

RECEIVED: February 3, 2011

REVISED: April 7, 2011

ACCEPTED: May 8, 2011

PUBLISHED: May 25, 2011

Geometry dependence of RMT-based methods to extract the low-energy constants Σ and F

Christoph Lehner,^{a,b} Jacques Bloch,^a Shoji Hashimoto^c and Tilo Wettig^a

^a*Institute for Theoretical Physics, University of Regensburg,
93040 Regensburg, Germany*

^b*RIKEN/BNL Research Center, Brookhaven National Laboratory,
Upton, NY-11973, U.S.A.*

^c*High Energy Accelerator Research Organization (KEK),
Tsukuba 305-0801, Japan*

E-mail: clehner@quark.phy.bnl.gov,
jacques.bloch@physik.uni-regensburg.de, shoji.hashimoto@kek.jp,
tilo.wettig@physik.uni-regensburg.de

ABSTRACT: The lowest-order low-energy constants Σ and F of chiral perturbation theory can be extracted from lattice data using methods based on the equivalence of random matrix theory (RMT) and QCD in the epsilon regime. We discuss how the choice of the lattice geometry affects such methods. In particular, we show how to minimize systematic deviations from RMT by an optimal choice of the lattice geometry in the case of two light quark flavors. We illustrate our findings by determining Σ and F from lattice configurations with two dynamical overlap fermions generated by JLQCD, using two different lattice geometries.

KEYWORDS: Lattice QCD, Chiral Lagrangians, QCD

ARXIV EPRINT: [1101.5576](https://arxiv.org/abs/1101.5576)

Contents

1	Introduction	1
2	The epsilon expansion at NNLO	1
3	Random matrix theory	4
4	Numerical results	8
5	Conclusions	12

1 Introduction

It is well known that QCD in a finite volume V at small quark masses m simplifies as the Compton wavelength of the pion, m_π^{-1} , becomes large compared to $V^{1/4}$ [1]. In this limit the space-time dependence of the low-energy effective theory is suppressed and the theory is dominated by the constant mode of the pions. The distribution of the low-lying eigenvalues of the Dirac operator can then be calculated in random matrix theory [2], see ref. [3] for a review. The low-energy constants (LECs) of chiral perturbation theory are used to map the dimensionful quantities of QCD (or the effective theory) to the dimensionless quantities of RMT, see, e.g., ref. [4]. Matching lattice data for the low-lying Dirac eigenvalues to RMT results then allows for a determination of phenomenologically important LECs.

The lowest-order LECs are Σ and F . While Σ can be determined rather easily from the distribution of the small Dirac eigenvalues, F can be determined only if one includes a suitable constant background gauge field [5, 6] such as isospin imaginary chemical potential [7, 8]. In the following we discuss the geometry dependence of these methods and show how to minimize systematic deviations from RMT by an optimal choice of the lattice geometry. We also compare our findings with lattice data of the two-flavor epsilon-regime run of JLQCD [9, 10] and extract Σ and F from these configurations.

The paper is structured as follows. In section 2 we briefly review the epsilon expansion of chiral perturbation theory at next-to-next-to-leading order (NNLO) which allows for a systematic discussion of the geometry dependence of RMT-based methods. In section 3 we summarize relevant results of RMT for the distribution of the lowest Dirac eigenvalues at small imaginary chemical potential. In section 4 we compare the analytic predictions of section 2 and 3 to lattice data of JLQCD. We conclude in section 5.

2 The epsilon expansion at NNLO

In this section we briefly review the epsilon expansion at NNLO with a small imaginary chemical potential $i\mu$, see ref. [11]. In the domain where the Compton wavelength of the

pion becomes large compared to $V^{1/4}$, chiral perturbation theory (χ PT) can be reordered according to the power counting [1]

$$V \sim \varepsilon^{-4}, \quad \partial_\rho \sim \varepsilon, \quad \pi(x) \sim \varepsilon, \quad m_\pi \sim \varepsilon^2, \quad \mu \sim \varepsilon^2 \quad (2.1)$$

with covariant derivative ∂_ρ , pion fields $\pi(x)$, pion mass m_π , and chemical potential μ .¹ The corresponding systematic expansion of χ PT is called epsilon expansion. To each order in ε^2 one can integrate out the space-time dependence and obtain a finite-volume effective theory in terms of the constant pion mode. The order in ε^2 then translates into the order in $1/(F^2\sqrt{V})$. At leading order the finite-volume effective action is given by

$$S_{\text{eff}}^{\text{LO}} = -\frac{1}{2}V\Sigma \text{Tr}(M^\dagger U_0 + U_0^{-1}M) - \frac{1}{2}VF^2 \text{Tr}(CU_0^{-1}CU_0) \quad (2.2)$$

with constant pion mode

$$U_0 = \exp[i\pi_0], \quad \pi_0 = \frac{1}{V} \int d^4x \pi(x), \quad (2.3)$$

quark mass matrix $M = \text{diag}(m_1, \dots, m_{N_f})$, and quark chemical potential matrix $C = \text{diag}(\mu_1, \dots, \mu_{N_f})$, where m_f is the quark mass and $i\mu_f$ is the imaginary chemical potential of quark flavor $f = 1, \dots, N_f$. We find that $S_{\text{eff}}^{\text{LO}}$ is identical to the RMT action with nonzero chemical potential [8]. Note that the pion decay constant F drops out for vanishing chemical potential. At next-to-leading order (NLO) in ε^2 the general form of eq. (2.2) remains unchanged with $\Sigma \rightarrow \Sigma_{\text{eff}}^{\text{NLO}}$, $F \rightarrow F_{\text{eff}}^{\text{NLO}}$, see refs. [12–14] for explicit expressions. In an actual lattice simulation we measure effective values Σ_{eff} and F_{eff} , and we need to include finite-volume corrections to recover the infinite-volume values Σ and F .

At NNLO and to leading order in the small chemical potential² the effective action has the form [11]

$$\begin{aligned} S_{\text{eff}}^{\text{NNLO}} = & -\frac{1}{2}V\Sigma_{\text{eff}}^{\text{NNLO}} \text{Tr}(M^\dagger U_0 + U_0^{-1}M) - \frac{1}{2}V(F_{\text{eff}}^{\text{NNLO}})^2 \text{Tr}(CU_0^{-1}CU_0) \\ & + \Upsilon_1 \Sigma(VF)^2 \text{Tr}(C)[\text{Tr}(U_0\{M^\dagger, C\}) + \text{Tr}(U_0^{-1}\{C, M\})] \\ & + \Upsilon_2 \Sigma(VF)^2 \text{Tr}(\{M^\dagger, C\}U_0C + \{C, M\}CU_0^{-1} \\ & \quad + \{U_0, C\}U_0^{-1}CU_0M^\dagger + CU_0\{C, U_0^{-1}\}MU_0^{-1}) \\ & + \Upsilon_3 \Sigma(VF)^2 \text{Tr}(U_0^{-1}CU_0C + C^2) \text{Tr}(MU_0^{-1} + M^\dagger U_0) \\ & + \Upsilon_4 \Sigma(VF)^2 \text{Tr}(U_0^{-1}CU_0C - C^2) \text{Tr}(MU_0^{-1} + M^\dagger U_0) \\ & + \Upsilon_5 \Sigma(VF)^2 \text{Tr}([M^\dagger, C]U_0C + [C, M]CU_0^{-1} \\ & \quad + [U_0, C]U_0^{-1}CU_0M^\dagger + CU_0[C, U_0^{-1}]MU_0^{-1}) \\ & + \Upsilon_6 (V\Sigma)^2 [\text{Tr}(MU_0^{-1} + M^\dagger U_0)]^2 + \Upsilon_7 (V\Sigma)^2 [\text{Tr}(MU_0^{-1} - M^\dagger U_0)]^2 \\ & + \Upsilon_8 (V\Sigma)^2 [\text{Tr}(MU_0^{-1}MU_0^{-1}) + \text{Tr}(M^\dagger U_0M^\dagger U_0)] \\ & + \mathcal{H}_1 VF^2 \text{Tr}(C^2) + \mathcal{H}_2 (V\Sigma)^2 \text{Tr}(M^\dagger M) + \mathcal{H}_3 VF^2 (\text{Tr } C)^2 \end{aligned} \quad (2.4)$$

¹In the chiral effective theory a nonzero chemical potential is introduced through the covariant derivative of the gauged flavor symmetry (see, e.g., ref. [12]), and therefore the power counting of μ is fixed by the power counting of ∂_ρ .

²There are also NNLO terms proportional to V^2C^4 that have been omitted in (2.4).

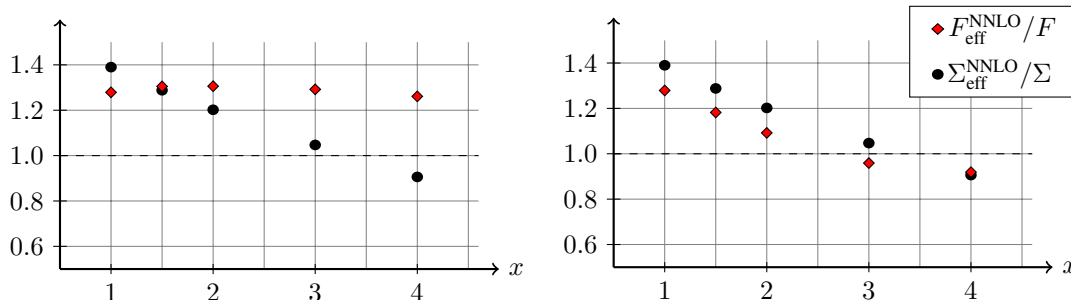


Figure 1. Finite-volume corrections to Σ and F for geometries (a_x) on the left and (b_x) on the right with parameters $F = 90 \text{ MeV}$, $L = 1.71 \text{ fm}$, and $m_\pi^2 \sqrt{V} = 1$. Taken from ref. [11].

with finite-volume effective coupling constants Υ_i and \mathcal{H}_i . The LECs Σ and F also receive further corrections, $\Sigma \rightarrow \Sigma_{\text{eff}}^{\text{NNLO}}$ and $F \rightarrow F_{\text{eff}}^{\text{NNLO}}$. The terms in eq. (2.4) that were not present in eq. (2.2) cannot be mapped to RMT. These terms are proportional to the Υ_i and \mathcal{H}_i . Therefore the magnitude of these coefficients determines the systematic deviations from RMT of, e.g., Dirac eigenvalue distributions. The coefficients \mathcal{H}_1 and \mathcal{H}_3 do not couple to U_0 or M and are therefore irrelevant for Dirac eigenvalue distributions (which involve derivatives with respect to M in the partially quenched theory). The coefficients Υ_i , \mathcal{H}_2 , $\Sigma_{\text{eff}}^{\text{NNLO}}$, and $F_{\text{eff}}^{\text{NNLO}}$ depend on the NLO LECs of χPT and on the geometry of the space-time box through finite-volume propagators. Explicit results are given in [11].

To be specific we discuss the following lattice geometries from now on,

$$(a_x) \quad L_0 = xL, \quad L_1 = L_2 = L_3 = L, \quad (2.5a)$$

$$(b_x) \quad L_3 = xL, \quad L_0 = L_1 = L_2 = L, \quad (2.5b)$$

where $x \in \{1, 3/2, 2, 3, 4\}$, and L_i is the extent of the space-time box in direction i ($i = 0$ denotes the temporal direction to which μ couples). In figure 1 we show the finite-volume corrections to Σ and F for the different geometries at NNLO for a set of parameters similar to the parameters of the JLQCD two-flavor epsilon-regime run [9, 10]. We note that the finite-volume corrections to Σ are invariant under $(a_x) \leftrightarrow (b_x)$, while the finite-volume corrections to F depend on the choice of geometry. The reason is that the permutation symmetry of the four space-time dimensions is broken by the chemical potential, to which F couples. For our choice of parameters, geometry (b_x) leads to smaller finite-volume corrections to F than geometry (a_x) . This was also observed in ref. [12] at NLO.

We continue our discussion with the finite-volume effective coupling constants Υ_i and \mathcal{H}_2 that are responsible for the systematic deviations from RMT. The high-energy constant \mathcal{H}_2 should not contribute to low-energy phenomenology (as was shown explicitly for the spectral density in [15]), and therefore we do not discuss it further. It is an interesting observation [11] that $\Upsilon_1, \Upsilon_2, \Upsilon_3$ do not depend on the NLO LECs of χPT and depend on the geometry only through a common coefficient γ , i.e.,

$$\Upsilon_1, \Upsilon_2, \Upsilon_3 \propto \gamma. \quad (2.6)$$

The coefficient γ changes under $(a_x) \leftrightarrow (b_x)$, while $\Upsilon_4, \dots, \Upsilon_8$ (and \mathcal{H}_2) are invariant under the same exchange [11]. This implies that for nonzero chemical potential a judicious

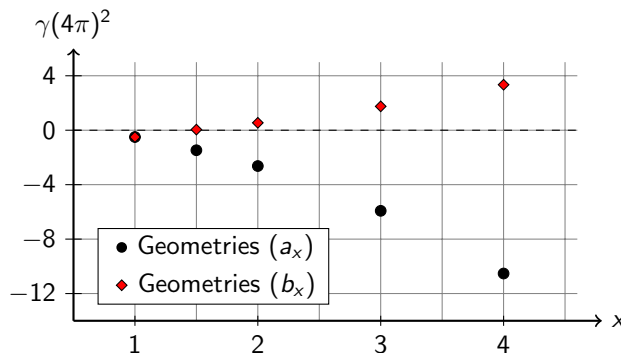


Figure 2. Geometry dependence of systematic deviations from RMT. Taken from ref. [11].

choice of geometry is possible which minimizes the systematic deviations from RMT due to $\Upsilon_1, \Upsilon_2, \Upsilon_3$ at a given volume. Of course, for zero chemical potential (a_x) and (b_x) are equivalent, so the choice of geometry has no impact.

We plot γ for different geometries in figure 2 for the same set of parameters used in figure 1.³ We note that the coefficient γ , and thus a part of the systematic deviations from RMT, can be reduced significantly by choosing the geometry (b_x) instead of (a_x) for the same value of the asymmetry x . Explicit numbers for the Υ_i and their impact on systematic errors will be discussed in section 4. Note that for large asymmetries x the coefficients Υ_i grow rapidly. Thus for too large values of x the epsilon expansion breaks down. This corresponds to the largest individual dimension $\max(L_i)$ being significantly larger than the Compton wavelength of the pion.

Note that we can project out a single topological sector by modifying the integration domain of the constant pion mode U_0 and including a volume-independent determinant term in the partition function, see, e.g., refs. [3, 12]. Therefore the discussion of systematic deviations from RMT presented in this section is also valid for fixed topological charge.

3 Random matrix theory

In this section we summarize some important results of random matrix theory that can be used to determine Σ and F from fits to Dirac eigenvalue distributions. We consider chiral random matrix theory with imaginary chemical potential defined by the partition function

$$Z_\nu = \int dV dW e^{-N \text{Tr}(W^\dagger W + V^\dagger V)} \prod_{f=1}^{N_f} \det(D(\mu_f^r) + m_f^r), \quad (3.1)$$

where $m_1^r, \dots, m_{N_f}^r$ ($i\mu_1^r, \dots, i\mu_{N_f}^r$) are the masses (imaginary chemical potentials) of the sea quarks, see refs. [8, 16], the latter for the case of real chemical potential. The integral is over the real and imaginary parts of the elements of the complex $N \times (N + \nu)$ matrices W and V with Cartesian integration measure. The random matrix Dirac operator

³Since the Υ_i are of order $1/(4\pi)^2$ we plot $\gamma(4\pi)^2$, which is then of order 1.

is defined by

$$D(\mu_f^r) = \begin{pmatrix} 0 & iV + i\mu_f^r W \\ iV^\dagger + i\mu_f^r W^\dagger & 0 \end{pmatrix}, \quad (3.2)$$

which has $|\nu|$ eigenvalues equal to zero. Therefore ν is interpreted as the topological charge. Note that m_f^r and μ_f^r are dimensionless quantities. They have to be mapped to physical quantities by comparison with the low-energy effective theory of QCD. It was shown in ref. [4] that in the limit $N \rightarrow \infty$ chiral random matrix theory can be mapped to chiral perturbation theory using

$$\hat{m}_f = m_f V \Sigma = 2N m_f^r, \quad \hat{\mu}_f^2 = \mu_f^2 F^2 V = 2N (\mu_f^r)^2, \quad (3.3)$$

where f denotes an arbitrary quark flavor, m_f is the physical quark mass, and μ_f is the physical chemical potential. Thus, the low-energy constants Σ and F appear in the conversion from physical units to dimensionless random matrix units. Note that refs. [4] and [8] use a different notation for the dimension of the random matrix Dirac operator. The quantities \hat{m}_f and $\hat{\mu}_f$ are often referred to as microscopic scaling quantities due to the limit $N \rightarrow \infty$.

The eigenvalue correlation functions for the random matrix model defined by eqs. (3.1) and (3.2) in the limit of $N \rightarrow \infty$ were calculated in ref. [8]. In this section we consider the case of $N_f = 2$ sea quarks with masses \hat{m}_u and \hat{m}_d at zero chemical potential. This setup corresponds to the two-flavor simulation of JLQCD [9, 10] that is described in more detail in section 4. We then compute Dirac eigenvalues \hat{x} at zero chemical potential and \hat{y} at imaginary chemical potential $i\hat{\delta}$. Note that we could equally well have used a setup with \hat{x} at imaginary chemical potential $-i\hat{\delta}/2$ and \hat{y} at imaginary chemical potential $+i\hat{\delta}/2$ since only the isospin component of the chemical potential is relevant for eigenvalue correlation functions [8].

We define the two-point correlator

$$\rho_{(1,1)}^{(2)}(\hat{x}, \hat{y}) = \left\langle \sum_{n,m} \delta(\hat{x} - \hat{\lambda}_n(\hat{\mu} = 0)) \delta(\hat{y} - \hat{\lambda}_m(\hat{\mu} = \delta)) \right\rangle, \quad (3.4)$$

where $\hat{\lambda}_n = 2N\lambda_n^r = \lambda_n V \Sigma$ and the sum is over all eigenvalues λ_n^r of the random matrix Dirac operator at chemical potential $\hat{\mu} = 0$ and $i\hat{\mu} = i\hat{\delta}$. This correlator allows for a discussion of the shift of Dirac eigenvalues due to the imaginary chemical potential $i\hat{\delta}$. Equation (3.4) is calculated in ref. [8]. The result is given by

$$\rho_{(1,1)}^{(2)}(\hat{x}, \hat{y}) = \hat{x} \hat{y} \det \begin{bmatrix} J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \end{bmatrix}^{-2} \det \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \quad (3.5)$$

where J_ν is the Bessel function of the first kind,

$$\Psi_{11} = \det \begin{bmatrix} \mathcal{I}^0(\hat{x}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^0(\hat{x}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^0(\hat{x}, \hat{x}) & J_\nu(\hat{x}) & \hat{x} J_{\nu+1}(\hat{x}) \end{bmatrix}, \quad (3.6a)$$

$$\Psi_{12} = \det \begin{bmatrix} \mathcal{I}^0(\hat{x}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^0(\hat{x}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ -\tilde{\mathcal{I}}^-(\hat{x}, \hat{y}) & e^{-\hat{\delta}^2/2} J_\nu(\hat{y}) & e^{-\hat{\delta}^2/2} G_\nu(\hat{y}, \hat{\delta}) \end{bmatrix}, \quad (3.6b)$$

$$\Psi_{21} = \det \begin{bmatrix} \mathcal{I}^+(\hat{y}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^+(\hat{y}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^+(\hat{y}, \hat{x}) & J_\nu(\hat{x}) & \hat{x} J_{\nu+1}(\hat{x}) \end{bmatrix}, \quad (3.6c)$$

$$\Psi_{22} = \det \begin{bmatrix} \mathcal{I}^+(\hat{y}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^+(\hat{y}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^0(\hat{y}, \hat{y}) & e^{-\hat{\delta}^2/2} J_\nu(\hat{y}) & e^{-\hat{\delta}^2/2} G_\nu(\hat{y}, \hat{\delta}) \end{bmatrix} \quad (3.6d)$$

with

$$\mathcal{I}^0(\hat{x}, \hat{y}) = \frac{1}{2} \int_0^1 dt J_\nu(\hat{x}\sqrt{t}) J_\nu(\hat{y}\sqrt{t}) = \frac{\hat{x} J_{\nu+1}(\hat{x}) J_\nu(\hat{y}) - \hat{y} J_{\nu+1}(\hat{y}) J_\nu(\hat{x})}{\hat{x}^2 - \hat{y}^2}, \quad (3.7a)$$

$$\mathcal{I}^\pm(\hat{x}, \hat{y}) = \frac{1}{2} \int_0^1 dt e^{\pm \hat{\delta}^2 t/2} J_\nu(\hat{x}\sqrt{t}) J_\nu(\hat{y}\sqrt{t}), \quad (3.7b)$$

$$\tilde{\mathcal{I}}^-(\hat{x}, \hat{y}) = \frac{1}{\hat{\delta}^2} \exp\left(-\frac{\hat{x}^2 + \hat{y}^2}{2\hat{\delta}^2}\right) I_\nu\left(\frac{\hat{x}\hat{y}}{\hat{\delta}^2}\right) - \mathcal{I}^-(\hat{x}, \hat{y}), \quad (3.7c)$$

$$G_\nu(\hat{y}, \hat{\delta}) = \hat{y} J_{\nu+1}(\hat{y}) + \hat{\delta}^2 J_\nu(\hat{y}), \quad (3.7d)$$

and I_ν is the modified Bessel function.

In the limit of small chemical potential $\hat{\delta}^2 \ll 1$ the term proportional to $\hat{\delta}^{-2}$ in $\tilde{\mathcal{I}}^-$ dominates. Furthermore, we can perform a large-argument expansion of the Bessel function in $\tilde{\mathcal{I}}^-$ and ignore all terms of order $\hat{\delta}^2$, so that

$$\rho_{(1,1)}^{(2)}(\hat{x}, \hat{y}) = H_\nu(\hat{x}, \hat{y}, \hat{m}_u, \hat{m}_d) \frac{1}{\sqrt{2\pi\hat{\delta}^2}} \exp\left(-\frac{(\hat{x} - \hat{y})^2}{2\hat{\delta}^2}\right) \quad (3.8)$$

with

$$H_\nu(\hat{x}, \hat{y}, \hat{m}_u, \hat{m}_d) = \sqrt{\hat{x}\hat{y}} \frac{\det \begin{bmatrix} \mathcal{I}^0(\hat{y}, i\hat{m}_u) & J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ \mathcal{I}^0(\hat{y}, i\hat{m}_d) & J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \\ \mathcal{I}^0(\hat{y}, \hat{x}) & J_\nu(\hat{x}) & \hat{x} J_{\nu+1}(\hat{x}) \end{bmatrix}}{\det \begin{bmatrix} J_\nu(i\hat{m}_u) & i\hat{m}_u J_{\nu+1}(i\hat{m}_u) \\ J_\nu(i\hat{m}_d) & i\hat{m}_d J_{\nu+1}(i\hat{m}_d) \end{bmatrix}}. \quad (3.9)$$

Note that the prefactor H_ν is independent of $\hat{\delta}$. Let us define a probability distribution that measures the shift \hat{d} of the eigenvalues due to the imaginary chemical potential $i\hat{\delta}$ up

to a cutoff \hat{x}_c ,

$$\begin{aligned} P(\hat{d}, \hat{x}_c) &= \frac{1}{\mathcal{N}(x_c)} \int_0^{\hat{x}_c} d\hat{x} \rho_{(1,1)}^{(2)}(\hat{x}, \hat{x} + \hat{d}) \\ &= \tilde{H}_\nu(\hat{d}, \hat{x}_c, \hat{m}_u, \hat{m}_d) \frac{1}{\sqrt{2\pi\hat{\delta}^2}} \exp\left(-\frac{\hat{d}^2}{2\hat{\delta}^2}\right) \end{aligned} \quad (3.10)$$

with

$$\begin{aligned} \tilde{H}_\nu(\hat{d}, \hat{x}_c, \hat{m}_u, \hat{m}_d) &= \frac{1}{\mathcal{N}(x_c)} \int_0^{\hat{x}_c} d\hat{x} H_\nu(\hat{x}, \hat{x} + \hat{d}, \hat{m}_u, \hat{m}_d), \\ \mathcal{N}(x_c) &= \int d\hat{d} \int_0^{\hat{x}_c} d\hat{x} \rho_{(1,1)}^{(2)}(\hat{x}, \hat{x} + \hat{d}). \end{aligned} \quad (3.11)$$

The Gaussian factor peaks strongly at $\hat{d} = 0$, and thus we can expand \tilde{H}_ν about $\hat{d} = 0$ to linear order in \hat{d} . The constant term in the expansion is fixed by the normalization

$$\int d\hat{d} P(\hat{d}, \hat{x}_c) = 1 \quad (3.12)$$

for $\hat{\delta}^2 \rightarrow 0$. Therefore we have

$$P(\hat{d}, \hat{x}_c) = \frac{1}{\sqrt{2\pi\hat{\delta}^2}} \exp\left(-\frac{\hat{d}^2}{2\hat{\delta}^2}\right) (1 + c_1 \hat{d} + \mathcal{O}(\hat{d}^2)), \quad (3.13)$$

where only c_1 depends on \hat{x}_c . Note that to first order in \hat{d} , $P(\hat{d}, \hat{x}_c)$ corresponds to a Gaussian distribution with width $\hat{\delta}$ and center $c_1 \hat{\delta}^2$.

In section 4 we use a small $\hat{\delta}$ for the numerical fits. Due to the Gaussian factor, \hat{d} is of order $\hat{\delta}$, and therefore the contribution of c_1 can be neglected (we have confirmed this for our numerical results in section 4). We define $P(\hat{d})$ to be the leading contribution to $P(\hat{d}, \hat{x}_c)$ in the limit of small $\hat{\delta}$, and therefore

$$P(\hat{d}) = \frac{1}{\sqrt{2\pi\hat{\delta}^2}} \exp\left(-\frac{\hat{d}^2}{2\hat{\delta}^2}\right). \quad (3.14)$$

This quantity is well-suited to determine $\hat{\delta}$ and therefore F from a fit to eigenvalue spectra obtained in lattice QCD simulations. Note that in the limit of small $\hat{\delta}$ the distribution does not depend on c_1 . For a related discussion with imaginary isospin chemical potential we refer to ref. [17].

In refs. [18, 19] the distribution of the lowest Dirac eigenvalue \hat{y} was calculated analytically, and in ref. [20] the calculation was extended to nonzero imaginary chemical potential $i\hat{\mu}$. We use the notation of ref. [20]. The distribution of the lowest eigenvalue is given by

$$P_1(\hat{y}) = -\partial_{\hat{y}} E_{0,0}^{(0+2)}(\hat{y}, 0) \quad (3.15)$$

with gap probability

$$E_{0,0}^{(0+2)}(\hat{y}, 0) = \frac{2 \det \begin{bmatrix} Q_S(\hat{y}, \hat{m}_u; t=1) & \partial_t Q_S(\hat{y}, \hat{m}_u; t)|_{t=1} \\ Q_S(\hat{y}, \hat{m}_d; t=1) & \partial_t Q_S(\hat{y}, \hat{m}_d; t)|_{t=1} \end{bmatrix}}{\hat{m}_d I_0(\hat{m}_u) I_1(\hat{m}_d) - \hat{m}_u I_0(\hat{m}_d) I_1(\hat{m}_u)} \exp\left(-\frac{1}{4}\hat{y}^2 - \hat{\delta}^2\right), \quad (3.16)$$

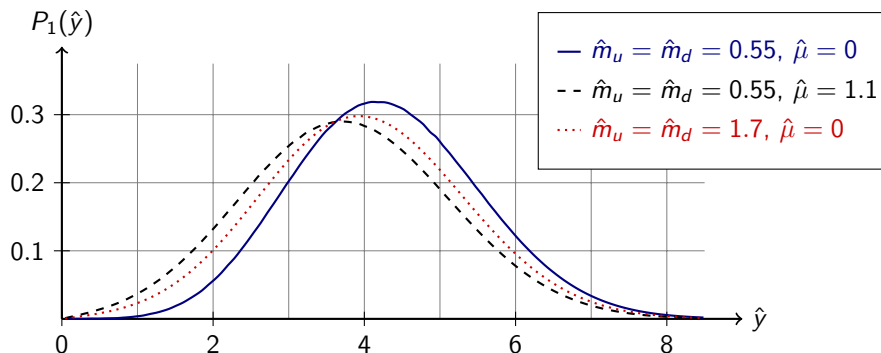


Figure 3. Distribution of the lowest Dirac eigenvalue for different quark masses $\hat{m}_u = \hat{m}_d$ and imaginary chemical potentials $i\hat{\mu}$.

where

$$Q_S(\hat{y}, \hat{m}; t) = \frac{1}{2} \int_0^1 dr e^{r(t/2)\delta^2} I_0(\sqrt{rt}\hat{m}) \sqrt{\frac{t}{1-r}} \hat{y} I_1(\sqrt{(1-r)t}\hat{y}) + e^{(t/2)\delta^2} I_0(\sqrt{t}\hat{m}). \quad (3.17)$$

In figure 3 we display $P_1(\hat{y})$ for different values of $\hat{\mu}$ and $\hat{m}_u = \hat{m}_d$. Note that the dependence on $\hat{\mu}$ and \hat{m}_u is strongly correlated, and therefore it is challenging to use this quantity to determine both Σ and F from a fit to numerical data. Nevertheless, the distribution of the lowest eigenvalue is well-suited to determine the scale of \hat{y} for $\hat{\mu} = 0$ and therefore Σ .

4 Numerical results

In this section we check the results of section 2 against the epsilon-regime run of JLQCD with two dynamical overlap fermions with masses $am_u = am_d = 0.002$ and $16^3 \times 32$ lattice points at lattice spacing $a = 0.1091(23)$ fm [9, 10]. For these parameters we have $m_\pi \min(L_i) \simeq 1$, $m_\pi \max(L_i) \simeq 2$, and $m_\pi^2 \sqrt{V} \simeq 1.34$ (using the GOR value $m_\pi = 110$ MeV, see below). The sea quarks are at zero chemical potential, and topology is fixed to $\nu = 0$. We compute the eigenvalues of the valence overlap Dirac operator on 460 configurations at zero and nonzero imaginary chemical potential.⁴ In this way the existing configurations can be used to extract Σ and F with low numerical cost.

We first fit the distribution $P_1(\lambda)$ of the lowest-lying Dirac eigenvalue at zero chemical potential in figure 4 in order to extract the finite-volume effective value $a^3 \Sigma_{\text{eff}} = 0.00208(2)$, where we cite the statistical error. This corresponds to the dimensionful value

$$\Sigma_{\text{eff}} = (231(1)(5) \text{ MeV})^3, \quad (4.1)$$

where we cite the statistical error (left) and the error propagated from the uncertainty in the lattice spacing (right). The dimensionless value is compatible with $a^3 \Sigma_{\text{eff}} = 0.00212(6)$ obtained in ref. [9] on the same configurations by a fit to the integrated Dirac eigenvalue distribution. Note that Σ is renormalization scheme dependent and that we give only the

⁴In order to introduce a nonzero imaginary chemical potential $i\mu$ in the overlap Dirac operator we multiply the forward (backward) temporal links by a factor of $e^{ia\mu}$ ($e^{-ia\mu}$), see [21].

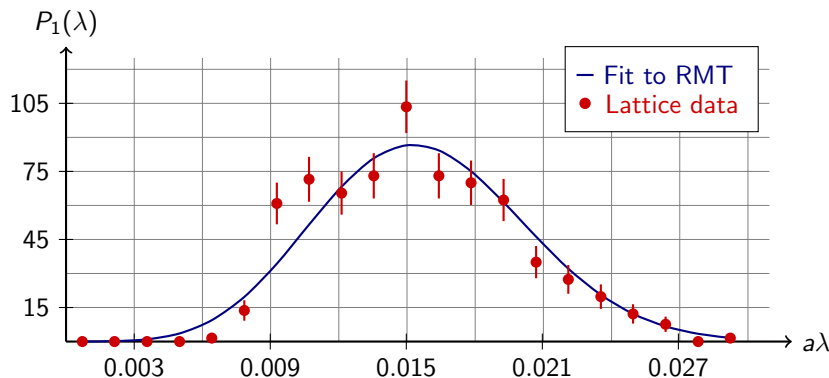


Figure 4. Fit to lowest-lying Dirac eigenvalue distribution $P_1(\lambda)$ with $\chi^2/\text{dof} = 2.9$, $a^3\Sigma_{\text{eff}} = 0.00208(2)$.

values for the lattice renormalization scheme here. Including finite-volume corrections at NLO gives an infinite-volume value

$$\Sigma = \Sigma_{\text{eff}}/1.1454 = (221(1)(5) \text{ MeV})^3. \quad (4.2)$$

We use only NLO finite-volume corrections here and in the remainder of this paper since the NNLO finite-volume corrections and the systematic deviations from RMT are of the same order in $1/F^2\sqrt{V}$. From figure 4 we see that the systematic deviations from RMT are significant. As discussed in section 2, they cannot be minimized by a judicious choice of lattice geometry. To further reduce the systematic errors in the fit for Σ , one would have to go to a larger volume, which is beyond the scope of this work. Alternatively, one could compute $P_1(\lambda)$ to NNLO, including all non-universal terms. This is a very difficult calculation that nobody has attempted yet.

Next we fit the shift of the lowest-lying Dirac eigenvalue due to a small imaginary chemical potential $i\mu$ in order to extract F as proposed in ref. [7]. As shown in section 3, RMT predicts a Gaussian distribution with $\sigma^2 = \mu^2 F^2 V$ for the distribution P of the difference d between the lowest Dirac eigenvalue at zero and at nonzero imaginary chemical potential, see also refs. [7, 8, 17]. In figure 5 we show the resulting fit for geometry (a_2) with finite-volume effective value

$$F_{\text{eff}}^{(a_2)} = 66(5)(1) \text{ MeV}, \quad (4.3)$$

where we cite the statistical error (left) as well as the error propagated from the uncertainty in the lattice spacing (right). We note that the quality of the fit is rather bad ($\chi^2/\text{dof} = 4.2$) and that this value is not compatible with the result from a fit to meson correlators obtained on the same configurations [22], $F_{\text{meson}} = 87.3(5.6) \text{ MeV}$. If we include finite-volume corrections at NLO we obtain the infinite-volume value

$$F^{(a_2)} = 50(4)(1) \text{ MeV} \quad (4.4)$$

so that the agreement is even worse. The bad $\chi^2/\text{dof} = 4.2$ suggests that the non-universal terms at NNLO, see eq. (2.4) and the subsequent discussion, affect the distribution in a non-trivial manner.

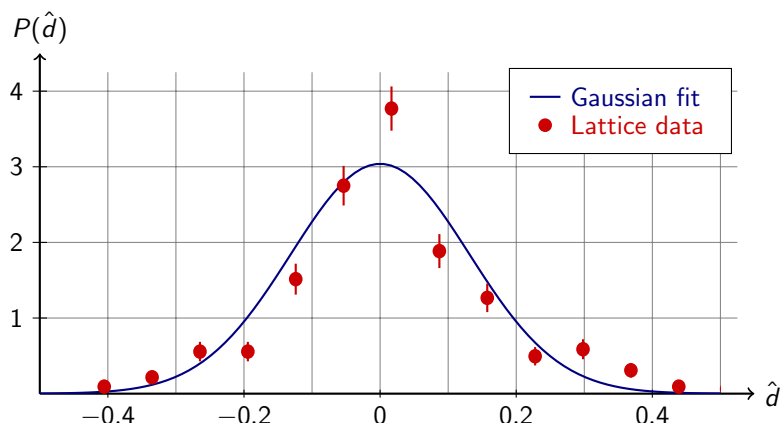


Figure 5. Fit to the distribution of Dirac eigenvalue shifts $P(\hat{d})$ due to imaginary chemical potential $a\mu = 0.01$ with $\hat{d} = d\Sigma V$ in geometry (a_2) . The result is given by $F_{\text{eff}} = 66(5)(1)$ MeV with $\chi^2/\text{dof} = 4.2$. We cite the statistical error (left) as well as the error propagated from the uncertainty in the lattice spacing (right).

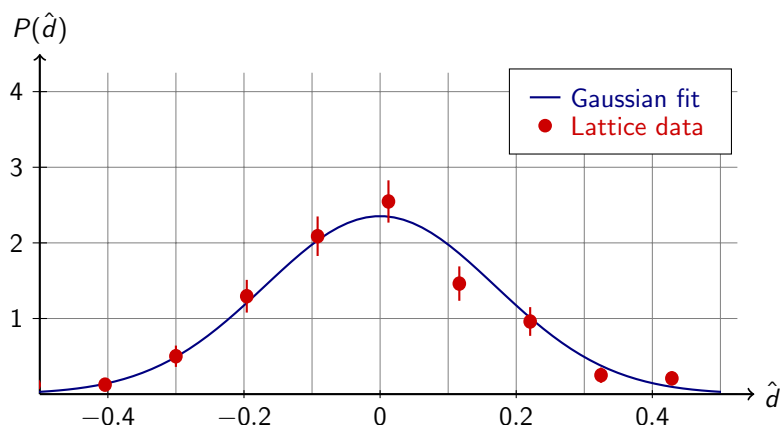


Figure 6. Fit to the distribution of Dirac eigenvalue shifts $P(\hat{d})$ due to imaginary chemical potential $a\mu = 0.01$ with $\hat{d} = d\Sigma V$ in geometry (b_2) . The result is given by $F_{\text{eff}} = 85(5)(2)$ MeV with $\chi^2/\text{dof} = 0.91$. We cite the statistical error (left) as well as the error propagated from the uncertainty in the lattice spacing (right).

From our discussion in section 2 we learned that we can significantly reduce these systematic deviations from RMT by choosing lattice geometry (b_2) instead of (a_2) . In practice this means that we should rotate the lattice by 90 degrees so that we have one large spatial dimension instead of a large temporal dimension. In figure 6 we show the resulting fit for geometry (b_2) with good $\chi^2/\text{dof} = 0.91$ and

$$F_{\text{eff}}^{(b_2)} = 85(5)(2) \text{ MeV}. \quad (4.5)$$

Including finite-volume corrections at NLO⁵ this gives

$$F^{(b_2)} = 80(5)(2) \text{ MeV}, \quad (4.6)$$

⁵If we use finite-volume corrections at NNLO the value is further reduced by 2%.

which agrees within errors with the result from the fit to meson correlators given above.

The value of χ^2/dof in our fits is stable under variations of the bin size. Specifically, if we keep the value of F fixed at the result of eq. (4.5) and vary the bin size from 0.05 to 0.25, the value of χ^2/dof only changes within about one standard deviation of the χ^2/dof distribution, as theoretically expected. Based on the good χ^2/dof in geometry (b_2), we make the hypothesis that in this geometry the non-universal terms in eq. (2.4) are small compared to the universal terms. Using the Cramér-von-Mises criterion to estimate the goodness of our fit in geometry (b_2) we obtain a T -value of 0.097, which implies that there is no reason to reject our hypothesis.

Nevertheless, it is instructive to compute the actual values of the Υ_i which determine the size of the systematic errors. One can show that for $N_f = 2$ the traces of the flavor matrices multiplied by Υ_4 and Υ_5 in (2.4) are identical, and therefore (2.4) only depends on the combination $\Upsilon_4 + \Upsilon_5$. Similarly, (2.4) only depends on the combinations $\Upsilon_6 + \Upsilon_8/2$ and $\Upsilon_7 + \Upsilon_8/2$, up to a term proportional to $\text{Re det}(MU_0^\dagger)$, which for $U_0 \in \text{SU}(N_f)$ reduces to $\text{Re det } M$ and can therefore be neglected for the same reason as for the term proportional to \mathcal{H}_2 .⁶ To compute the Υ_i we use eq. (2.26) of [11] and find

$$\Upsilon_1 = -2\Upsilon_2 = -2\Upsilon_3 = \frac{P_4^r + 4P_5^r}{2F^4V} = \begin{cases} -0.033 & \text{for geometry } (a_2), \\ 0.0069 & \text{for geometry } (b_2), \end{cases} \quad (4.7a)$$

$$\Upsilon_4 + \Upsilon_5 = -\frac{P_4^r + l_4^r}{2F^4V} = -0.017, \quad (4.7b)$$

$$\Upsilon_6 + \Upsilon_8/2 = -\frac{\frac{3}{4}P_4^r + l_3^r + l_4^r}{4F^4V} = -0.0074, \quad (4.7c)$$

$$\Upsilon_7 + \Upsilon_8/2 = \frac{l_7^r}{4F^4V} = 0.0025, \quad (4.7d)$$

see [11] for the notation and the values of $P_{4,5}^r$. To arrive at the formulas for the Υ_i in (4.7) we used the relation between the three-flavor NLO LECs L_i and the two-flavor NLO LECs l_i [23, eq. (11.6)]. To compute the numbers, we took F from (4.6) and $l_7^r = l_7 = 0.005$ [24, eq. (19.21)]. For l_3^r and l_4^r we used the running given in [24, eq. (10.18)] with $M = m_\pi$ and $\mu = V^{-1/4}$ and took $\bar{l}_3 = 3.13$ and $\bar{l}_4 = 4.43$ [25, table X]. For m_π we used the GOR relation, resulting in $m_\pi = \sqrt{2m\Sigma}/F = 110 \text{ MeV}$, where we took Σ from (4.2). For the combinations of the Υ_i in (4.7) the dependence on the renormalization scale drops out so that the numbers given there are scale independent.

Note that in geometry (b_2) all coefficients Υ_i are quite small, which is consistent with our hypothesis that in this geometry the systematic errors due to the non-universal terms are under control. In geometry (a_2) the $\Upsilon_{1,2,3}$ are larger by a factor of ~ 5 . These terms appear to be the reason for the systematic deviations observed in figure 5. As in the case of Σ , the systematic errors can be reduced either by going to a larger volume, or by computing the spectral correlation functions corresponding to the effective action of eq. (2.4) including all non-universal terms. Again, this is beyond the scope of this work.

⁶For fixed topology we have $U_0 \in \text{U}(N_f)$, and the contribution proportional to $\Upsilon_8 \text{Re det}(MU_0^\dagger)$ in (2.4) cannot be neglected. However, for two flavors Υ_8 does not depend on the geometry [11] and is proportional to L_8 , which in turn is a combination of two-flavor LECs and HECs that are all of order $1/(4\pi)^2$ [24].

To estimate the systematic errors on our determination of Σ and F , we recall that the NNLO finite-volume corrections and the systematic deviations from RMT are of the same order in $1/F^2\sqrt{V}$, see the comment after (4.2). The systematic errors due to the NNLO finite-volume corrections can be estimated by comparing the NLO and NNLO values given in [11, table 2], and we obtain $\approx 2\%$ for F and $\approx 5\%$ for Σ . Assuming that for geometry (b_2) the systematic errors due to non-universal contributions in eq. (2.4) are roughly of the same numerical size, we assign a total systematic error of $\approx 4\%$ to F and of $\approx 10\%$ to Σ .

5 Conclusions

We have shown that the geometry dependence of the Dirac eigenvalue distributions at nonzero chemical potential strongly influences the determination of F from RMT fits. Making a judicious choice of the lattice geometry (in our case, by exchanging the temporal axis with one of the spatial axes), this dependence can be significantly reduced such that the systematic error on F is kept under control. This makes the RMT-based method proposed in ref. [7] a useful alternative to other lattice methods.

Our final results for Σ and F obtained from the two-flavor epsilon-regime run of JLQCD are given by

$$\Sigma^{\overline{\text{MS}}}(2\text{ GeV}) = Z_S^{\overline{\text{MS}}}(2\text{ GeV}) \Sigma = (230(1)(14)\text{ MeV})^3, \quad F = 80(5)(5)\text{ MeV}, \quad (5.1)$$

where both values include finite-volume corrections at NLO and $Z_S^{\overline{\text{MS}}}(2\text{ GeV}) = 1.14(2)$ [9, 10]. The left bracket gives the statistical error, the right bracket gives the combined systematic error (including the systematic error of the conversion to $\overline{\text{MS}}$ and the uncertainty in the lattice spacing). The individual systematic errors were added linearly.

To reduce the systematic errors due to the finite volume, it would be beneficial to repeat this study at a larger simulation volume. Also, to eliminate the contamination caused by the non-universal terms, one could attempt, within the framework of the finite-volume effective theory of ref. [11], to calculate Dirac eigenvalue distributions beyond RMT including the systematic deviations at NNLO in the epsilon expansion. Work in this direction is in progress.

Acknowledgments

This work was supported in part by BayEFG and the RIKEN FPR program (CL), the Grant-in-Aid (No. 21674002) of the Japanese Ministry of Education (SH), DFG grant SFB-TR 55 (JB and TW), and a KEK fellowship (TW). The numerical calculations were carried out on the IBM System Blue Gene Solution at the High Energy Accelerator Research Organization under support of its Large Scale Simulation Program (No. 09/10-09).

References

- [1] J. Gasser and H. Leutwyler, *Thermodynamics of chiral symmetry*, *Phys. Lett. B* **188** (1987) 477 [SPIRES].
- [2] E.V. Shuryak and J.J.M. Verbaarschot, *Random matrix theory and spectral sum rules for the Dirac operator in QCD*, *Nucl. Phys. A* **560** (1993) 306 [hep-th/9212088] [SPIRES].
- [3] J.J.M. Verbaarschot and T. Wettig, *Random matrix theory and chiral symmetry in QCD*, *Ann. Rev. Nucl. Part. Sci.* **50** (2000) 343 [hep-ph/0003017] [SPIRES].
- [4] F. Basile and G. Akemann, *Equivalence of QCD in the ϵ -regime and chiral random matrix theory with or without chemical potential*, *JHEP* **12** (2007) 043 [arXiv:0710.0376] [SPIRES].
- [5] C.T. Sachrajda and G. Villadoro, *Twisted boundary conditions in lattice simulations*, *Phys. Lett. B* **609** (2005) 73 [hep-lat/0411033] [SPIRES].
- [6] T. Mehen and B.C. Tiburzi, *Quarks with twisted boundary conditions in the ϵ -regime*, *Phys. Rev. D* **72** (2005) 014501 [hep-lat/0505014] [SPIRES].
- [7] P.H. Damgaard, U.M. Heller, K. Splittorff and B. Svetitsky, *A new method for determining F_π on the lattice*, *Phys. Rev. D* **72** (2005) 091501 [hep-lat/0508029] [SPIRES].
- [8] G. Akemann, P.H. Damgaard, J.C. Osborn and K. Splittorff, *A new chiral two-matrix theory for Dirac spectra with imaginary chemical potential*, *Nucl. Phys. B* **766** (2007) 34 [hep-th/0609059] [SPIRES].
- [9] JLQCD collaboration, H. Fukaya et al., *Two-flavor lattice QCD simulation in the ϵ -regime with exact chiral symmetry*, *Phys. Rev. Lett.* **98** (2007) 172001 [hep-lat/0702003] [SPIRES].
- [10] H. Fukaya et al., *Two-flavor lattice QCD in the ϵ -regime and chiral random matrix theory*, *Phys. Rev. D* **76** (2007) 054503 [arXiv:0705.3322] [SPIRES].
- [11] C. Lehner, S. Hashimoto and T. Wettig, *The ϵ -expansion at next-to-next-to-leading order with small imaginary chemical potential*, *JHEP* **06** (2010) 028 [arXiv:1004.5584] [SPIRES].
- [12] C. Lehner and T. Wettig, *Partially quenched chiral perturbation theory in the epsilon regime at next-to-leading order*, *JHEP* **11** (2009) 005 [arXiv:0909.1489] [SPIRES].
- [13] P.H. Damgaard, T. DeGrand and H. Fukaya, *Finite-volume correction to the pion decay constant in the ϵ -regime*, *JHEP* **12** (2007) 060 [arXiv:0711.0167] [SPIRES].
- [14] G. Akemann, F. Basile and L. Lellouch, *Finite size scaling of meson propagators with isospin chemical potential*, *JHEP* **12** (2008) 069 [arXiv:0804.3809] [SPIRES].
- [15] P.H. Damgaard and H. Fukaya, *The chiral condensate in a finite volume*, *JHEP* **01** (2009) 052 [arXiv:0812.2797] [SPIRES].
- [16] J.C. Osborn, *Universal results from an alternate random matrix model for QCD with a baryon chemical potential*, *Phys. Rev. Lett.* **93** (2004) 222001 [hep-th/0403131] [SPIRES].
- [17] P.H. Damgaard, U.M. Heller, K. Splittorff, B. Svetitsky and D. Toublan, *Microscopic eigenvalue correlations in QCD with imaginary isospin chemical potential*, *Phys. Rev. D* **73** (2006) 105016 [hep-th/0604054] [SPIRES].
- [18] T. Wilke, T. Guhr and T. Wettig, *The microscopic spectrum of the QCD Dirac operator with finite quark masses*, *Phys. Rev. D* **57** (1998) 6486 [hep-th/9711057] [SPIRES].
- [19] S.M. Nishigaki, P.H. Damgaard and T. Wettig, *Smallest Dirac eigenvalue distribution from random matrix theory*, *Phys. Rev. D* **58** (1998) 087704 [hep-th/9803007] [SPIRES].

- [20] G. Akemann and P.H. Damgaard, *Individual eigenvalue distributions of chiral random two-matrix theory and the determination of F_π* , *JHEP* **03** (2008) 073 [[arXiv:0803.1171](#)] [[SPIRES](#)].
- [21] J.C.R. Bloch and T. Wettig, *Overlap Dirac operator at nonzero chemical potential and random matrix theory*, *Phys. Rev. Lett.* **97** (2006) 012003 [[hep-lat/0604020](#)] [[SPIRES](#)].
- [22] JLQCD collaboration, H. Fukaya et al., *Lattice study of meson correlators in the ϵ -regime of two-flavor QCD*, *Phys. Rev. D* **77** (2008) 074503 [[arXiv:0711.4965](#)] [[SPIRES](#)].
- [23] J. Gasser and H. Leutwyler, *Chiral perturbation theory: expansions in the mass of the strange quark*, *Nucl. Phys. B* **250** (1985) 465 [[SPIRES](#)].
- [24] J. Gasser and H. Leutwyler, *Chiral perturbation theory to one loop*, *Ann. Phys.* **158** (1984) 142 [[SPIRES](#)].
- [25] RBC-UKQCD collaboration, C. Allton et al., *Physical results from 2+1 flavor domain wall QCD and SU(2) chiral perturbation theory*, *Phys. Rev. D* **78** (2008) 114509 [[arXiv:0804.0473](#)] [[SPIRES](#)].