $\mu=1$ GeV is

$$\mathcal{H}_{\mathrm{eff}}^{\Delta S=1}=\tilde{G}_F\sum_{i=1}^{10} c_i(\mu)Q_i(\mu), \tag{12}$$

where $\tilde{G}_F=G_F V_{us}^* V_{ud}/\sqrt{2}$, $c_i(\mu)=z_i(\mu)+\tau y_i(\mu)$, $\tau=-(V_{ts}^* V_{td})/(V_{us}^* V_{ud})$, and $V_{ud}, \ldots,$ are the CKM matrix elements. [Direct CP violation is a measure of $\mathrm{Im}(\tau)$.] The coefficients $c_i(\mu)$, at next-to-leading order are quoted in Ref. [2], as are the operators: Q_i. We reproduce the coefficients in the Appendix, Eq. (A11), but not the operators and note only that $Q_{3,4,5,6}$ are the QCD penguin operators; e.g.,

$$Q_6=\bar{s}_i O_\mu^- d_j \sum_{q=u,d,s} \bar{q}_j O_\mu^+ q_i, \tag{13}$$

and $Q_{7,8,9,10}$ are the ew penguin operators; e.g.,

$$Q_8=\frac{3}{2}\bar{s}_i O_\mu^- d_j \sum_{q=u,d,s} e_q \bar{q}_j O_\mu^+ q_i, \tag{14}$$

where e_q is the quark's electric charge (in units of the positron charge). The expectation value of the operators in Eq. (12); i.e., the long-distance contributions, are the primary source of theoretical uncertainty in the estimation of w and ϵ'/ϵ, Eqs. (4) and (10).

Herein we calculate the expectation values of the operators in Eq. (12) using the Dyson-Schwinger equation (DSE) model of Ref. [5]. The DSE's are reviewed and some of their phenomenological applications are described in Refs. [6–14]. In this approach mesons are bound states of a dressed-quark and -antiquark with Bethe-Salpeter amplitudes describing their internal structure. It has already been used to explore CP violation in hadrons [15]. We describe the calculation and its elements in Sec. II, and present and discuss our results in Sec. III. Section IV is a brief recapitulation.

II. OPERATOR EXPECTATION VALUES

To calculate the expectation values in Eq. (12) we employ an impulse approximation that is consistent with the rainbow-ladder truncation of the DSE's. It yields real decay amplitudes, which ensures no conflict with the explicit factorization of the strong $\pi\pi$ phase shifts in Eq. (10). The fidelity of this approximation relies on it yielding the dominant contribution to the amplitudes' magnitude while providing none of the strong phase. It has proven efficacious; e.g., in analyzing $\pi\pi$ scattering [12,13] and the electromagnetic pion form factor [13,14], and in the latter case calculated corrections are small [16]. However, such successes do not preclude the possibility that unitarizing corrections; e.g., Refs. [17–20], may be quantitatively important in the present application. Anomalies in our analysis could signal this.

A. Charged kaon decay

The impulse approximation to the meson–meson transitions mediated by $\mathcal{H}_{\mathrm{eff}}^{\Delta S=1}$ is straightforward to evaluate; e.g., in the absence of ew penguins only the operators $Q_{1,2}$ contribute to $K^+\to\pi^+\pi^0$ transitions and

$$\langle\pi^+(p_1)\pi^0(p_2)|Q_1|K^+(p)\rangle=\frac{1}{\sqrt{2}}\sum_{i=1,2} N_c^i T_i(p_1,p_2), \tag{15}$$

$$T_1(p_1,p_2)=i\sqrt{2}\,\mathrm{tr}\,Z_2\int_{k_2}^{\Lambda} O_\mu^-\chi_\pi(k_2;-\tfrac{1}{2}p_2,-\tfrac{1}{2}p_2)2\,\mathrm{tr}\,Z_2 \times\int_{k_1}^{\Lambda} O_\mu^-\chi_K(k_1;p_2,p_1)\Gamma_\pi(k_1;-p_1)S_u(k_1), \tag{16}$$

$$iT_2(p_1,p_2)=2\sqrt{2}\,\mathrm{tr}\,Z_2^2\int_{k_1}^{\Lambda}\int_{k_2}^{\Lambda} O_\mu^-\chi_\pi(k_2;-\tfrac{1}{2}p_2,-\tfrac{1}{2}p_2) \times O_\mu^-\chi_K(k_1;p_2,p_1)\Gamma_\pi(k_1;-p_1)S_u(k_1), \tag{17}$$

with the trace over Dirac indices only, and

$$\chi_\pi(k;l_1,l_2)=S_u(k+l_1)\Gamma_\pi(k;l_1+l_2)S_u(k-l_2), \tag{18}$$

$$\chi_K(k;l_1,l_2)=S_s(k+l_1)\Gamma_K(k;l_1+l_2)S_u(k-l_2). \tag{19}$$

Here we use a Euclidean formulation with

$$\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu},\quad \gamma_\mu^\dagger=\gamma_\mu,\quad p\cdot q=\sum_{i=1}^{4} p_i q_i. \tag{20}$$

$\int_k^\Lambda := \int^\Lambda d^4k/(2\pi)^4$ is a mnemonic representing a translationally invariant regularization of the integral, with Λ the regularization mass scale that is removed ($\Lambda\to\infty$) as the final stage of any calculation, and $Z_2(\mu,\Lambda)$ is the quark wave function renormalization constant. $S_{f=u,s}$ are the dressed-quark propagators (we assume isospin symmetry) and $\Gamma_{H=K,\pi}$ are the meson Bethe-Salpeter amplitudes, both of which we discuss in detail in Sec. II B.

Using the Fierz rearrangement property,

$$\mathrm{tr}[O_\mu^- G_1 O_\mu^- G_2]=-\mathrm{tr}[O_\mu^- G_1]\mathrm{tr}[O_\mu^- G_2], \tag{21}$$

where $G_{1,2}$ are any Dirac matrices, it is clear that $T_1\propto T_2$. Furthermore, the analysis for Q_2 is similar and the result is identical so that

$$\langle\pi^+(p_1)\pi^0(p_2)|(c_1Q_1+c_2Q_2)|K^+(p)\rangle$$
$$=\frac{c_1+c_2}{\sqrt{2}}N_c(N_c+1)T_1(p_1,p_2). \tag{22}$$

This can be simplified using [21]

$$f_\pi p_\mu=-\sqrt{2}N_c\,\mathrm{tr}\,Z_2\int_k^\Lambda O_\mu^-\chi_\pi(k;-\tfrac{1}{2}p,-\tfrac{1}{2}p) \tag{23}$$

and [22]

$$-(p+p_1)_\mu f_+^{K^+}(p_2^2)-p_{2\mu}f_-^{K^+}(p_2^2)$$
$$=2N_c\,\mathrm{tr}\,Z_2\int_{k_1}^\Lambda iO_\mu^-\chi_K(k_1;p_2,p_1)\Gamma_\pi(k_1;-p_1)S_u(k_1), \tag{24}$$

where $f_\pm^{K^+}$ are the K_{l3} semileptonic transition form factors, to yield

$$\langle\pi^+(p_1)\pi^0(p_2)|\mathcal{H}_{\rm eff}^{\Delta S=1}|K^+(p)\rangle$$
$$=\frac{N_c+1}{\sqrt{2}N_c}\tilde{G}_F(c_1+c_2)\mathcal{M}_1(p_1,p_2), \tag{25}$$

$$\mathcal{M}_1(p_1,p_2)=f_\pi[p_2\cdot(p+p_1)f_+^{K^+}(p_2^2)+p_2^2f_-^{K^-}(p_2^2)] \tag{26}$$

$$\approx f_\pi(m_K^2-m_\pi^2). \tag{27}$$

The last line follows from [22] $f_+^{K^+}(-m_\pi^2)\approx-1.0$ and $m_\pi^2f_-^{K^-}(-m_\pi^2)\approx 0$.

We can compare our result with the contemporary phenomenological approach to $K\rightarrow\pi\pi$ decays, which employs a parametrization of $\mathcal{M}_1$:

$$\mathcal{M}_1=f_\pi(m_K^2-m_\pi^2)B_1^{(3/2)}, \tag{28}$$

with the parameter $B_1^{(3/2)}$ fixed by fitting the experimental width. One historical means of estimating $\mathcal{M}_1$ is to employ the vacuum saturation ansatz, which gives $B_1^{(3/2)}=1$. It is clear from Eqs. (25) and (27) that our impulse approximation is equivalent to this ansatz. Equation (27) is an exact algebraic constraint, which has been overlooked by other authors and consequently violated in fitting $\Gamma_{K^+\rightarrow\pi^+\pi^0}$; e.g., Ref. [23] and references therein. (The connection can also be made via a bosonization of four-fermion interaction models [24], which illustrates an equivalence between that approach and the rainbow-ladder DSE truncation.)

Agreement with the experimental value of $\Gamma_{K^+\rightarrow\pi^+\pi^0}$, however, requires $B_1^{(3/2)}\approx\frac{1}{2}$, as can be seen using Eq. (A12). Thus, while the impulse approximation is reliable for estimating the order of magnitude, it appears that an accurate result requires additional contributions.[2] Unitarizing corrections arising from $\pi\pi$ final state interactions are a plausible candidate. However, a contemporary estimate [20] indicates that they can provide no more than one-fifth of the reduction required in this channel; i.e., $B_1^{(3/2)}>0.9$ still.

Nevertheless, our primary goal is to identify a plausible mechanism for an enhancement of $\pi\pi_{I=0}$ transitions and this level of accuracy is sufficient for that purpose. Hence we proceed by adopting the contemporary artifice and use

$$\mathcal{M}_1(p_1,p_2):=f_\pi(m_K^2-m_\pi^2)B_1^{(3/2)},\quad B_1^{(3/2)}=\tfrac{1}{2}. \tag{29}$$

In doing this we bypass the calculation of $B_1^{(3/2)}$, which our elucidation of the impulse approximation has identified as a real challenge for models whose basis is kindred to ours, and also for other approaches.

B. Propagators and Bethe-Salpeter amplitudes

Although the matrix element discussed above was expressed in terms of dressed u- and s-quark propagators, and π- and K-meson Bethe-Salpeter amplitudes, we obtained a model independent result without introducing specific forms. That is an helpful but uncommon simplification only encountered before in the study of anomalous processes; e.g., $\pi^0\rightarrow\gamma\gamma$ [13,14], $K\bar{K}\rightarrow\pi^+\pi^0\pi^-$ [25], and $\gamma\pi\rightarrow\pi\pi$ [26].

In general these quantities can be obtained as solutions of the quark DSE and meson Bethe-Salpeter equation [6]. However, the study of an extensive range of low- and high-energy light- and heavy-quark phenomena has led to the development of efficacious algebraic parametrizations, and we employ them herein.

The dressed-quark propagator is

$$S_f(p)=-i\gamma\cdot p\,\sigma_V^f(p^2)+\sigma_S^f(p^2), \tag{30}$$

$$=[i\gamma\cdot pA_f(p^2)+B_f(p^2)]^{-1}, \tag{31}$$

$$\bar{\sigma}_S^f(x)=2\bar{m}_f\mathcal{F}(2(x+\bar{m}_f^2))+\mathcal{F}(b_1^fx)\mathcal{F}(b_3^fx)[b_0^f+b_2^f\mathcal{F}(\varepsilon x)], \tag{32}$$

$$\bar{\sigma}_V^f(x)=\frac{1}{x+\bar{m}_f^2}[1-\mathcal{F}(2(x+\bar{m}_f^2))], \tag{33}$$

with $\mathcal{F}(y)=(1-\mathrm{e}^{-y})/y$, $x=p^2/\lambda^2$, $\bar{m}_f=m_f/\lambda$, $\bar{\sigma}_S^f(x)=\lambda\,\sigma_S^f(p^2)$ $\bar{\sigma}_V^f(x)=\lambda^2\,\sigma_V^f(p^2)$. The mass scale, $\lambda=0.566$ GeV, and parameter values

[2] As observed already, the impulse approximation has proven reliable in a range of applications [5–15], and in some cases corrections have been calculated and shown to be small [16,22]. One new feature here is that the calculation is not self-contained; i.e., we rely on external input: the c_i in Eq. (12).

	$\bar{m}$	b_0	b_1	b_2	b_3
u	0.009 48	0.131	2.94	0.733	0.185
s	0.210	0.105	3.18	0.858	0.185

(34)

were fixed [5] in a least-squares fit to light- and heavy-meson observables, with these dimensionless u,s current-quark masses corresponding to[3]

$$m_u^{1\ \mathrm{GeV}}=5.4\ \ \mathrm{MeV},\quad m_s^{1\ \mathrm{GeV}}=119\ \ \mathrm{MeV}. \tag{35}$$

This algebraic parametrization combines the effects of confinement and dynamical chiral symmetry breaking with free-particle behavior at large spacelike p^2 [8].

The dominant component of the π- and K-meson Bethe-Salpeter amplitudes is primarily determined by the axial-vector Ward-Takahashi identity [21,27]:

$$\Gamma_H(k^2)=i\gamma_5\frac{\sqrt{2}}{f_H}B_H(k^2),\quad H=\pi,K, \tag{36}$$

where $B_H:=B_u|^{\bar{m}_u\to 0}_{b_0^u\to b_0^P}$ and [5]

$$b_0^\pi=0.204,\quad b_0^K=0.319, \tag{37}$$

i.e., B_H is the quark–quark mass function obtained from Eqs. (30)–(33) with $\bar{m}_f=0$ and b_0^f replaced by the values indicated. With these dressed-propagators and Bethe-Salpeter amplitudes one obtains (in GeV)

	f_π	m_π	f_K	m_K
Calc.	0.146	0.130	0.178	0.449
Obs. [1]	0.131	0.138	0.160	0.496

(38)

and $\langle\bar{q}q\rangle^{1\ \mathrm{GeV}}=(0.220\ \ \mathrm{GeV})^3$.

C. Neutral kaon decay

We now consider the transitions $K_S^0\to\pi^+\pi^-,\ \pi^0\pi^0$. In comparison with $K^+\to\pi^+\pi^0$ there is a significant qualitative difference: all effective operators contribute to these transitions and furthermore the QCD penguin operators: $Q_{5,6}$, and ew penguin operators: $Q_{7,8}$, can direct the transition through 0^{++} intermediate states. This may have material consequences.

1. Light scalar meson?

A contemporary analysis of $\pi\pi$ data identifies a scalar–isoscalar s-channel pole with [28]

$$m_{0^{++}}\approx 0.93m_K,\quad \Gamma_{0^{++}}\sim 0.22\text{–}0.47\ \ \mathrm{GeV}, \tag{39}$$

and, it is argued, because $\Gamma_{0^{++}}/m_{0^{++}}\gtrsim 0.5$ a simple Breit-Wigner form is inadequate as a model for this pole's contribution to the scattering amplitude. The decay $\tau\to\nu_\tau\pi^-\pi^0\pi^0$ also exhibits a broad scalar resonance [29], which, however, has only been characterized by Breit-Wigner parameters:

$$m_{0^{++}}\approx 1.12m_K,\quad \Gamma_{0^{++}\to\pi\pi}\approx 0.54\ \ \mathrm{GeV}. \tag{40}$$

The small quantitative discrepancy is explicable via parametrization dependence.

As summarized in Ref. [28], $\gamma\gamma\to\pi\pi$ data are consistent with the interpretation of this pole as a $(u\bar{u}+d\bar{d})$ scalar meson. However, this interpretation is not universally accepted. It is an experimental fact that $\pi\pi$ scattering is very attractive in the scalar–isoscalar channel and this provides for another perspective; i.e., that the 0^{++} resonance is merely a $\pi\pi$-rescattering effect [17–19]. In either event, the implications in the present context are similar: spectral strength in the 0^{++} channel, located in the neighborhood of m_K, can significantly enhance nonleptonic K^0 decays [17,18,25,26].

Dyson-Schwinger equation studies can contribute somewhat to this discussion. The rainbow-ladder approximation is the lowest order in an axial-vector Ward-Takahashi identity preserving truncation scheme [30], and a light $(u\bar{u}+d\bar{d})$ meson is a feature of this approximation. However, there is some model sensitivity and combining the results of four independent studies [31–34] yields

$$m_{0^{++}}=0.64\pm 0.07\ \ \mathrm{GeV}. \tag{41}$$

This is a simple pole mass. The rainbow-ladder truncation of the quark–antiquark scattering kernel ignores the coupling to the $\pi\pi$ loop, which would provide a width. That defect is not significant for the ρ-meson, where the same collection of models yields $m_\rho=0.75$ GeV, with a standard deviation of $<2\%$, and calculations show that the loop contribution can be included perturbatively [10,35], reproducing the experimental value of $\Gamma_\rho/m_\rho=0.2$. However, that is not necessarily a reliable guide to the importance of these effects in the 0^{++} channel because the width-to-mass ratio is so much larger in this case.

That the calculated mass in Eq. (41) lies between that in Eq. (39) and that of the isovector $a_0(980)$ is unsurprising because the rainbow-ladder truncation yields degenerate isoscalar and isovector bound states, and ideal flavor mixing in the three-flavor case. However, this degeneracy signals another weakness of the ladder-rainbow truncation in the 0^{++} channel. The truncation is reliable for flavor-nonsinglet pseudoscalar mesons because of cancellations between vertex corrections and crossed-box contributions at each higher order in the quark-antiquark scattering kernel [30]. However, these cancellations do not take place in the 0^{++} channel [36]. In our view, this is inextricably linked with the difficulties encountered in understanding the composition of sca-

[3] $\varepsilon=10^{-4}$ in Eq. (32) acts only to decouple the large- and intermediate-p^2 domains. The study used Landau gauge because it is a fixed point of the QCD renormalization group and $Z_2\approx 1$, even nonperturbatively [21].

lar resonances below 1.4 GeV [28,35,37]. For the isoscalar–scalar vertex the problem is exacerbated by the presence of timelike gluon exchange contributions to the kernel, which are the analogue of those diagrams expected to generate the η-η' mass splitting in BSE studies [38].

Hitherto no model bound-state description escapes these deficiencies and developing an improved kernel is an important current focus. In the meantime, our discussion indicates that the leading order, chiral symmetry preserving DSE truncation supports an existing view that the low-mass spectral strength in 0^{++} channel has a $(u\bar{u}+d\bar{d})$-meson component. The truncation also admits that the properties of this component are materially modified by $\pi\pi$-rescattering effects, which are an additive, nonperturbative contribution to the quark–antiquark scattering kernel. In quantifying the admixtures some aspects will be truncation and/or model dependent [28].

2. The decay

To proceed we explore the hypothesis that there is a low-mass scalar–isoscalar meson that can be represented as a quark-antiquark s-channel pole characterized by a Bethe-Salpeter amplitude. Since the absence of a DSE truncation reliable in this channel prevents an accurate determination of its mass and Bethe-Salpeter amplitude, we parametrize the amplitude as

$$\Gamma_\sigma(k;p)=I_D\frac{1}{\mathcal{N}_\sigma}\frac{1}{1+(k^2/\omega_\sigma^2)^2}, \tag{42}$$

where $I_D=\gamma_4^2$ and ω_σ is a width parameter to be determined. (This is analogous to our treatment of the π and K meson amplitudes.) Γ_σ is normalized canonically and consistent with the impulse approximation; i.e., $\mathcal{N}_\sigma$ is fixed via ($q_\pm=q\pm\frac{1}{2}p$)

$$\begin{aligned}p_\mu=N_c\,\mathrm{tr}\int_q^\Lambda\bigg[&\Gamma_\sigma(q;-p)\frac{\partial S(q_+)}{\partial p_\mu}\Gamma_\sigma(q;p)S(q_-)\\&+\Gamma_\sigma(q;-p)S(q_+)\Gamma_\sigma(q;p)\frac{\partial S(q_-)}{\partial p_\mu}\bigg]\bigg|_{p^2=-m_\sigma^2}.\end{aligned} \tag{43}$$

We separate the Q_6 contribution to the $K_S^0\to\pi\pi$ transition into two parts and consider first the new class of contributions, which introduce the putative σ intermediate state:

$$\begin{aligned}\langle\pi(p_1)\pi(p_2)|Q_6|K^0(p)\rangle=&\langle\pi(p_1)\pi(p_2)|\sigma(p)\rangle D_\sigma(p^2)\\&\times\langle\sigma(p)|Q_6|K^0(p)\rangle,\end{aligned} \tag{44}$$

where we represent σ propagation by

$$D_\sigma(p^2)=1/[p^2+m_\sigma^2], \tag{45}$$

with m_σ a parameter to be determined, and employ the impulse approximation for the $\sigma\pi\pi$ coupling

$$\begin{aligned}M_{\sigma\pi\pi}(p_1,p_2)&:=\langle\pi(p_1)\pi(p_2)|\sigma(p)\rangle\\&=2N_c\,\mathrm{tr}\int_k^\Lambda\Gamma_\sigma(k;p)S_u(k_{++})i\Gamma_\pi(k_{0+};-p_1)\\&\quad\times S_u(k_{+-})i\Gamma_\pi(k_{-0};-p_2)S_u(k_{--}),\end{aligned} \tag{46}$$

$k_{\alpha\beta}=k+(\alpha/2)p_1+(\beta/2)p_2$, which provides the basis for the calculation of $\Gamma_{0^{++}\to\pi\pi}$: $g_{\sigma\pi\pi}:=M_{\sigma\pi\pi}(-m_\pi^2,-m_\pi^2)$. This combination of simple-pole propagator plus impulse approximation coupling to the dominant decay channel is phenomenologically efficacious; e.g., Ref. [10], and necessary to avoid overcounting of final-state interactions [26].

In impulse approximation

$$\begin{aligned}\langle\sigma(p)|Q_6|K^0(p)\rangle=&\sqrt{2}N_c^2\,\mathrm{tr}\,Z_4^2\int_{k_1}^\Lambda\int_{k_2}^\Lambda i\chi_K(k_1;\tfrac{1}{2}p,\tfrac{1}{2}p)\\&\times O_\mu^+\chi_\sigma(k_2;-\tfrac{1}{2}p,-\tfrac{1}{2}p)O_\mu^-,\end{aligned} \tag{47}$$

with $Z_4(\mu,\Lambda)$ the mass renormalization constant and $\chi_\sigma(k;l_1,l_2)$ an obvious analogue of $\chi_\pi(k;l_1,l_2)$ in Eq. (18). Using

$$\mathrm{tr}[G_1O_\mu^+G_2O_\mu^-]=2\,\mathrm{tr}[G_1(1-\gamma_5)]\mathrm{tr}[G_2(1+\gamma_5)] \tag{48}$$

this yields

$$\begin{aligned}-\frac{1}{\sqrt{2}}&\langle\sigma(p)|Q_6|K^0(p)\rangle\\&=\left(\sqrt{2}N_c\,\mathrm{tr}\,Z_4\int_{k_1}^\Lambda i\gamma_5\chi_K(k_1;\tfrac{1}{2}p,\tfrac{1}{2}p)\right)\\&\quad\times\left(\sqrt{2}N_c\,\mathrm{tr}\,Z_4\int_{k_2}^\Lambda\chi_\sigma(k_2;-\tfrac{1}{2}p,-\tfrac{1}{2}p)\right).\end{aligned} \tag{49}$$

From Refs. [21,27] we identify the first parenthesized term as the residue of the kaon pole in the pseudoscalar vertex:

$$ir_K:=\sqrt{2}N_c\,\mathrm{tr}\,Z_4\int_{k_1}^\Lambda\gamma_5\chi_K(k_1;\tfrac{1}{2}p,\tfrac{1}{2}p)=\frac{f_Km_K^2}{m_u+m_s}. \tag{50}$$

The second term is the scalar meson analogue in the scalar vertex but the vector Ward-Takahashi identity, which is relevant in this case, does not make possible an algebraic simplification. The integral and its μ-dependence must therefore be calculated. That is straightforward when the renormalization-group-improved rainbow-ladder truncation is accurate; e.g., Refs. [11,21], but not yet for scalar mesons. This is where the simple ansatz of Eq. (42) is useful: it yields a finite integral and we therefore suppress Z_4 to obtain

$$-\frac{1}{\sqrt{2}}\langle\sigma(p)|Q_6|K^0(p)\rangle=r_K\sqrt{2}N_c\,\mathrm{tr}\int_{k_2}^{\Lambda}\chi_\sigma(k_2;-\tfrac{1}{2}p,-\tfrac{1}{2}p)$$
$$=:-r_K r_\sigma(p^2). \quad (51)$$

The result for Q_5 is similar, but suppressed by a factor of $1/N_c$, and the contribution of the ew penguins, $Q_{7,8}$, can be obtained similarly.

The other class of contributions, which do not involve a 0^{++} intermediate state, can be evaluated following the explicit example of Q_1 presented above. Only two additional three-point functions arise:

$$\mathcal{G}_\pi^S(p_1,p_2)=\langle\pi(p_1)\pi(p_2)|(\bar{u}u+\bar{d}d)|0\rangle, \quad (52)$$

$$-\mathcal{G}_{K\pi}^S(p_1,p_2)=\langle\pi^-(p_1)|\bar{s}u|K^0(p)\rangle. \quad (53)$$

They are the scalar pion form factor and the scalar $K\pi$ transition form factor, respectively, and $\mathcal{G}_{K\pi}^S(p_1,p_2)$ can be expressed without additional calculation in terms of the K_{l3} form factors [22]:

$$\mathcal{G}_{K\pi}^S(p_1,p_2)=\frac{p_1^2-p_2^2}{m_s-m_d}\left[f_+^K(-p_2^2)+\frac{p_2^2}{p_2^2-p_1^2}f_-^K(-p_2^2)\right], \quad (54)$$

a result which follows from the vector Ward-Takahashi identity. A preliminary result is available for $\mathcal{G}_\pi^S(p_1,p_2)$ [39], which takes the form anticipated from current algebra. That is to be expected because correctly truncated DSE models provide a good description of chiral symmetry and its dynamical breakdown, as illustrated in a study of $\pi\pi$ scattering [12,13]. This makes a calculation of $\mathcal{G}_\pi^S(p_1,p_2)$ unnecessary for our present analysis because we can adopt the form [40] [$(r_\pi^S)=3.76\,\mathrm{GeV}^{-1}$]:

$$\mathcal{G}_\pi^S(p_1,p_2)=-4\frac{\langle\bar{q}q\rangle}{f_\pi^2}[1-\tfrac{1}{6}(r_\pi^S)^2(p_1+p_2)^2]. \quad (55)$$

The matrix elements for the $K\to\pi\pi$ transitions can all be written

$$M_{K\to\pi\pi}=M_{K\to\pi\pi}^{\mathrm{QCD}}+\alpha_{\mathrm{em}}M_{K\to\pi\pi}^{\mathrm{ew}}, \quad (56)$$

with the explicit forms given in the Appendix and the pure isospin amplitudes defined in Eqs. (2) and (3).

III. RESULTS AND DISCUSSION

Everything required for our calculation of the widths is now specified. There are two parameters: ω_σ in Eq. (42); and m_σ in Eq (45). We determine them in a least-squares fit to $\Gamma_{K_S^0\to\pi^+\pi^-}$, $\Gamma_{K_S^0\to\pi^0\pi^0}$, taken from Ref. [1]; and $\Gamma_{\sigma\to(\pi\pi)}$ in Eq. (40), and obtain (in GeV)

	Obs.	Calc.
m_σ	$1.12m_K$	$1.14m_K$
ω_σ		0.611
$\Gamma_{\sigma\to(\pi\pi)}$	0.54	0.54
$\Gamma^{\times 10^{-15}}_{K_S^0\to\pi^+\pi^-}$	5.055 ± 0.025	5.16
$\Gamma^{\times 10^{-15}}_{K_S^0\to\pi^0\pi^0}$	2.305 ± 0.023	2.11
$\Gamma^{\times 10^{-15}}_{K^+\to\pi^+\pi^0}$	0.0112 ± 0.0001	0.0116

(57)

which is a relative error on fitted quantities of $<4\%$.[4] The value of ω_σ corresponds to an *intrinsic* σ-meson size: $r_\sigma^I:=1/\omega_\sigma$, which is $0.84\,r_\rho^I$; i.e, 84% of that of the ρ-meson determined in Ref. [5]. With $\Gamma_\sigma(k;p)\propto\exp(-k^2/\omega_\sigma^2)$ instead of Eq. (42), we find that $m_\sigma=1.12m_K$, $\omega_\sigma=0.694$ GeV yield exactly the same results for the calculated quantities. Furthermore, a value of $m_{0^{++}}=0.7\,\mathrm{GeV}\approx1.4m_K$ [24] is excluded in our analysis: in a description of $K\to\pi\pi$ decays it requires $\Gamma_\sigma=1.41$ GeV, which is ~3-times the value in Eq. (40) and ~8 standard-deviations larger than the mean-width estimated from Ref. [28].

The parameter values in Eq. (57) also yield $r_\sigma(-m_\sigma^2)=(0.51\ \mathrm{GeV})^2$, which is comparable with an estimate [41]: $r_\sigma(-m_\sigma^2)=(0.58\ \mathrm{GeV})^2$, obtained using the separable BSE model of Ref. [32], and $g_{\sigma\pi\pi}/m_\sigma=6.4$, cf. a renormalization-group-improved rainbow-ladder estimate [33]: $g_{\sigma\pi\pi}/m_\sigma=4.1$. These results are an *a posteriori* justification of the parametrization of the σ-meson Bethe-Salpeter amplitude: it yields results consistent with contemporary bound state calculations and the $\lesssim30\%$ differences enable agreement with current data analyses. Hence our working assumption is internally consistent.

We anticipate that with only small modifications of the parameters: ω_σ, $m_{0^{++}}$, a description of the quality in Eq. (57) would still be obtained after the inclusion of unitarizing corrections [20] to the impulse approximation. Such corrections appear able to magnify the enhancement from a σ-meson intermediate state (by a factor of $\lesssim1.5$) but not replace it. The impulse approximation, as we have formulated it in terms of a 0^{++} intermediate meson state, is reliable at this level. An improved calculation would be an internally consistent combination of a 0^{++}-pole and $\pi\pi$ final-state interactions, with the relative strengths allowed to vary in order to explore the necessity, rather than just the sufficiency, of the contributions.

The widths in Eq. (57) are obtained from the calculated amplitudes [in GeV with m_K from Eq. (38)]

$$|M_{K^0\to\pi^+\pi^-}|=2.7\times10^{-7}=5.9\times10^{-7}m_K, \quad (58)$$

$$|M_{K^0\to\pi^0\pi^0}|=2.4\times10^{-7}=5.4\times10^{-7}m_K, \quad (59)$$

[4]We used $G_F=1.166\times10^{-5}\,\mathrm{GeV}^{-2}$, $V_{ts}=0.0385$, $V_{td}=0.0085$, $V_{us}=0.220$, $V_{ud}=0.975$, $\mathrm{Im}(V_{ts}^*V_{td})=0.000133$, and c_i obtained from Eq. (A11).

$$|M_{K^+\to\pi^+\pi^0}| = 1.8\times 10^{-8} = 4.0\times 10^{-8} m_K. \tag{60}$$

For the pure isospin amplitudes we find (in GeV)

$$\mathrm{Re}(A_0) = 31.7\times 10^{-8}, \quad \mathrm{Re}(A_2) = 1.47\times 10^{-8}, \tag{61}$$

which yield

$$1/w = 21.6. \tag{62}$$

Our analysis also yields values of the parameters: $B_i^{(1/2),(3/2)}$, used in phenomenological analyses to express the operator expectation values [2]. Of course, $B_1^{(3/2)} = 0.5$, as discussed in connection with Eq. (29) and, using the formulas in the appendix, we obtain algebraically, $B_1^{(1/2)} = B_2^{(1/2)} = B_3^{(1/2)} = B_4^{(1/2)} = B_2^{(3/2)} = B_1^{(3/2)}$. We also calculate

$$B_5^{(1/2)} = B_6^{(1/2)} = 1.43 + (17.9)_\sigma, \tag{63}$$

where the second term is the contribution of the σ meson. The non-σ contribution is necessarily large because of the strength of the $K\pi$ transition form factor. If the vacuum saturation ansatz is used to estimate the operator expectation values they are all $\equiv 1$. That method does not admit a σ-meson contribution nor the effect of $\pi\pi$ final state interactions.

Eliminating the ew penguin contributions yields a $<1\%$ reduction in $1/w$, which is consistent with the the magnitude of α_{em}. Suppressing instead the σ-meson contribution, while not affecting $\Gamma_{K^+\to\pi^+\pi^0}$ of course [see Eq. (A1)], yields $\Gamma_{K_S^0\to\pi^+\pi^-} = 1.3\times 10^{-16}$ GeV, $\Gamma_{K_S^0\to\pi^0\pi^0} = 1.1\times 10^{-17}$ GeV, and $1/w = 2.9$.

The value of ϵ'/ϵ follows from Eq. (10). Suppressing the σ-meson and ew penguin contributions we obtain $\epsilon'/\epsilon = 128\times 10^{-3}$, which is ~ 60 times larger than the experimental average in Eq. (11). Including the σ-meson we find 31.3×10^{-3}. To understand these results we note that Eq. (10) can be written

$$\frac{\epsilon'}{\epsilon} = -\frac{1}{\sqrt{2}}\frac{w}{|\epsilon|}\frac{\mathrm{Im}\,A_0}{\mathrm{Re}\,A_0}\left\{1 - \frac{1}{w}\frac{\mathrm{Im}\,A_2}{\mathrm{Im}\,A_0}\right\}, \tag{64}$$

which makes clear that the ratio is determined by $\mathrm{Im}(A_0)/\mathrm{Re}(A_0)$ *unless* $\mathrm{Im}(A_2)\neq 0$. Noting that $c_{1,2}$ are real, Eq. (A11), then it follows from Eq. (A1) that $\mathrm{Im}(A_2) = 0$ in the absence of ew effects. Hence our calculated results are large because the prefactor in Eq. (64) is large. The dependence on the σ contribution is easily understood. The prefactor is $\propto \mathrm{Im}(A_0)/\mathrm{Re}(A_0)^2$, which is large in the absence of the σ contribution even though $\mathrm{Im}(A_0)$ and $\mathrm{Re}(A_0)$ are individually small. The σ contribution adds simultaneously to $\mathrm{Im}(A_0)$ and $\mathrm{Re}(A_0)$ with a magnitude ~ 100 times larger than the original values. Hence the final ratio is sensitive only to the relative strength of the σ contributions, which is determined by the coefficients $c_{5,6}$.

Including both the σ and ew penguin contributions we obtain

$$\epsilon'/\epsilon = 31.7\times 10^{-3}, \tag{65}$$

from which it is clear that the ew penguins are a correction of order α_{em} as one would naively expect. In this case $\mathrm{Im}(A_2)\neq 0$. However, as observed above, the σ-meson enhancement responsible for the $\Delta I = \frac{1}{2}$ rule affects the real and imaginary parts of A_0 simultaneously so that $(1/w)\mathrm{Im}\,A_2/\mathrm{Im}\,A_0$ remains negligible.

If we employ the artifice of an *ad hoc* suppression of the σ contribution to $\mathrm{Im}(A_0)$ while retaining it in $\mathrm{Re}(A_0)$; i.e., make the replacement

$$c_i\mathcal{M}_3 \to \mathrm{Re}(c_i\mathcal{M}_3), \quad i = 5,6,7,8 \tag{66}$$

in Eqs. (A1)–(A6), we find

$$\epsilon'/\epsilon = 2.7\times 10^{-3}. \tag{67}$$

This artifice is implicit in the phenomenological analyses reviewed in Ref. [2] and that is why Eq. (67) reproduces their order of magnitude. The small value is only possible because in this case $\mathrm{Im}(A_0)$ is not σ-enhanced and is therefore of the same magnitude as $\mathrm{Im}\,A_2/w \propto \alpha_{\mathrm{em}}/w$, due to the $1/w$ enhancement factor. That factor survives because $\mathrm{Re}(A_0)$ is still magnified as required in order to satisfy the $\Delta I = \frac{1}{2}$ rule. Currently we cannot justify this procedure. (NB. If this procedure is followed then $m_u \neq m_d$ isospin symmetry breaking effects also contribute significantly to ϵ'/ϵ.)

IV. EPILOGUE

We have demonstrated that estimating the $K\to\pi\pi_{I=2}$ matrix element using the impulse approximation is algebraically equivalent to using the vacuum saturation ansatz and yields a result that is ~ 2 times too large. The identification of a compensating mechanism that can remedy this overestimate is a contemporary challenge. $\pi\pi$ final state interactions in the $I=2$ channel act to ameliorate the discrepancy [20].

We have also shown that the contribution of a light scalar meson mediated by the QCD penguin operators: $Q_{5,6}$, is a plausible candidate for the long-range mechanism underlying the enhancement of $K\to\pi\pi_{I=0}$ transitions.[5] Our description of that enhancement requires a mass and width for this 0^{++} resonance that agree with those recently inferred [28,29]: $m_{0^{++}} \approx m_K$, $\Gamma_{0^{++}\to\pi\pi} \lesssim m_{0^{++}}$, and the analysis is not sensitive to details of the Bethe-Salpeter amplitude. However, this same mechanism yields a value of ϵ'/ϵ that is ~ 15-times larger than the average of contemporary experimental results unless a means is found to suppress its contribution to $\mathrm{Im}(A_0)$.

If a light scalar resonance exists it will contribute in the

[5] $Q_{5,6}$ mediated scalar diquark transitions: $(us)_{0^+}^{I=1/2} \to (ud)_{0^+}^{I=0}$, are the $s\to t$-channel interchange of the interaction that herein produces the σ-meson. They are a viable candidate for the mechanism that produces the $\Delta I = \frac{1}{2}$ enhancement for baryons. This was explored in Ref. [42], however, the requirement therein that diquarks also explain the enhancement for mesons appears unnecessarily cumbersome.

manner we have elucidated and should be incorporated in any treatment of $K\to\pi\pi$. Even in its absence strong $\pi\pi$ final state interactions play a material role [17,18,20].

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APPENDIX: COLLECTED FORMULAS

The matrix elements for the $K\to\pi\pi$ transitions are all of the form in Eq. (56) with

$$M^{\rm QCD}_{K^+\to\pi^+\pi^0}=\frac{1}{\sqrt{2}}\tilde{G}_F\left(1+\frac{1}{N_c}\right)(c_1+c_2)\mathcal{M}_1, \tag{A1}$$

$$M^{\rm QCD}_{K^0_S\to\pi^+\pi^-}=\tilde{G}_F\left\{\left[c_2+c_4+\frac{1}{N_c}(c_1+c_3)\right]\mathcal{M}_1 +2\left(\frac{1}{N_c}c_5+c_6\right)\left(\mathcal{M}_2+\frac{1}{\sqrt{2}}\mathcal{M}_3\right)\right\}, \tag{A2}$$

$$M^{\rm QCD}_{K^0_S\to\pi^0\pi^0}=\tilde{G}_F\left\{\left[c_4-c_1-\frac{1}{N_c}(c_2-c_3)\right]\mathcal{M}_1 +2\left(\frac{1}{N_c}c_5+c_6\right)\left(\mathcal{M}_2+\frac{1}{\sqrt{2}}\mathcal{M}_3\right)\right\}, \tag{A3}$$

$$M^{\rm ew}_{K^+\to\pi^+\pi^0}=-\frac{1}{\sqrt{2}}\tilde{G}_F\left\{\frac{3}{2}\left[c_7+\frac{1}{N_c}c_8-\left(1+\frac{1}{N_c}\right)\times(c_9+c_{10})\right]\mathcal{M}_1+3\left(\frac{1}{N_c}c_7+c_8\right)\mathcal{M}^b_2\right\}, \tag{A4}$$

$$M^{\rm ew}_{K^0_S\to\pi^+\pi^-}=\tilde{G}_F\left\{\left(\frac{1}{N_c}c_9+c_{10}\right)\mathcal{M}_1-\left(\frac{1}{N_c}c_7+c_8\right)\times\left(\mathcal{M}^a_2+2\mathcal{M}^b_2+\frac{1}{\sqrt{2}}\mathcal{M}_3\right)\right\}, \tag{A5}$$

$$M^{\rm ew}_{K^0_S\to\pi^0\pi^0}=\tilde{G}_F\left\{\left[c_7+\frac{1}{N_c}c_8-\left(1+\frac{1}{2N_c}\right)c_9 -\frac{1}{2}\left(1+\frac{2}{N_c}\right)c_{10}\right]\mathcal{M}_1-\left(\frac{1}{N_c}c_7+c_8\right)\times\left(\mathcal{M}_2+\frac{1}{\sqrt{2}}\mathcal{M}_3\right)\right\}, \tag{A6}$$

$$\mathcal{M}^a_2=r_K\mathcal{G}^S_\pi(p_1,p_2),\quad \mathcal{M}^b_2=r_\pi\mathcal{G}^S_{K\pi}(p_1,p_2), \tag{A7}$$

$$\mathcal{M}_2=\mathcal{M}^a_2-\mathcal{M}^b_2, \tag{A8}$$

$$\mathcal{M}_3=-r_Kr_\sigma(p^2)D_\sigma(p^2)M_{\sigma\pi\pi}(p_1,p_2), \tag{A9}$$

with $p^2=(p_1+p_2)^2=-m_K^2$, $p_1^2=p_2^2=-m_\pi^2$. These formulas make clear the operators that would be suppressed if N_c were large. Note that

$$M^{\rm QCD}_{K^+\to\pi^+\pi^0}=\frac{1}{\sqrt{2}}(M^{\rm QCD}_{K^0_S\to\pi^+\pi^-}-M^{\rm QCD}_{K^0_S\to\pi^0\pi^0}). \tag{A10}$$

This is not true of the complete amplitude.

In our calculations we use values of the coefficients that correspond to our choice of $\Lambda_{\rm QCD}\sim 0.2\,{\rm GeV}$: $c_i=z_i+\tau y_i$, $\tau=-(V^*_{ts}V_{td})/(V^*_{us}V_{ud})$ with [2]

	z_i	y_i
1	−0.407	0.0
2	1.204	0.0
3	0.007	0.023
4	−0.022	−0.046
5	0.006	0.004
6	−0.022	−0.076
7	0.003	−0.033
8	0.008	0.121
9	0.007	−1.479
10	−0.005	0.540

(A11)

Using the alternative set listed in Ref. [2] then, with $m_\sigma=1.06m_K$ and $\omega_\sigma=0.670\,{\rm GeV}$, we obtain results that differ from those in Eq. (57) by $\lesssim 1\%$, and $\epsilon'/\epsilon=69.0\times10^{-3}$ primarily because y_5 in the alternative set is 2.6 times as large.

From the complete matrix elements: Eq. (56) and Eqs. (A1)–(A6), we obtain the widths

$$\Gamma_{K^+\to\pi^+\pi^0}=\mathcal{C}(m_K)|M_{K^+\to\pi^+\pi^0}|^2, \tag{A12}$$

$$\Gamma_{K^0_S\to\pi^+\pi^-}=2\mathcal{C}(m_K)|M_{K^0_S\to\pi^+\pi^-}|^2, \tag{A13}$$

$$\Gamma_{K^0_S\to\pi^0\pi^0}=\mathcal{C}(m_K)|M_{K^0_S\to\pi^0\pi^0}|^2, \tag{A14}$$

$$\mathcal{C}(x)=\frac{1}{16\pi x}\sqrt{1-\frac{4m_\pi^2}{x^2}}, \tag{A15}$$

while the matrix element of Eq. (46) features in

$$\Gamma_{\sigma\to(\pi\pi)}=\tfrac{3}{2}\mathcal{C}(m_\sigma)|M_{\sigma\pi\pi}(-m_\sigma^2;-m_\pi^2,-m_\pi^2)|^2. \tag{A16}$$

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